Solid Dynamics
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Lecture - 20
Wave Propagation in An Infinite and Semi-Infinite Elastic Medium

Hello everyone, welcome to the course soil dynamics. Today we will start first we will continue our discussion of Wave Propagation in An Infinite Elastic Medium, then we will start discussion on Wave Propagation in a Semi Infinite Elastic Medium.

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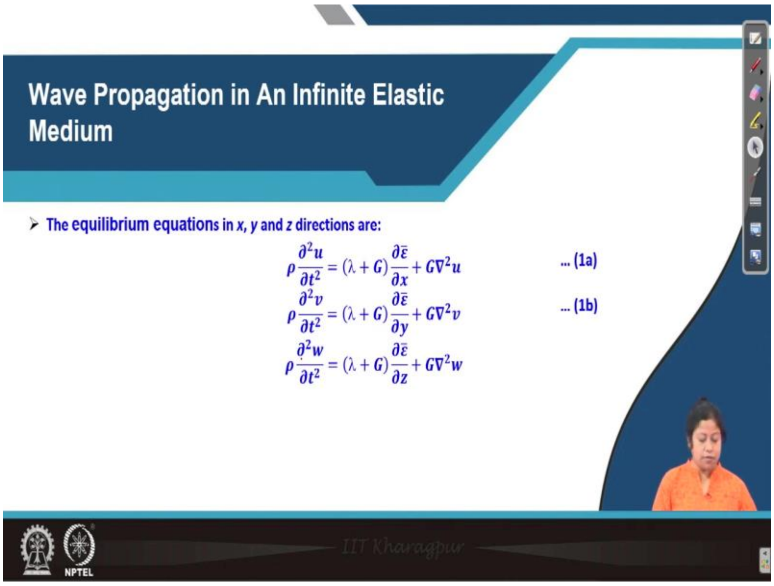

So, in last class, we have seen that equilibrium equations in three mutually perpendicular directions you can say x, y and z directions, which are rho times del 2 u of del t 2 is equal to lambda plus G times del of epsilon bar by divided by del x plus G times grad 2 u. So, what is lambda and G here? Lambda and G are the lame's constant that we have already seen. What is epsilon bar? That is volumetric strain. u is the displacement in x direction.

Now, the equilibrium equation in y direction is rho times del to v of del t 2 is equal to lambda plus G times del epsilon bar by del y plus G times grad 2 v. So, here v is the displacement in y direction and rho is the material density. The third equilibrium equation in z direction is rho times del 2 w of del t 2 is equal to lambda plus G times del epsilon bar by del z plus G times grad 2 w. Here w is the displacement in z direction.

So, we can see here that means, the three term on the left hand side that is del 2 u of del t 2 del 2 v of del t 2 and del 2 w of del t 2, we present the accelerations in x, y and z directions.

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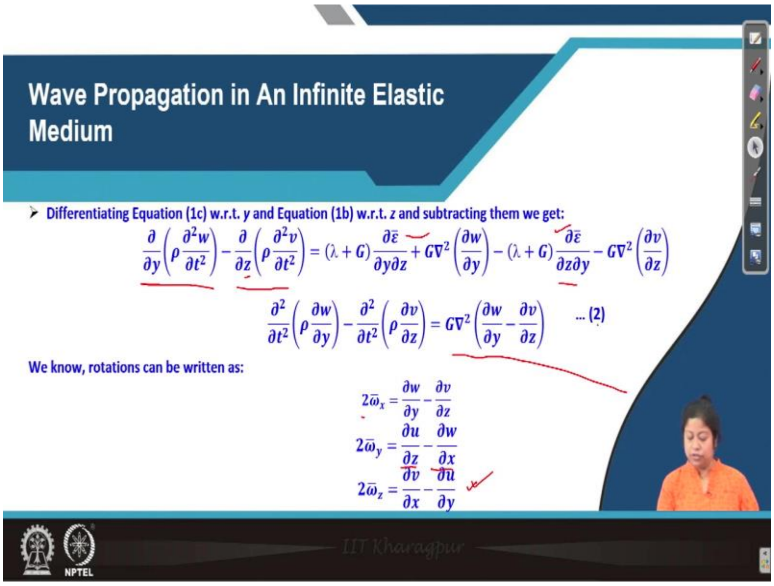

So, now, if we differentiate equation 1 c with respect to the y and equation 1 b with respect to z and then subtract them from each other what we will get? We will get del of del y because, one c is differentiating with respect to y, so, del of del y times rho times del 2 w of del t 2 this is the part of the left hand side of the equation 1 c when we differentiating it with respect to y. Next term is the left hand side of the equation 1 b when we differentiating it with respect to z.

So, when we are subtracting these then the right hand side becomes lambda plus G times del epsilon bar of del y del z plus G times grad to del w of del y minus lambda plus G times del epsilon bar of del z del y you can see here minus G times grad 2 of del v of del z. So, here you can see this term and these terms are common and they are subtracting from each other so, they can cancel each other.

