Mathematical Geophysics
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Hello everyone, welcome to the SWAYAM NPTEL course on mathematical geophysics. We continue with the third module, Mathematical Modeling Part 1. Today, we are going to discuss Lecture Number 2, which is the gravitational field. In this lecture, we will cover some very basic elementary concepts related to the gravitational field. The gravitational field, as we have discussed in earlier lectures, forms one of the most important fields in geophysics.

Thus, the following components will be discussed in the present lecture. First, Newton's laws of motion, Newton's law of universal gravitation, and then Newton's law of attraction. Then, we will have a discussion comparing Newton's laws of motion and gravitation. Then, we will discuss the concept of elementary mass. Lastly, we will discuss various applications of the gravitational field as inherent in geophysical studies.

So, let us begin. First, Newton's laws of motion. Newton's laws of motion are a familiar concept for many. Newton's laws of motion describe the motion of an object, the forces acting on that object, and the resulting interaction between the body and the forces. The fundamental laws are three in number.

First, the law of inertia states that an object will remain at rest in the absence of any external forces and will continue to move at a constant velocity unless an external force disturbs it. The second law is the law of force and acceleration. The acceleration of an object depends on the mass of the object and the net force acting on it. This is mathematically represented as:

$$\vec{a} = \frac{\vec{F}}{m}$$

In other words:

$$\vec{F} = m\vec{a}$$

This is the common form of Newton's second law, as we have used in many previous applications and earlier studies. The third is the law of action and reaction, which states that for every action, there is an equal and opposite reaction. Mathematically, it can be represented as:

$$\vec{F}_{12} = -\vec{F}_{21}$$

This can be read as the force acting on 1 by agent 2. It equals the negative of the force acting on agent 2 by agent 1. Mathematically, this can be read as the force acting on body 1 by agent 2 equals minus the force acting on body 2 by agent 1. Thus, Newton's laws of motion, a familiar concept, are reviewed here for completeness. What is the focus?

Next, we discuss the focus and applications. Newton's laws of motion generally apply to all types of motion, such as the motion of cars, projectiles, falling objects, etc. This forms the basis of classical mechanics. Next, we consider Newton's law of universal gravitation.

Newton's law of universal gravitation forms the fundamental basis of gravity and other associated field studies in geophysics. Newton's law of attraction explains the gravitational attraction between two masses located at a particular distance. For example, have a look at this picture. A and B are two locations where the centers of masses, M and m, are located.

O is the origin. R is the distance between point A and point B, which are the centers of the masses. R is the distance between point A and point B, which are the centers of the masses. Newton's law of universal gravitation states that any two objects with masses attract each other. The force of attraction is proportional to the product of the masses and inversely proportional to the distance between their centers.

This is mathematically represented as:

$$\vec{F} = -G\frac{Mm}{R^2}\hat{R}$$

As the force is directed along the line joining the centers, that direction is denoted by the radial vector $\vec{R}$. The $\vec{R}$ vector is directed along the centers of the two points. $\vec{F}$ is the gravitational force.

G is the gravitational constant, which is universal for all materials. $G = 6.67\times10^{-11}$ m^3 kg^{-1} s^{-2}. M and m are the two masses, and R is the distance between them. This is the very familiar law of gravitation proposed by Newton for two objects which have their centers at a distance of R between them. This law specifically deals with gravitational interactions.

And this law has been used to explain planetary orbits, satellite motion, tidal forces, and many other applications in planetary sciences as well as geophysical studies. Next, we look at Newton's law of attraction. Now, Newton's law of attraction combines two forces. As planetary bodies have mass, but they also undergo rotation. So, the net force at any point is the combination of the gravitational force and the force generated through rotation.

Thus, the gravitational method is a study of the distribution of Newton's attraction force, which is $\vec{F}$, caused by all masses of the Earth under the influence of rotation. One part of Newton's attraction force is the uniform motion around the rotational axis z. The other part is generated by the weight itself. So, the total Newtonian force is given by the $\vec{F}$ vector, which is a combination of centripetal force and the force generated by gravity, that is, the weight of the object. The centripetal force is directed toward the rotation axis.

In this figure, we can see this clearly. The z-axis is the axis of rotation, which occurs at a rate of ω. The centripetal force on any object is directed toward the rotational axis and perpendicular to it. This is the direction of the centripetal force. Note that the centripetal force does not act in the same direction as the gravitational force, which is along the blue line.

This is the total Newtonian force. P is the weight of the object. Vectorially, this diagram can be represented as:

$$\vec{BO} = \vec{AO} + \vec{BA}$$

The centripetal force $\vec{F}_A$ can be calculated and removed very accurately, and the remaining force, which is P, is used for further geophysical analysis. Thus, Newton's law of attraction encompasses the effect of rotation on top of gravitational attraction for geophysical applications.

Let us have a detailed comparison between the fundamental effects of Newton's laws of motion and Newton's law of gravitation. Newton's laws of motion and the law of gravitation are compared on the basis of four characteristics. One, the general focus. Newton's laws of motion deal with general forces and interactions between them. They are not restricted to gravitational fields only.

They include electromagnetic forces, frictional forces, and all other types of forces, which may be conservative or non-conservative in nature. However, Newton's law of gravitation deals with gravitational force only. As far as Newton's laws of motion are concerned, they govern the motion of various bodies, which may or may not follow conservative field characteristics. But Newton's law of gravitation only deals with the conservative field, which is the gravitational field. Next is the mathematical formula.

The mathematical representation of Newton's laws of motion is elegant and general:

$$\vec{F} = m\vec{a}$$

The acceleration can be generated by any force field, such as a gravitational field, electromagnetic field, frictional field, or any other force field, and the action on the mass m will be given by this equation. Newton's law of gravitation is restricted to gravitational interactions only, where:

$$\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{r}$$

For Newton's laws of motion, the direction of the force and acceleration is obtained from the vector sum of all applied forces.

