Mathematical Geophysics
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Lecture - 13

Hello everyone, welcome to the SWAYAM NPTEL course on mathematical geophysics. We continue with module number 3, mathematical modeling part 1. This is the third lecture, gravitational field continued. In this lecture, the concepts covered are related to the gravitational field and the concepts we discussed in the previous lecture. Here, the components of this lecture are the gravitational field of an elementary mass, a spherical shell, and inside and outside the spherical shell.

So, this forms the total content of the present lecture. So, let us begin. First, we will have a look at the fundamental concept of the gravitational field of an elementary mass. In previous lectures, we have seen Newton's laws of gravity and Newton's laws of attraction, which were used for determining the gravitational force between two points. Here, we are looking at elementary masses.

The gravitational field for a point mass located at Q at the point P is given by:

$$\mathbf{G}_P = -G\frac{M_1}{L_{QP}^2}\frac{\mathbf{L}_{QP}}{L_{QP}}$$

This is directed along the line joining Q and P locations. Note that Q and P are position vectors.

So in XYZ plane, which is shown here, let us consider a point Q as the origin (0, 0). This is the origin of the Cartesian coordinate system which has been used in this diagram. Thus:

$$\mathbf{L}_{QP} = \mathbf{L}_{OP} = x\hat{\imath} + z\hat{k}$$

The length L_{OP} can also be given by:

$$L_{OP} = \sqrt{x^2 + z^2}$$

This is nothing but $\sqrt{AB^2 + OA^2}$. This is also equal to $\sqrt{x^2 + h^2}$, where h is the gap between the two plates. Thus, the gravitational field can be simplified as:

$$\mathbf{G}_P = -G\frac{M}{x^2 + h^2}\frac{x\hat{\imath} + z\hat{k}}{\sqrt{x^2 + h^2}}$$

Note that this gravitational field is a function of x and z spatial coordinates. This makes it a two-dimensional field. The variation of this gravity field along the spatial coordinates are:

$$\frac{x\hat{\imath}}{(x^2 + h^2)^{3/2}} + \frac{z\hat{k}}{(x^2 + h^2)^{3/2}}$$

These are the dependencies of this gravitational field, which is depending on x and z coordinates. This is clearly shown in the following expressions.

The g_x is the x-directional component of the gravitational field, which is:

$$g_x = -G\frac{Mx}{(x^2 + h^2)^{3/2}}$$

Similarly, the axial gravitational field component g_z equals:

$$g_z = -G\frac{Mh}{(x^2+h^2)^{3/2}}$$

Next, having understood the gravitational field of an elementary mass, we apply it to get the gravitational field caused by a thin spherical shell. This thin spherical shell has a density σ. The thin spherical shell is shown as a grey-shaded area. The inner shell, shown by the deep greyish colour, is hollow.

The density σ is an area density, which means it is the density over which mass per unit area is distributed. Now, the field inside the shell at any point P, which is caused by two elementary masses, is discussed below. Now, have a look at this diagram, which makes things clearer. dS_1 and dS_2 are the elementary areas on this outer shell. Now, if we multiply the density by this elementary area, we will get the elementary mass.

The elementary masses are given by σdS_1 and σdS_2, respectively. These masses are located at positions Q_1 and Q_2. We are considering the gravitational field at point P. This point P is located inside this thin spherical shell, and this point P lies on the line joining Q_1 and Q_2. Now, if Q_1 and Q_2 are two points on the circumference of this spherical body, then P, which lies at the midpoint of Q_1 and Q_2, is equidistant from both Q_1 and Q_2.