So, what is left in the next line? Next line we can write del 2 of del t 2 for rho times del w of del y minus del 2 of del t 2 del t 2 of rho times del v of del z equal to G times grad 2 of del y del w by del y minus del v by del z you can see here. So, here we have seen two new terms, one is del w of del y and the second one is del v by del z, what are these two terms?

Let us see, how we can take care of these two terms, we already know that rotations can be written as 2 times of omega x bar is equal to del w by del y minus del v by del z when rotation about x axis is considered. Likewise, rotation about y axis that time we can write 2 times omega y bar is equal to del w by del z minus sorry del u by del z minus del w by del x. Similarly, for rotation about z axis, we can write the third equation. Let us give this equation and number 2.

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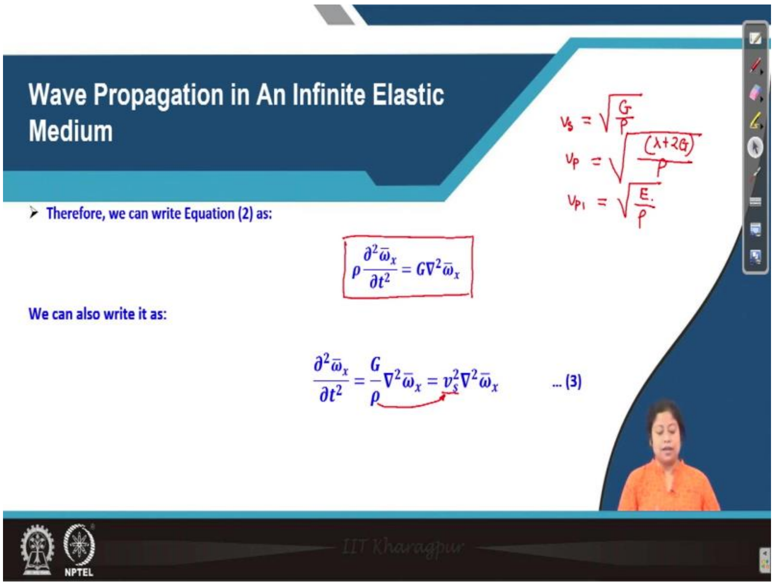

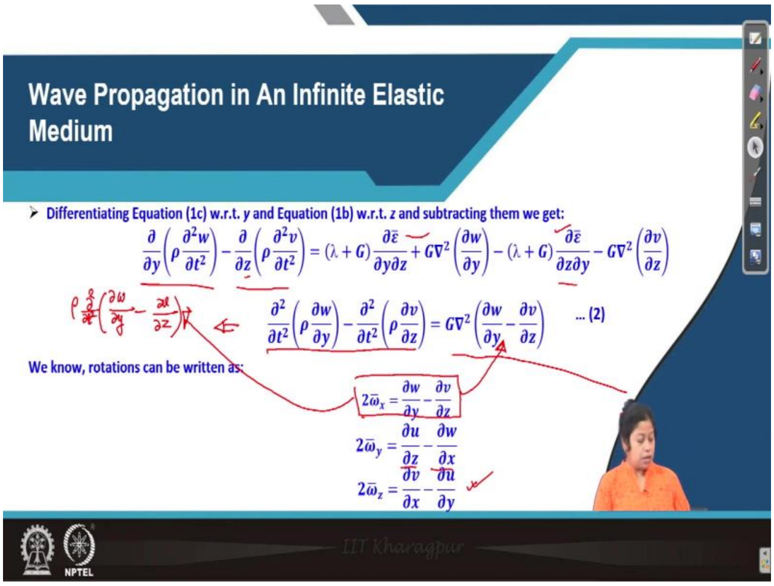

So, now, this equation 2 can be rewritten like rho times how it is coming just once again going back to the previous slide. Here now, what we can see there is a term del w by del y minus del v by del z, what it is exactly? It is omega x, 2 times of omega x bar you can see here this one basically. So, here we can write 2 times of omega x bar.

Likewise, for the left hand side what we can write actually, we can write this left hand side also as I am just writing it the side rho times del 2 by del t 2 times del w del y minus del v del z. If we can write it this way, so, here also we can use 2 times of omega x bar. Thus, we can write equation 2 as rho times of del 2 omega x bar of del t 2 which is equal to G times grad 2 of omega x bar, this one.

We can also write it as left hand side we will keep as it is, for the right hand side sorry from the actually left and right hand sides can be divided by the mass density of the material which is rho here. So, then left hand side becomes del 2 omega x bar of del t 2 which is equal to g divided by rho times grad 2 of omega x bar. Now, what is g by rho that we have already seen that is the square of the shear velocity or torsional velocity, velocity due to the torsional wave. So, in place of g by rho, what I have written here is v s square.

