Mathematical Geophysics

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Week - 01
Lecture - 01

Hello everyone, this is the SWAYAM NPTEL course on Mathematical Geophysics. Welcome, I am Professor Swarandeep Sahoo from the Department of Applied Geophysics, Indian Institute of Technology (ISM) Dhanbad. The course starts with the first module, termed Basic Concepts. It consists of five lectures. Today we will be conducting the first lecture, Lecture Number One, on Scalars and Vectors.

In this first lecture, we will be covering some basic concepts. First, we will look at the link between mathematics and geophysics in general. Then we will have a discussion on scalars and vectors. These are the building blocks of various mathematical concepts used in geophysics. Finally, we will look at the internal structure of the Earth, which is the hallmark of geophysical study.

Thus, look at this picture. Have a look at this flowchart. This indicates that mathematical geophysics links various geophysical aspects to mathematical concepts. The geophysical aspects are listed on the left, and the mathematical concepts are listed on the right. The geophysical aspects, such as core dynamics and mantle dynamics, occur within the deep interior of our planet.

The geomagnetic field is a magnetic covering that acts as a shield from solar irradiation, protecting our planet Earth from harmful solar rays. The gravitational field is inherent to our planet Earth. Geodynamics is the study of various geophysical processes occurring above, at, and below the Earth’s surface. Seismology is a widespread study with applications in geophysical aspects, covering earthquakes, tectonic motions, oceanic circulations, and more. Flows in rivers and glaciers cover various aspects that control and affect the weather and climate. Finally, remote sensing is another geophysical investigation that helps us to obtain data on various geophysical processes.

Corresponding and complementing the study of such geophysical aspects are various mathematical concepts. Such as differential equations, vector calculus, special functions, signal processing, inversion, optimization, functional analysis, and data analysis. Differential equations and vector calculus are used in various physical applications. Special functions help to describe geophysical data. Signal processing helps to understand and interpret various time series data obtained from geophysical processes.

Inversion and optimization are interrelated and help us to obtain models from geophysical data for better understanding and generality. Functional analysis and data analysis are at the forefront of today's geophysical analysis and are advanced concepts. Thus, it can be understood that mathematics and geophysics have a symbiotic relationship, furthering our understanding of mathematical geophysics. So, we begin with the concept of scalars. What is a scalar?

Scalar is a category of physical quantity that has magnitude but does not have a direction. Such physical quantities can have positive or negative values, but they do not have any direction. These are denoted by the letter A. The second type of quantities are vectors. Vectors are the category of physical quantities that have magnitude as well as direction. Such physical quantities are described with a numerical value and a direction.

Given in any of the various coordinate systems, which we will discuss in further lectures. These vector quantities are denoted by $\vec{A}$. The symbols $\overrightarrow{A}$ and $\vec{B}$ denote that the vector has a direction and it represents a vector quantity. Examples of scalars and vectors abound in geophysics. For example, temperature is a scalar quantity, while velocity, displacement, angular velocity, momentum, and forces are vector quantities.

We will now discuss the algebraic operations for vectors. A vector $\vec{A}$ can be denoted by its components a_1, b_1, c_1 along three mutually perpendicular directions $\hat{\imath}, \hat{\jmath}, \hat{k}$. Similarly, any other vector can be represented by its components along three mutually perpendicular directions. This can be pictorially depicted as shown in this diagram. The vectors $\overrightarrow{A}$ and $\vec{B}$ are shown in red and blue colors, respectively.

Next, we have the scalar product of two vectors $\vec{A}$ an $\vec{B}$. The scalar product is a measure of the similarity between the two vectors $\overrightarrow{A}$ and $\vec{B}$. It can be obtained by the sum of the products of the corresponding components of the vectors. The scalar product results in a scalar quantity even though it consists of two vectors at the beginning. The vector product is defined as

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ A_1 & B_1 & C_1 \end{vmatrix}$$

This matrix is evaluated as follows. For the *i*-th component, B_1 is multiplied by C_2 minus B_2 is multiplied by C_1. Similarly, other components can be obtained by the determinant value neglecting the respective column and rows for a given component. In particular, there are properties of vector products for similar vectors which result in a zero vector product.

While for mutually perpendicular vectors, it produces a cyclic structure. The cyclic structure is as follows $\hat{\imath}\ x\ \hat{\jmath} = \hat{k}$, while $\hat{\jmath}\ x\ \hat{k} = \hat{\imath}$ and $\hat{k}\ x\ \hat{\jmath} = \hat{\imath}$.

Addition and subtraction operations: The addition and subtraction operations for vector quantities $\vec{A}$ and $\vec{B}$ are given by the following example. Here, the vector$\vec{A}$ is added to or subtracted from the vector$\vec{B}$ by the corresponding addition and subtraction of the components. These components correspond to the respective components of the individual $\vec{A}$ and $\vec{B}$ vectors. The adjacent diagrams show or explain these scalar and vector products in a clearer manner. First, let us have a look at the scalar product. The scalar product can be obtained as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the cosine of the angle θ between them. Overall, it can be understood that the scalar product defines the similarity of the $\vec{A}$ and $\vec{B}$ vectors.