This means that $\mathbf{L}_{PQ1}$ is equal and opposite in direction to $\mathbf{L}_{PQ2}$. Following the previous slide, we can get the elementary gravitational field caused by these elementary masses as:

$$d\mathbf{g}_1 = -G\frac{\sigma dS_1}{L_{PQ1}^3}\mathbf{L}_{PQ1}$$

Similarly:

$$d\mathbf{g}_2 = -G\frac{\sigma dS_2}{L_{PQ2}^3}\mathbf{L}_{PQ2}$$

Now, since P lies at the midpoint of the Q_1Q_2 line, it makes a right angle with the origin, which is the center of the sphere. The Q_1Q_2 line is shown as a blue dotted-dashed line, and OP makes a right angle with this dashed-dotted line shown in blue color, since P is the midpoint of Q_1Q_2. Thus, we have:

$$\frac{dS_1}{L_{PQ1}^3} = \frac{dS_2}{L_{PQ1}^3}$$

We can also write this in terms of ω, which is the solid angle. So, ω is the solid angle subtended by dS_1 or dS_2 on P. Since dS_1 is an elementary surface area, it is more appropriate to call this an elementary solid angle.

We recall that a solid angle is given by area divided by distance squared. Here, the area is the elementary area dS_1, and the distance between the points Q_1 and P is L_{PQ1}^2. Also:

$$L_{PQ} = r\cos\alpha$$

where α is shown here. It is the angle between the radial vector or the radial line shown in red and the blue line. So, the radial line R has its component given by:

$$R\cos\alpha = L_{PQ}$$

So we can make use of these relations to simplify and express the equality in terms of solid angle. Thus, we have the equation:

$$\frac{dS_1}{L_{PQ1}^3} = \frac{dS_2}{L_{PQ2}^3} = \frac{d\omega}{r\cos\alpha}$$

This can be used to find the field in this diagram. Thus, the total field is obtained by taking an integral over the entire spherical shell. The gravitational field for a spherical shell of radius r is obtained by integrating over the elementary fields, which is given by:

$$\mathbf{g}_R = \int d\mathbf{g}_1 + \int d\mathbf{g}_2$$

Substituting the relations, we get:

$$d\mathbf{g}_1 = -G\sigma \frac{dS_1}{L_{PQ1}^3} \mathbf{L}_{PQ1}$$

$$d\mathbf{g}_2 = -G\sigma \frac{dS_2}{L_{PQ2}^3} \mathbf{L}_{PQ2}$$

Now, using the relation with the solid angle expression, we can substitute this to obtain:

$$\mathbf{g}_R = -\int \frac{d\omega}{r\cos\alpha} \left(\mathbf{L}_{PQ1} + \mathbf{L}_{PQ2}\right)$$

Since we have the relation that $\mathbf{L}_{PQ1} = -\mathbf{L}_{PQ2}$, we get the bracketed term equals 0. This renders the gravitational field:

$$\mathbf{g}_R = 0$$

for a thin spherical shell with density σ.

So we come to the nice conclusion that the total field caused by two elementary masses is equal to zero. Now, what are these two elementary masses? These are the elementary masses located at Q_1 and Q_2. Now, for any point such as P, the entire shell can be obtained as a combination of such pairs of points Q_1, Q_2. For example, these two points.

These two points are equal and opposite to the point which is shown as the midpoint. So like this, we can understand that if we consider the spherical shell as a system of such pairs, we can conclude that the field inside a uniform spherical shell is zero for all the points. This occurs because any point, if you consider for example this, it will have two points which will make the gravitational field zero at this point. Thus, the gravitational field caused by a thin spherical shell inside it is zero.

Next, we go to discuss the gravitational field which is caused by a thin spherical shell of density σ, and we consider the point P located outside this shell instead of inside it. Have a look at this diagram. This diagram shows a thin spherical shell with origin O, the radius R, and the P point is located outside this shell. The P point is at a distance of L from a ring which is shown here.