So, similar so, here what we have seen? We have seen finally, the velocity of the shear wave that is, when wave is propagating in an infinite elastic medium that time the velocity of the shear wave is square root of G by rho. If you recall in previous class, we have seen the velocity of the P--wave when it is propagating through your infinite elastic media is different.

It was square root of lambda plus 2 G divided by rho. So, in this way, we can see that the velocity of the shear wave does not change when wave is propagating through and infinite elastic media in comparison to the case for wave propagation in a elastic rod because that time also we get the same expression for v s. However, when wave is propagating through one elastic rod, what we have seen for v p?

We have seen v p 1 is equal to square root of E divided by rho. So, the velocity of the P-wave is different when wave is propagating in an infinite elastic medium.

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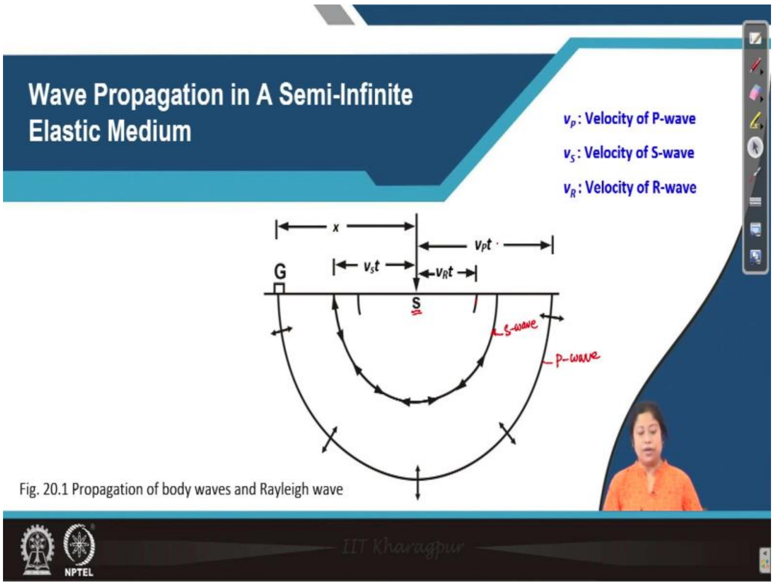

Now, what is happen when wave propagation occurring in a semi-infinite elastic medium? What is the example of semi-infinite elastic medium below the ground surface? If you see that upper limit is defined which is the ground surface, but if you go below the ground surface you do not know how much we have to what is the extension of the domain below the ground surface.

So, in the direction vertically downward when we are going that direction is extended to we can see infinity. So, that is the reason we consider it as semi-infinite elastic medium. Now, what is happened here is the source where we are creating some kind of impulse. Now, if we will go away from the x, what we will get? We will see at some point we will note, what we will see here that you can see that this is the, this is your P-wavefront or wavefront for the P-wave. So, I am writing this is for P-wave.

This is for S-wave. So, the interesting point is that both body waves that is P and S waves are propagating in a following is a hemispherical wavefront here. Whereas, if you see the Rayleigh wave it is propagating following one, it is propagating basically radially following one cylindrical wavefront.

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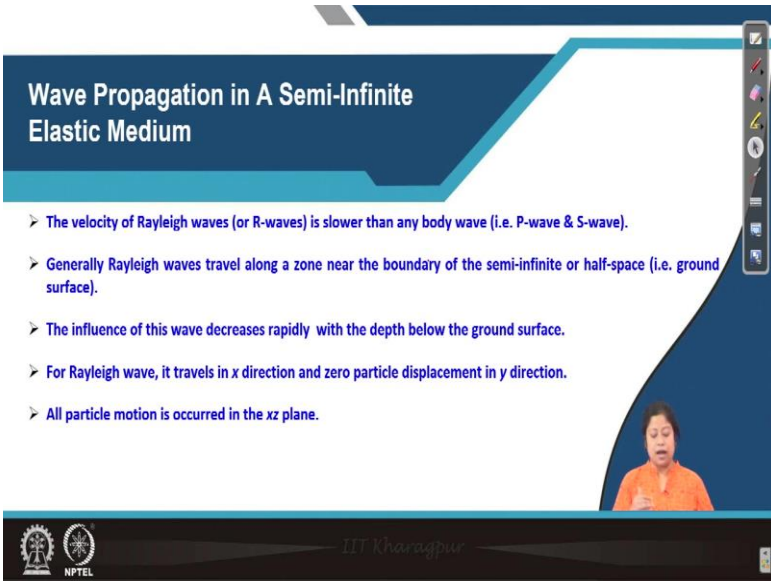

And these times what is happening basically, you can see really waves propagates along the ground surface or close to the ground surface. So, the velocity of Rayleigh wave we can call it also as R-waves is slower than the velocity of the body wave feature P-wave and S-wave then, generally Rayleigh waves travel along a zone near the boundary of semi-infinite or hub space that means close to the ground surface.

