Mathematical Geophysics

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Week - 02

Lecture - 10

Hello everyone, welcome to the SWAYAM NPTEL course on mathematical geophysics. We will continue with module 2, fundamental theorems. This is the fifth lecture in this module, titled Green's function and theorem. In this lecture, the following concepts are covered. The overall idea of Green's function and related theorem.

The lecture is divided into five components. First, the basic concept of Green's function, followed by Green's theorem. Then we will look at how to derive Gauss's theorem and Stokes's theorem from the generalized Green's theorem. Finally, we will look at various applications of Green's function and theorem in geophysical studies. So let us begin.

What is the basic concept of Green's function? Physically, the Green's function encapsulates the response of a physical system to a unit source. It relates a point source to the field around it. It provides the fundamental behavior of a system as a response to a unit impulse. For example, if this is a point source, the field may diverge, converge, or circulate around this point.

The behavior of the field With respect to a unit source is captured by the Green's function. Mathematically, the Green's function can be denoted as g(x, x'). The marking as a superscript of x is called prime, and it denotes the location of the source. The coordinate x is the position vector of any point.

The Green's function g(x, x') is an impulsive response to any inhomogeneous linear differential operator L. When such a linear differential operator operates on the Green's function, it gives the impulse response of the linear differential operator in the form of a delta function. The delta function, commonly known as the Dirac delta function, is defined as follows. Delta(x - x') is equal to 1 when x equals x' and equals 0 when x is not equal to x'. In fact, when x is not equal to x', the delta function is undefined.

In fact, the delta function as defined here denotes the unit impulse function. The Dirac delta function also has alternative definitions, such as the delta function going to infinity at the source location while being undefined at any other point. Essentially, the Dirac delta function is a proxy for an impulse source. The general application of Green's function. The Green's function is used to solve partial differential equations such as Poisson's equation, Laplace's equation, Helmholtz's equation, and other equations.

These equations will be discussed in much more detail in the next module, which consists of the modeling of geophysical systems. But before that, we will discuss the Green's function as it is used in geophysical applications. For better understanding, we will take the help of this diagram. The delta function can be schematically represented as this left figure. It indicates an impulsive source located at x prime and denoted by delta x minus x prime.

The Green's function is the response of the differential operator to this impulse function. Note that the Green's function is spreading over a wider region, whereas the source function is highly localized. Now we come to the Green's theorem. The Green's theorem is a generalized theorem relating double integrals to line integrals around a closed curve. We will shortly see that this is in line with Gauss's theorem and Stokes' theorem, and we will be able to derive Gauss's theorem and Stokes' theorem from the Green's theorem.

We will consider the mathematical form of the Green's theorem as shown here. A continuously differentiable vector field given by F vector is equal to PQ in a region S can be utilized in the form as shown here on the left-hand side and right-hand side to form the Green's theorem. F vector is the field vector which has two components, P and Q. The left-hand side denotes the line integral of the component P along X and the component Q along Y, respectively. The right-hand side is a surface integral. The surface integral is taken for the partial derivatives of the field components with respect to x and y for the Q and P components, respectively.

The integral is then formed as the difference between these partial derivatives. This can be interpreted as the flux of the directional derivatives of the field components. The line integral P dx plus Q dy integrated along the closed loop is shown here. This is the surface integral. This equation represents Green's theorem.

We can take the help of the following diagram for a better understanding of Green's theorem. We have the Cartesian coordinate system denoted by the x and y axes. The closed curve is denoted by C, whereas the interior is denoted by A, which is the area enclosed by the curve C. YL and YU denote the two paths from point 1 to point 2 going below and above the loop. Similarly, XL and XR are two parts of the loop with respect to the X axis. The elemental area is shaded as shown here.

If the field F has a component P, it has to be integrated along the differential element dx, and Q has to be integrated along the differential y elements. These are differential x elements. These are the differential y elements. These elements are obtained as a projection from the dl, which is the elemental line, to the respective coordinate axis. Using the line integral, Green's function can be evaluated on the left-hand side as well as the right-hand side.

