Mathematical Geophysics

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Week - 03

Lecture - 11

Hello, welcome to the SWAYAM NPTEL course on Mathematical Geophysics. Today, we start module number 3, named Mathematical Modeling Part 1. Today, we will begin with the first lecture, System of Equations, of this module. In this lecture, we cover the system of linear equations in general. The concepts covered under this system of linear equations are divided into five components.

First, we will look at the general formal representation of the set of linear equations. Next, we will look at three methods of solution for linear equations which are commonly used. First, the graphical method. Second, the substitution method, and third, the matrix method. Then, we will look into applications in geophysical studies of linear equations and their solutions.

So, let us begin with the set of linear equations. A set of linear equations is a set of two or more linear equations which are algebraic in nature involving the same set of variables. For example, this represents a set of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3. \end{cases}$$

By linear equations, we mean that the maximum exponent of all the variables is equal to 1. In this set, the variables are x_1, x_2, and x_3.

The coefficients are given by a_{11}, a_{12}, etc. The system of linear equations with three variables x_1, x_2, x_3, nine coefficients, and three on the right-hand side is as given here. This is the general mathematical representation of a set of three linear equations. The set of linear equations for any number of variables can be obtained by further extension. For example, if we consider a set of four equations, then we would have four equations with four variables.

The variables would be x_1, x_2, x_3, and x_4, and the coefficients will range from a_{11} to a_{44} and b_1 to b_4. Similarly, any number of equations that are linear in nature can be combined to form a set of linear equations. This set of linear equations will be used to model various linear phenomena in geophysics. There are various methods to solve a set of linear equations. The most common of them are the graphical method, substitution method, matrix method, and elimination method.

We will have a detailed discussion of these methods next. First, the graphical method. The graphical method is designed to be visually appealing and gives an insight into the solution of a system of equations. For example, consider this as a set of two linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases}$$

There are two variables, x_1 and x_2, and four coefficients, a_{11} and a_{12}, a_{21} and a_{22} and two coefficients on the right-hand side b_1 and b_2. Since these equations are represented as a function of two variables, these equations geometrically represent a set of two lines. These two lines lie on a two-dimensional plane. For example, in the Cartesian coordinate system, we can represent these two lines as given in the following diagrams. Consider this diagram.

Here, x_1 and x_2 are the two coordinates with O as the origin. m_1 and m_2 are the slopes of the two lines, which are given by the set of two linear equations. P is the point of intersection. Now, how these two lines can be solved to obtain the point P is explained here. The condition for these two lines to intersect is given here.

The coefficients and the ratio are arranged such that:

$$\frac{a_{11}}{a_{21}} \neq \frac{a_{12}}{a_{22}}.$$

This condition has to be followed so that a solution exists. The solution is unique and is given as $P(x_1, x_2)$. Now, consider the next possible case. If the coefficients are such that the ratios given by:

$$\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} \neq \frac{b_1}{b_2},$$

then the solution does not exist. This is because the equations represent two parallel lines which do not intersect at any point. Next, the condition given by:

$$\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} = \frac{b_1}{b_2},$$

renders both equations as identical. Thus, the two equations are similar and represent the same single line. So this indicates that the coincident line represents an infinite number of solutions.

Thus, we understand from these details that for a solution to exist, the condition to be followed is given by this ratio, and if the condition fails to follow that, two situations arise: either there are parallel lines which lead to no solution, or there are coincident lines which have infinitely many solutions. Thus, the system of linear equations in general also displays a similar behavior where, under certain conditions, only a unique solution exists, while for various other conditions, there may be no solution or infinitely many solutions. In general, no solution exists where one equation can be represented as a multiple of the next equation. And when all the equations become incidentally equal, similar, or identical, then infinitely many solutions can exist.

Next, we move on to the substitution method. The substitution method is more commonly used as a mathematical way to analyze and get the solution of linear equations. The substitution method is also used in many applications to get solutions. Let us consider a system of two linear equations which we have used earlier:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases}$$

From equation 2, it can be understood that we can simplify one of the variables. Here, x_1 is considered as the independent variable, while x_2 is considered as the dependent variable. x_2 is simply expressed in terms of x_1 as:

$$x_2 = -\frac{a_{21}}{a_{22}}x_1 + \frac{b_2}{a_{22}}.$$

Thus, we can substitute equation 3 into equation 1 to get a single equation in terms of one variable. Once we have this simplified form of the equation, it can be easily seen that the solution x_1 can be obtained by suitable algebraic additions, subtractions, divisions, and multiplications. Once the solution for the first variable x_1 is obtained, the second variable can be readily obtained using the solution of the first variable.

This gives the second variable solution. We can have a quick look at one of the simple examples. Consider these two sets of linear equations:

$$\begin{cases} x_1 - 2x_2 = 8, \\ x_1 + x_2 = 5. \end{cases}$$

From the second equation, we get $x_1 = 5 - x_2$, and this, when substituted into equation 1, results in $x_1 - 2x_2 = 8$.

This implies $5 - x_2 - 2x_2 = 8$. This gives $-3x_2 = 3$, which results in the solution $x_2 = -1$. Thus, using $x_2 = -1$, x_1 can be obtained as 6. Thus, we have both the solutions as 6 and -1 for x_1 and x_2, respectively. The substitution method is very suitable when the number of equations is somewhat smaller.