Next, the influence of these waves decreases rapidly with the depth below the ground surface that means that the ground surface or just below the grounds close to the immediately close to the ground surface. We will identify Rayleigh wave significantly in comparison to some finite depth below the ground surface.

Next thing is that for Rayleigh wave, it travels in y direction and zero particle displacement sorry it travels in x direction and zero particle displacement in y direction. So, I can say all particle motion for Rayleigh wave is occurred in xz plane.

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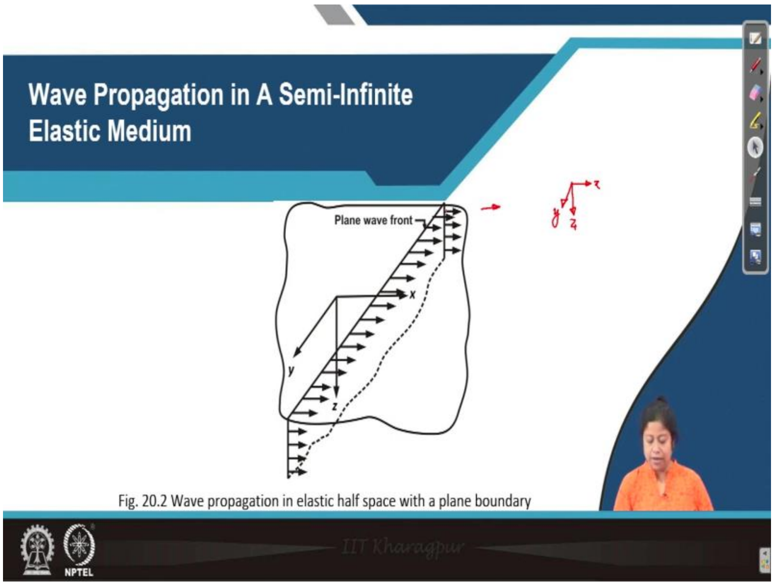

So, here you can see the plane wavefront, what we can see here this is our x direction I can just separately show x and y direction and this is our z direction. So, you can see here, what it is says? It says that there is no displacement in y direction that we can see here only it propagates in x direction.

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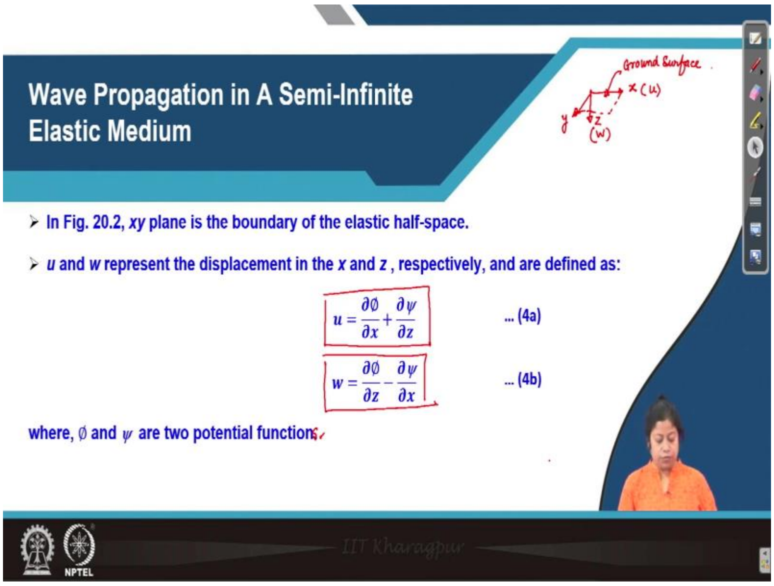

Now, in the previous xy plane is the boundary of the elastic half-space, so, in the previous figure xy plane is the boundary of the elastic half-space. I can draw here also for all of you this is x direction, this is y direction, this is z direction. So, this xy plane is the boundary of the elastic half-surface which is eventually your ground surface.

Now, u and w represent the displacement in x and z direction that means, in x direction u is the displacement and in z direction, w is the displacement, in y direction displacement is zero. So, for Rayleigh wave, u and w represent the displacement in x and z direction and these are defined by two potential functions phi and psi, you can see here u is equal to del phi by del x plus del psi by del z.

Similarly, we can define w which represents displacement in z direction as the w is equal to del phi by del z minus del psi by del x. It is one is missing. So, phi and psi are two potential functions, let us give two numbers for these two relations. So, first one 4 a, second one 4 b.