If the angle θ goes to zero, it means that the $\vec{A}$ and $\vec{B}$ vectors are aligned along the same direction and are very similar. This would result in the cosine of θ going to one, and the scalar product will reduce to the simple multiplication of the magnitudes of the $\vec{A}$ and $\vec{B}$ vectors. Now, coming to the vector product, the vector product of $\vec{A}$ and $\vec{B}$ can be given by the vector. $\vec{C}$It can be calculated as the product of the magnitudes of $\vec{A}$ the magnitude of $\vec{B}$ and the sine of the angle between them, that is θ. The vector product is also known as the cross product between $\vec{A}$ and $\vec{B}$.

Now, we will discuss various examples of scalar quantities that occur in the Earth's and geophysical context. First, the scalar quantity we will be discussing is the distance between any two depth points inside the Earth's interior. The Earth's interior is a very important aspect that we should be familiar with from the very beginning. Here, a cut section of the Earth's interior is depicted for a clear understanding. The Earth's surface is shown at the top, while the center of the Earth is at the bottom. The radius of the Earth is 6378 kilometres.

From the center, as we move upward, we have the inner core, the outer core, and the mantle. This is followed by the crust. The crust is the surface of the Earth where we live. We also have the lithosphere, which consists of the crust and the upper mantle. The core is the combination of the solid inner core and the liquid outer core. The mantle is made up mostly of rocky materials. So, the depth of any point in the Earth's interior can be considered as a scalar quantity. Also, the volume of objects can be considered as a scalar quantity.

An interesting fact worth discussing is the volume of the inner core. The inner core lies at the center of the Earth and is around 1200 kilometres in radius. It is roughly the size of our Moon. However, the volume of the inner core is just 1% of the total volume of our planet. Likewise, the temperature of the Earth at various locations is also considered a scalar. Like the temperature of the Earth's outer core or mantle or at any other point inside the Earth. This diagram shows the variation of pressure and temperature with depth in the Earth's interior.

Like temperature, pressure is also a scalar quantity. We can see that the temperature rises as we go toward the center of the Earth. Likewise, the pressure also increases as we go toward the center of the Earth. The hottest region of the Earth is at its core, which is roughly around 6000 to 7000 Kelvin, while the pressure can rise as high as 320 to 350 gigapascals.

Next, we consider the various examples of vector quantities. The most fundamental vector quantity is the position vector. The position vector indicates the location of any point with respect to the origin. In this diagram, let us consider the center of the Earth as the origin, and any point on the surface or any other point in the volume of the Earth can be located using position vectors. For example, the position vector of a point at this location can be indicated by this position vector. Similarly, any point in the interior of the Earth can also be indicated using a position vector.

Apart from position vectors, it is also useful to understand some geographical notations and terms, such as latitude and longitude, which are very important for geophysical applications. We also have the equator of the Earth as the mid-central line around the belly of the planet. The North Pole is the farthest and topmost point, with a corresponding South Pole, which is not visible in this diagram. The South Pole may be located somewhere around this point. Apart from the position vector, another example of a vector quantity is the angular velocity.

As we know, the Earth rotates around its axis every 24 hours, causing day and night. The angular velocity of the Earth is a vector quantity. The Earth rotates from west to east, and this is considered the rotation vector, denoted by ω. The curved arrow in red indicates the rotation direction.

We also have the magnetic field of the Earth. This magnetic field is generated deep inside the planet and can be measured on the surface of the Earth. Consider this point as a location above the surface of the Earth where the magnetic field is being measured. Being a vector quantity, it has various components. It has components along the northern direction, the southern direction, or any other direction it can be measured. This is the northern component of the magnetic field. H is the notation for the magnetic field, and the components H_x, H_y, H_z are the various mutually perpendicular components of the magnetic field. The angle which these components make with the horizontal and vertical are given as declination and inclination angles.

Overall, it can be understood that the examples of vector quantities in geophysical applications are above. Now, we move on to discuss the elemental quantities, which are vectors and scalars. Consider a small volume dV which has a small area dS on its three faces. The edges are dx, dy, dz. The dx-vector, d -vector, and dz - vector are the vector quantities, which are unit vectors along these line elements. Thus, the line element dL can be represented in terms of its components: $dx\hat{\imath}$, $dy\hat{\jmath}$, $dz\hat{k}$. The elemental area can also be likewise represented in terms of its components: dS_x, dS_y, dS_z. The elemental area dS_x can be represented as $dy \cdot dz$ along the i-direction. These quantities, line element and area element, are vector quantities. However, the total volume element which can be computed as dx, dy, dz, is a scalar quantity. Thus, these elemental quantities, in terms of line, area, and volume, are very useful to calculate various heat, energy, fluid flow, and other transport phenomena, which occur in geophysical applications.

It is very useful to remember the distinction of vector and scalar among these quantities. Thus, we come to the conclusion of the first lecture. The conclusions which we can derive from these ideas are as follows: Geophysics, in general, is inherently linked with mathematics, and their combination leads to improved understanding and accurate interpretation. Scalars and vectors are very important mathematical concepts and are building blocks for various geophysical calculations. Scalars and vectors are also useful for representing geophysical quantities in a more suitable manner. Geophysical quantities need to be represented using geometrical aspects, which involve scalars and vectors. These are the references from which we can obtain more detailed ideas and discussions on scalars, vectors, and their applications in geophysics.

Thank you.