For example, the substitution method also works fine for 3 or up to 4 variables. For a higher number of variables, like 10 or more, the substitution method becomes very tedious and, instead of providing insights into the solution, it becomes very lengthy and prone to mistakes. However, the substitution method gives very useful insights as long as the number of variables is not too large. For example, even up to five variables, the substitution method can provide insights into the dependencies of various variables in the equations and the behavior of the solutions. For example,

Looking at the expression for x_2 or x_1, for that matter, we can understand the coefficients and the relation with other constants in framing the solution x_2. For example, x_2 can be considered as the sum of two terms. The first term is a bit complicated, but the second term is wholly dependent on only two coefficients, that is b_2 and a_{22}. The relative importance of each of these terms may have significant physical interpretations if the set of equations represents a physical phenomenon. Thus, the substitution method is advisable to be used when the number of variables is smaller.

If the number of variables is higher, we should look into the next method, which is the matrix method. The matrix method is nothing but the substitution method but in a more elegant and artistic mathematical way. The matrix is a rearrangement, and constructing the matrix is a construction of the arrangement of various numbers. For the set of two linear equations which we have discussed earlier, the matrix form reads as equation number 3. Note that the matrix involves the coefficients a_{11}, a_{12}, etc.

As the left-hand side coefficient matrix for the variables x_1 and x_2. The variables are arranged in a column matrix, and the right-hand side is also arranged in a column matrix. Thus, we denote $\mathbf{A}$ as the coefficient matrix, $\mathbf{X}$ as the variable vector, and $\mathbf{B}$ as the right-hand side vector. From this equation, the matrix form can be concisely represented as $\mathbf{AX} = \mathbf{B}$. Which implies the solution $\mathbf{X}$ can be obtained as $\mathbf{A}^{-1}\mathbf{B}$. Here, $\mathbf{A}^{-1}$ is the inverse matrix of $\mathbf{A}$.

It is well known that if the determinant of $\mathbf{A}$ is not equal to zero, then the $\mathbf{A}^{-1}$ can be obtained as the adjoint of $\mathbf{A}$ by the determinant of $\mathbf{A}$. The adjoint of $\mathbf{A}$ can be obtained from the cofactor matrix. To calculate the adjoint of $\mathbf{A}$, we require the cofactor matrix. The transpose of this cofactor matrix gives the adjoint of $\mathbf{A}$. For example, interchanging the diagonal elements with a minus sign in the off-diagonal elements gives the cofactor matrix. The transpose of the cofactor matrix is the adjoint of $\mathbf{A}$. This gives the adjoint of $\mathbf{A}$. Next, we look at the various applications of a set of linear equations in geophysics. We will look at the specific example of seismic tomography.

Seismic tomography aims to determine the velocity structure of the Earth, which is inside the surface of the Earth. Let us consider the internal structure of the Earth as a cut section, as shown in the sketch. On the top lies the surface, below which lies the crust. Below that is the mantle, followed by the liquid outer core and the solid inner core. Seismic tomography aims to determine the velocity structure at any depth in this interior structure.

For example, These two points denote a location in the interior of the structure where seismic tomography may be employed to get the velocity of seismic waves that can pass through these points. First, we will introduce the concept of slowness and travel time. The travel time indicates the time taken for a seismic wave to cross a certain region from the source of the earthquake or an explosion, for example, to the observation point. This shows the propagation of seismic waves through the Earth's structure to the observatory.

Now, the travel time would depend on the velocity of the seismic waves in the material that lies between the source and the observation point. So, the travel time is the total time taken for the wave to emanate from the source and reach the observatory.

Next is the concept of slowness, which is $s = \frac{1}{v}$, which is nothing but the reciprocal of the velocity of the material, the velocity of seismic waves through the material between the source and the observatory. Thus, the travel time is given by the summation of d_{ij} multiplied by s_{ij}. Here, t_i is one of the components of the travel time vector.

Whereas D_{ij} is one of the elements of the matrix of path distance. S_{ij} is one of the components of the slowness matrix, which is nothing but the reciprocal of the individual components corresponding to the elements of the velocity matrix. Thus, what we have to understand from here is how to represent this travel time expression, which depends on the matrix of path distance and the slowness vector, in the form of a set of linear equations. This is the matrix representation of the set of linear equations. It can also be written as $\mathbf{Ds} = \mathbf{t}$, where $\mathbf{D}$ is the matrix, $\mathbf{s}$ is the variable which is the slowness vector, and $\mathbf{t}$ is on the right-hand side which is the travel time.

The matrix of path distance and the travel time are known quantities, whereas the slowness vector is a variable that must be determined. The slowness vector can be determined using the matrix method for solving the above linear set of equations, and from the solution, we obtain the values of $\mathbf{s}$, we can interpret the various properties that are inherent in the propagation of seismic waves through the subsurface structure. For example, a higher velocity, which means that s is slow, indicates a dense or hard rock, whereas a lower velocity or effectively a high slowness indicates less dense rock, like fracture zones, etc. So, this example illustrates the use of a set of linear equations in a specific example, such as seismic tomography, which results in the solution for the slowness vector.

Thus, this discussion on the set of linear equations indicates that such equations are widely applicable in various geophysical domains. Although we have prominently used seismic studies as an illustration. Besides that, geodetic modeling is another sphere of geophysical applications where a set of linear equations is widely used. The set of linear equations provides a structured approach to interpret observational data and infer critical subsurface properties. Advanced computational tools have the ability to solve a large set of linear equations.

Even if they are complex, the coefficients vary in their magnitude and other complications. These computational tools can contribute to more precise geophysical and geological models. Today, with the last set of geophysical data in hand, we can have the right-hand side like B_1, B_2, B_3, etc. and various theoretical models can give us the matrix $\mathbf{D}$. From this, we can infer important and necessary details by solving the geophysical phenomena and its model using the solution method for linear equations. Thus, the set of linear equations and their solution with appropriate representations form the basis of much geophysical understanding.

One can refer to the following references for a better understanding of sets of linear equations. Thank you.