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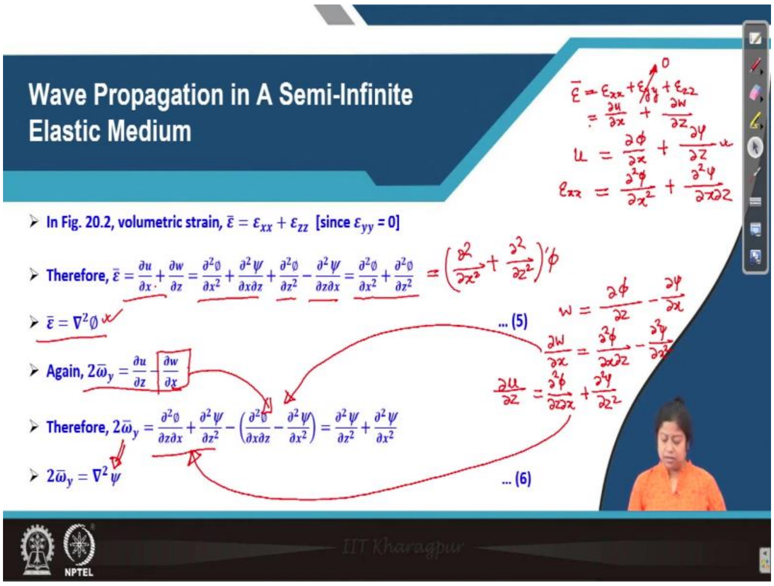

So, in figure 2, what we can write for volumetric strain, we already know volumetric strain, esilon bar is equal to epsilon x or epsilon xx plus epsilon yy plus epsilon zz. Now, in this case epsilon yy is 0. So, it is epsilon xx plus zz. Then epsilon bar epsilon xx means del u by del x, epsilon zz means del w by del z. So, epsilon xx del u by del x, epsilon zz del w by del z. So, that I have written here.

Now, we also know u is equal to del phi by del x plus del psi by del z. So, in place of epsilon xx we can write then del 2 phi by del x 2 plus del 2 psi by del x del z so that is written in first two terms. For the next two terms that means for del w by del z, what we have written you can see del 2 phi by del z 2 minus del 2 dsi by del z del x. Same way actually I have done it so, I am not showing it here. You can give it a try, very simple thing.

Then what we can write, here you can see del 2 psi by del x del z can cancel each other because one is with positive sign and the other one is with the negative sign. So, finally, which is left is del 2 psi by del x 2 plus del sorry del 2 phi by del x 2 plus del 2 phi by del z 2. So, that is nothing but del 2 del x 2 plus we can write it as like this. So, this is grad 2 that is not in the next step I have written a test epsilon bar is equal to grad 2 times grad 2 of phi.

Now, again if we consider the rotational component rotation about y axis then what we can write? 2 times omega y bar is equal to del u by del z minus del v sorry del the w by del x like this. Then what we can write in place of u and w? Already here you can see what we can write for u. So, and for the w we can write, here you can see del phi by del z minus del psi by del x. This is for the w.

So, what will be del w by del x. These will be del 2 phi by del x del z minus del 2 psi by del x 2. So, this is written so, actually this part is this one which is coming from here and the first part is this one which you can see as you can see from this that del u by del z is equal to del 2 phi by del z del x plus del 2 psi by del z 2. So, these you can directly write here.

So, what we can write from these? We can write them this one that 2 times del sorry 2 times omega y bar is equal to grad 2 of psi. So, the volumetric strain is a function of phi, you can see here whereas the rotational component is a function of psi here. Let us give two number for these two new equations.

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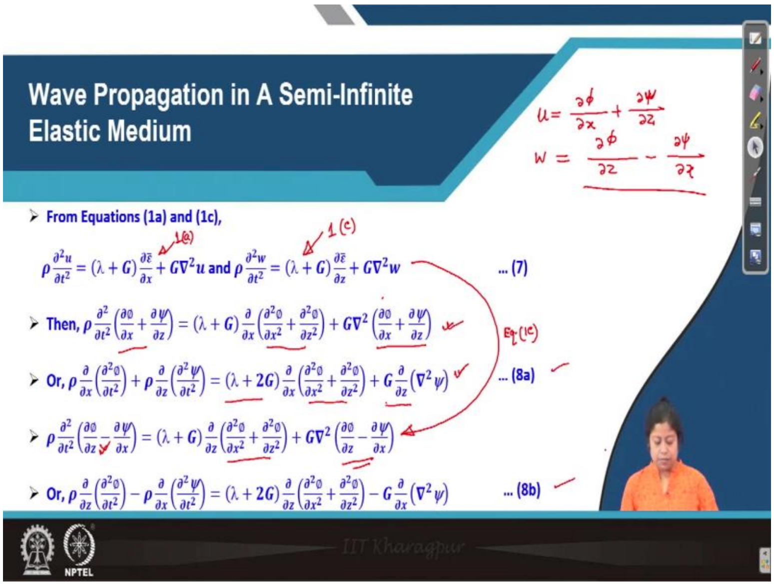

So, now, from equation 1 a and 1 c this is our 1 a and this is our 1 c. So, these are the two equations. So, here what we can do? We can write in place of u, the expression which already we have assumed that u is equal to del phi by del x plus del psi by del z. This is for our u and for displacement in z direction w for that we can write in these step it is not required.