On the right-hand side, Green's theorem has to be integrated over this entire area A for every differential area as shown here. Thus, Green's theorem indicates a relationship between line integrals and double integrals of a field with two components enclosed by the curve C. Now, let us have a look at the derivation of Gauss's theorem from Green's theorem. We have looked at Gauss's theorem earlier. It relates volume integrals to surface integrals or fluxes.

Consider this surface S with an elemental surface area dS. The elemental area is dS. And it is pointing along the direction in the hat. Setting p equals minus f and q equals f, and dr denoted by dx i-cap plus dy j-cap, and the normal vector denoted by dy i-cap minus dx j-cap, the above equations of Green's theorem result in pdx plus qdy equals f dot n dr, and the right-hand side results in the divergence of f dx dy.

Going from the line integral and double integral of these equations, we will obtain Gauss's theorem. The left-hand side is the double integral of f dot n dr, which is nothing but the divergence of f dx dy dz, which is obtained from the second relation. This gives Gauss's theorem. Next, we consider the derivation of Stokes's theorem from Gauss's theorem. For Stokes's theorem, we set P equals Fx and Q equals Fy.

This simplifies to the following relations. The left-hand side of Green's theorem, that is Pdx plus Qdy, becomes the dot product between the field f and the line element dr. This is the field vector f. It has a component p and a component q. Thus p dx and q dy is nothing but f dot dr. Thus, the first relation. In the second relation, del q by del x

minus del P by del y dx is nothing but the flux of the curl of F. This can be obtained by simple algebraic calculations. The K component can be obtained as so this forms the curl of F. The kth component is denoted by k hat. The ds vector in Cartesian coordinates becomes dx dy. Now, using this, we have Green's theorem given by equality among these two relations. So, substituting the right-hand side into Green's theorem, we have the first relation and the second relation. This gives the Stokes theorem. Here, n dot ds is nothing but k cap dx dy, as this dx dy is the area of the plane while the z direction is the direction along n hat, which is perpendicular to the surface. Now, let us have a look at the application of Green's function and theorem in geophysical studies.

First, in seismological studies, the Green's function is used to solve the seismic wave equation for a point source. If this is a schematic of the crust, mantle, and core denoting the layers of the Earth's interior, then an earthquake source can be located as a point where tectonic plates break or a shear occurs, leading to a massive release of energy which propagates as waves. So, this point is considered as a source, and the seismic waves which are propagating can be formulated in terms of Green's function. The Green's function, as shown here, is represented as G(R vector, T) and G(R prime vector, T). Here, R is the point of measurement. The point of measurement can be any seismic observatory or seismological station on the Earth's surface.

R prime vector is the point source or the epicenter of the earthquake. So, this model of the earthquake-generated seismic waves is ideally represented using Green's function, and the waves are the solution of the equation using Green's function. Next, the geophysical studies involving gravity and magnetic studies for geophysical exploration. In gravity and magnetic studies, Green's function is used to model gravitational and magnetic potentials originating from the subsurface mass or magnetized rocks. Let us consider this as the Earth's surface.

Consider the scenario where a high-density material is located with the ambient density being low. This localized high-density material can act as a point source for the gravitational field, and the gravitational field can be modeled using Green's function. The field profile and the field lines can be calculated using Green's function. Analogously, if there is localized highly electrically conductive material among the ambient low electrical conductivity material, the signature in terms of magnetic and electromagnetic fields can be obtained using Green's function. So, these are the various applications of Green's function and theorem in geophysics.

Thus, from this lecture, we understand that Green's function is an important tool to characterize point sources and point source-driven geophysical phenomena. In mathematical terms, Green's function and theorem provide a robust foundation for solving partial differential equations related to geophysical phenomena. These geophysical phenomena range from seismic wave propagation, gravitational potential, and electromagnetic fields. In seismic wave analysis, Green's function is an important tool for modeling seismic wave propagation and data interpretation. This is a very useful tool for understanding natural hazards like earthquake studies.

In general, it can be said that Green's function can solve phenomena that arise due to point sources or localized disturbances. One can have a look at the references as shown here for various aspects of Green's function. Thank you.