This is a ring which can be obtained from the spherical shell shown by dotted line. Now this ring has a radius A. Now this ring, its axis which is OP makes an angle ψ with the radial vector which is R joining the origin to the rim of this spherical shell ring. We are interested in calculating the field at point P which is given by the elementary field $d\mathbf{G}$. The elementary field $d\mathbf{G}$ is the first step towards calculation of the total gravitational field. Now this is:

$$d\mathbf{G} = -G \frac{\sigma R d\psi\, dl}{L^3} \mathbf{L}$$

σ is the density, whereas the elementary surface area is dS, which is equal to $Rd\psi\, dl$. Capital L is the distance from the mass to the observation point. The elementary area is obtained by multiplying $Rd\psi$ by an elementary length dl. This is the area on this ring. The radial component of the gravitational field is given by taking the cosine of α multiplied by dG, which is the magnitude of the elementary gravitational field as obtained above.

In this diagram, capital R represents OP. And $R\cos\psi$ represents this region. Hence:

$$\cos\alpha = \frac{R - r\cos\phi}{L}$$

This is simplified as dG multiplied by:

$$\frac{R - r\cos\phi}{L}$$

This is further simplified to:

$$-G\sigma \frac{d\psi\, dl}{L^2}(R - r\cos\phi)$$

Now, this statement gives the crux idea of deriving the gravitational field in total. All the elements of the ring are located at the same distance from the observation point, which means we can only consider the $\cos\alpha$ component, and the vertical components cancel out if we consider the sum over the entire ring. Thus, we have the expression for the radial component due to the ring mass as:

$$dg_r = -G\sigma R d\psi \frac{2\pi x}{L^3}(R - R\cos\psi)$$

where $x = R\sin\phi$ and:

$$L = \sqrt{r^2 + R^2 - 2Rr\cos\psi}$$

Now, if we replace capital L and $x = R\sin\phi$ in the above expression and integrate, we can obtain the field caused by all the masses of the shell. This is because ψ will cover the entire sphere in terms of small thin rings as it goes from 0 to π.

This is $\psi = 90°$ and a ring like this is $\psi = 0$. So from this point, if we consider ψ moving from 0 to π, we will consider all the rings which will cover the entire sphere. Thus, we are integrating from 0 to π:

$$-G\sigma 2\pi r^2 \int_0^\pi \sin\psi \frac{R - r\cos\psi}{L^3}\, d\psi$$

This integral is evaluated as follows:

$$-G\sigma 2\pi r^2 (RI_1 - rI_2)$$

where:

$$I_1 = \int_0^\pi \frac{\sin\psi}{(A - B\cos\psi)^{3/2}}\, d\psi$$
$$I_2 = \int_0^\pi \frac{\sin\psi\cos\psi}{(A - B\cos\psi)^{3/2}}\, d\psi$$

Now, if we consider:

$$A = r^2 + R^2$$
$$B = 2Rr$$

for simplification, we get:

$$I_1 = \frac{2}{R(r^2 - R^2)}$$

$$I_2 = -\frac{2}{rR^2} + \frac{2}{r(R^2 - r^2)}$$

Upon substitution into this expression, which we had earlier, we can obtain the gravitational field as:

$$\mathbf{g} = -G\frac{4\pi\sigma r^2}{R^2}\hat{R}$$

This equals:

$$\mathbf{g} = -G\frac{M}{R^2}\hat{R}$$

So, this is the gravitational field at a point P, which is located outside the spherical shell with density σ. This equals:

$$\mathbf{g} = -G\frac{M}{R^2}\hat{R}$$

In this lecture, we have discussed the gravitational field due to an elementary mass and a spherical shell. We conclude that an elementary mass provides a foundational understanding at a fundamental level of the gravitational interactions among various points and masses, and this is used in simple and large-scale problems. The spherical shell demonstrates a geometry that is very useful in the context of geophysical systems, and the gravitational field due to such spherical shells upon masses which are located inside and outside are very important for geophysical studies. It provides a very important symmetrical study for understanding hollow objects and layered Earth systems, as the Earth is made up of various layers. The spherical shell and the corresponding gravitational effects are very useful. One can refer to this reference for more details. Thank you.