So, later next step I will write it so, for equation 1 in place of u here you can see u is replaced by the function del phi by del x plus del psi by del z. Right hand side also in right hand side also in place of epsilon bar we can use phi already we have seen how to use phi, so, I am not writing it once again and plus G times grad 2 of u again u is written in terms of two potential functions phi and psi. Similarly, then in the next step, what we can do?

We can simplify this equation in this form. So, left hand side is very clear. Come to the right hand side, what is done in the right hand side, just I have taken these del phi by del x terms out of these bracket and added to the first term in the left hand side. So, that is the reason here I have written it as lambda plus 2 G of del lambda plus 2 G times del by del x of del 2 phi by del x 2 plus del 2 phi by del z 2 and the remaining term is G times grad 2 of del psi by del z.

So, G times grad 2 of del psi by del z is written here as G times del by del z of gad 2 psi. Now, these comes from equation c. I can write here equation 1 c. The same way equation 1 c is also expanded in terms of phi and psi you can see here the first term the w is replaced by this term because we know that w is equal to del phi by del z minus del psi by del x that we have already assumed and expressed w this way.

So, that is the reason here in equation 1 c w is replaced by this term. Likewise, in the right hand side I have replaced epsilon bar by this term, how it is coming that we have already seen and the last term was G times grad 2 w, here also w is replaced by the two potential functions phi and psi. Then we can do some orientation some arrangement of the different terms and can write it as in this form just like the previous steps we have done here. So, after this what we can see. So, finally, from this let us give some number this is 8 a and this is 8 b.

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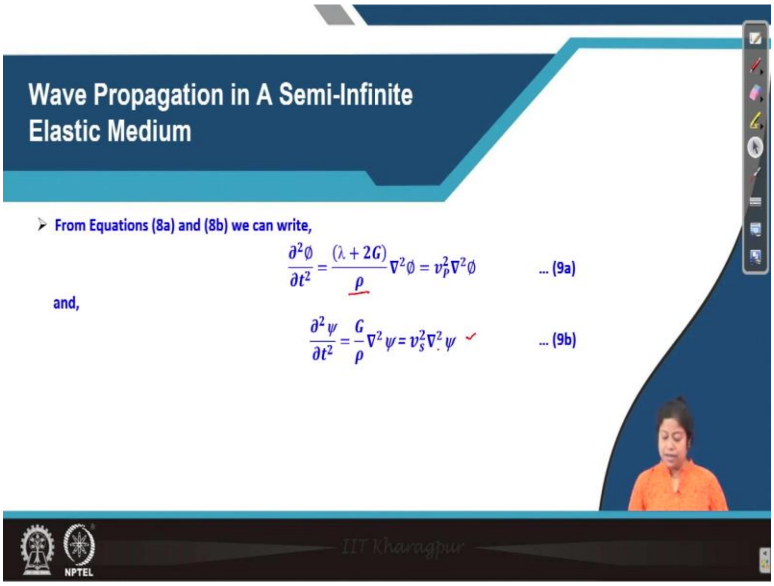

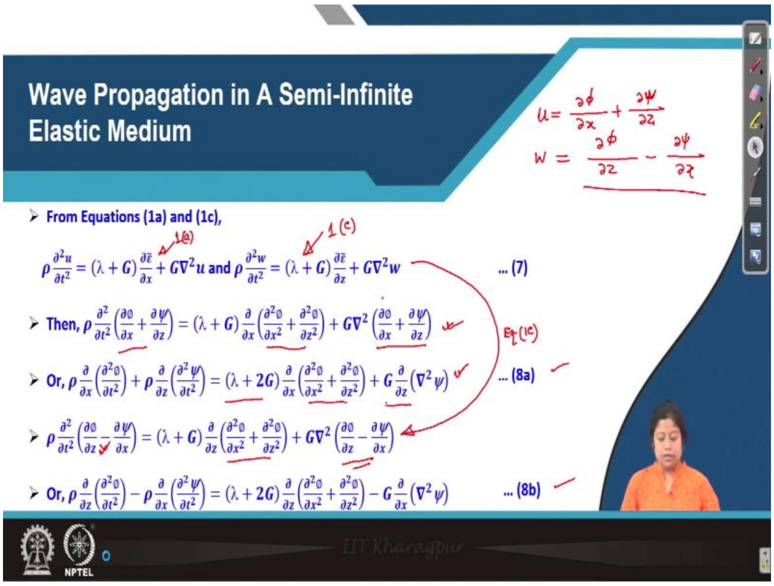

So, now, from equations 8 a and 8 b, what we can write? We can write del 2 phi by del t 2 is equal to lambda plus 2 G divided by rho times grad 2 phi this rho because, if you go back to the previous equation for both the terms rho was present, you can see rho is present on the left hand side. That is the reason we have written it this way.