Next, we consider the nature of the law. The Newton's law of motion explains motion due to forces, whereas the Newton's laws of attraction explains the attraction between masses. So, the nature of Newton's law of motion is more general compared to the Newton's law of attraction. The applicability of Newton's laws of motion is vast. It covers the motions of vehicles, falling objects, any microscopic object undergoing classical mechanics.

The Newton's laws of attraction covers the motion of planets, weight, tides, etc. Next, we look into the concept of elementary force. Consider two elementary masses M_1 and M_2 positioned at Q and P respectively. The distance between these two points is L_{PQ}. Consider an elementary force $d\vec{F}$ acting at a point P.

The masses are m_1 for Q and m_2 for P. The dimensions of volumes of two elementary masses are much smaller than the distance between them. Hence, they can be understood as point masses. The elementary masses M_1 acts on the elementary mass M_2 with the force $d\vec{F}_p$ along the unit vector $\hat{L}_{QP}$. This unit vector is directed from P to Q and is proportional to the product of the two and the force is proportional to the product of elementary masses and inversely proportional to L_{PQ}^2.

Thus, Newton's law of attraction gives us this relation: the elementary force is proportional to $\frac{M_1 M_2}{L_{PQ}^2}$. The proportionality constant is G, and since the masses attract each other, there is a minus sign. Note that to indicate the direction, the $\vec{L}_{PQ}$ vector is introduced, with the denominator now being L_{PQ}^3. G is the universal gravitational constant and the proportionality constant for the force and the masses, where $G = 6.67 \times 10^{-11}\,\mathrm{N} \cdot \mathrm{m}^2/\mathrm{kg}^2$.

Now, the properties of this elementary force. The mass M_2 exerts an elementary force on M_1, given by $d\vec{F}_Q$, which is $-G\frac{M_1 M_2}{L_{PQ}^2}\hat{L}_{PQ}$. Note the change in notation from $\vec{L}_{QP}$ to $\vec{L}_{PQ}$, as $d\vec{F}$ is now related to Q. $\vec{L}_{PQ}$ and $\vec{L}_{QP}$ are equal in magnitude but directed antiparallel. Hence, the forces are also antiparallel to each other.

These forces represent the action of one elementary mass on another and vice versa. Thus, the elementary forces between the elementary masses are antiparallel forces. Next, we consider the various applications of the gravitational field in geophysics. First, we examine the concept of the gravitational field for an elementary mass. This is required to understand the fundamental characteristics of the gravitational field in geophysical applications.

The force $d\vec{F}_P$ is proportional to the elementary mass m_{2P}. This is subject to the action of this force. Next, we discuss the concept of the gravitational field for an elemental mass. This concept of gravitational field for an elemental mass is very useful to understand the fundamentals of the gravitational field theory in geophysical applications. We take help of the same picture which we had discussed in previous slide.

The force $d\vec{F}_P$ is proportional to the elementary mass m_{2P}. Thus we consider the ratio $d\vec{g}_P$ which is given by:

$$d\vec{g}_P = \frac{d\vec{F}_P}{m_{2P}}$$

This is also given by $-G\frac{m_{2P}}{L_{QP}^2}\hat{L}_{QP}$. Thus the $d\vec{g}_P$ acts somewhat like a potential. This potential is nothing but the gravitational field at point P. This gravitational field is caused by the elementary mass m_{2P}.

These are the field lines of the gravitational field around M_{2P}. Next, we consider the concept of gravitational field for a distribution of masses. The distribution of masses is a very important concept in geophysics as the planet and other bodies in the solar system are not homogeneous material. A planet or any other geophysical rock samples is composed of various different type of elementary and compounds, various different kinds of elements and compounds which have different densities. Hence the masses have various and hence the distribution of masses.

Thus assuming that instead of elementary masses we have a volume V which is nothing but a distribution of masses. These are the masses which construct the volume V. The total mass can be obtained as an integral:

$$M = \int_V \rho(Q)\, dV$$

The gravitational field caused by the elementary mass is given by $d\vec{g}_P$. Integrating the gravitational field for all the masses using the integration gives the total gravitational field caused by the distribution of masses.

Thus we have the gravitational field $\vec{G}$ due to the point P as:

$$\vec{G} = -G \int_V \frac{\rho(Q)}{L_{PQ}^2} \hat{L}_{QP}\, dV$$

This is integrated over the entire volume V. So this gives the overall gravitational field for a distribution of masses instead of a point mass. Using this concept, the gravitational field of subsurface elements or subsurface distribution of various masses can be obtained for geophysical applications. For example, if this is the surface

And these are various point masses. The combination can be understood as a distribution of mass. If the cross mass is the ambient low density material with the square denoting high density material, the combined effect of all this point elementary masses results in the gravitational field due to a distribution of masses. This is extremely important for various geophysical exploration applications which we will consider in further lectures. Thus, in this lecture we obtain the following conclusions.

The gravitational field, in particular, can be used to study Earth's internal structure and composition, as well as various geological processes such as mountain building, earthquakes, and volcanic activity. These processes somewhat give a distribution of masses that vary with space and time. Hence, this leads to a gravitational field that is spatially complicated and varies temporally. The interaction of these individual components, along with their masses, results in complicated gravitational fields. Although the processes are very slow, the gravitational field does not change rapidly.

The gravitational field is also affected by global environmental changes, sea level rise, and ice sheet dynamics, which are very exciting topics in geophysics related to the gravitational field. One can refer to the following book for further understanding. Thank you.