And for the second term is del 2 psi by del t 2 is equal to G divided by rho times grad 2 psi so, we already know what is v p and what is v s. So, here I have written v p in place of lambda plus I have written v p squared actually in place of lambda plus 2 G by rho. Similarly, in the second equation here in this line in place of G by rho I have used v s square, give these two equations, two numbers 9 a and 9 b.

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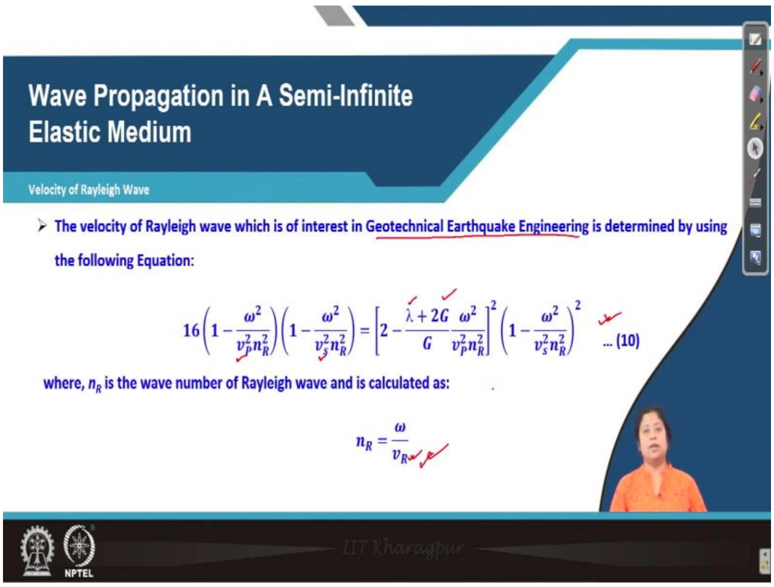

So, now, the interesting thing is that the velocity of Rayleigh wave, what is the velocity of how we will find out the velocity of Rayleigh wave because it is required for most of the geotechnical earthquake engineering problem, that is a reason we should know how we can get the velocity of Rayleigh wave. First we will solve this equation. Here v p and v s are the velocity of the P-wave and shear wave.

What is n R? n R is wave number of Rayleigh wave and that can be calculated by this equation omega divided by v R where v R is the velocity of Rayleigh wave. So, we can see here in this equation v s is known, v p is known from the material property, we also know the value of lambda and capital G. From these first two, we will find out n R, from n R we will find out v R and then we will get the velocity of the Rayleigh wave.

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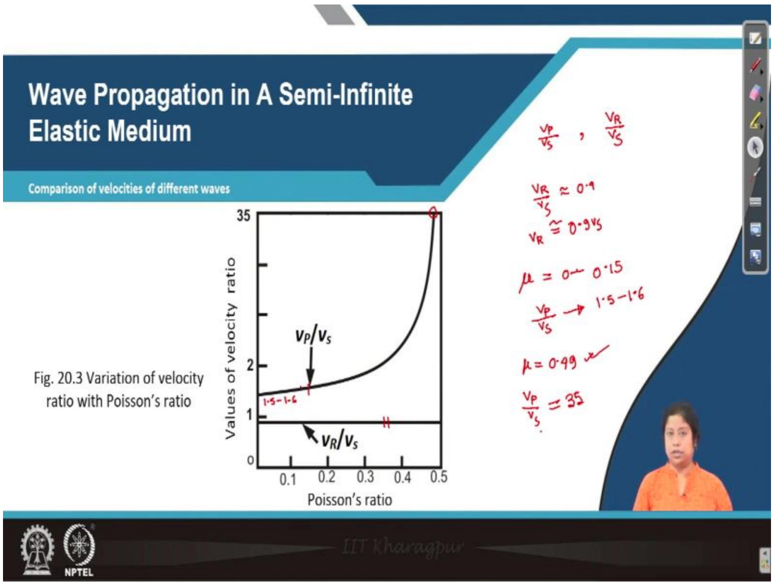

Now, here in this slide, you can see how the velocity ratio, velocity ratio here is expressed by V P divided by V S, give me one-minute time, so, V P divided by V S. This is one velocity ratio and the other velocity ratio is V R by V S. So, V S is the velocity of the shear wave, whereas, V P and V R are the velocities of P-wave and Rayleigh wave respectively.

So, what we can see here the variation of the velocity ratio with Poisson's ratio. So, you can see here for most of the cases for most of the Poisson's ratio, the velocity ratio of V R divided by V S that means, this curve always is below 1 and it is very close to 0.9. So, let us just write not for all the values it is not exactly 0.9 but close to 0.9.

So, from this what we can write V R is approximately equal to 0.9 times of V S. However, the digit in second and third decimal place depends upon the Poisson's ratio. So, generally if it is 0.4, it may be little bit different. So, now, what about the V P by V S? If the Poisson's ratio is low then we see V P by V S is close to 1, but definitely it is greater than 1 always.

Up to this point, we can see it is more or less 1.5 to 1.6. So, for Poisson's ratio you can take it mu also or you can take it mu also so, I am writing here mu 0.02 I can write it as 0 to 0.15 we can see this V P by V S varies 1.5 to 1.6 approximately. After that you can see this ratio increases rapidly and here when mu is close to 0.5, it is approximately 0.49.

So, you can see the ratio of V P by V S is here, which is 35 that means, the velocity of the P-wave is 35 times faster than the velocity of the shear wave or S-wave that we can say when mu is equal to 0.49. I think you have some idea for type of soil, we can see mu is equal to 0.49. So, for that type of soil, we will get these V P by V S ratio is 35 or close to 35.

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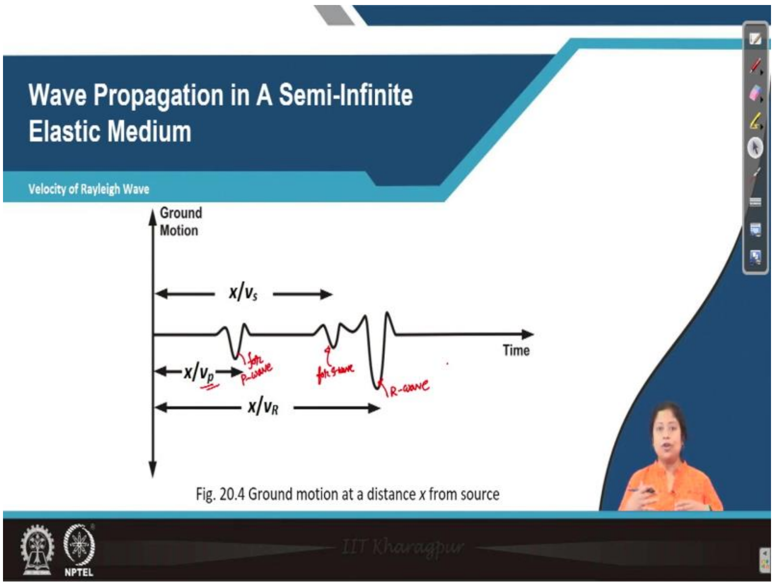

Now, in this slide you can see how the ground will deform. So, at shorter time or when just we created impulse and after some time of that, what we will see? We will note first we will note some displacement of the ground, which is because of the velocity or propagation of the P-wave. This is for P-wave. The second one is for the S-wave.

So, this is for S-wave. Interestingly, these two waves causes relatively low ground motion. You can see the displacement is low compared to the ground motion or displacement of the ground because of this Raleigh wave. So, what we can see, although the velocity of the

Raleigh wave is small or least, but its impact is more vulnerable for the ground, because it travels close to the ground surface.

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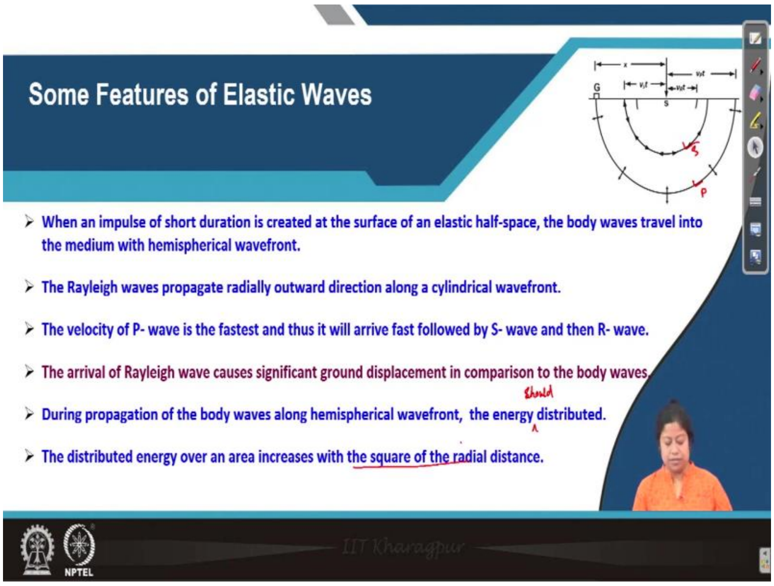

So, these are the some features of elastic waves. So, when an impulse of short duration is created at the surface of an elastic half-space, which is the boundary of the elastic half-space, the body waves travel into the medium with hemispherical wavefront that you can see here, this is for the P-wave and this is for the S-wave. However, for the Rayleigh wave how it will propagate? It propagates radially outward direction along a cylindrical wavefront.

Then the velocity of the P-wave is fastest and thus it will arrive first followed by S-wave and then R-wave. The arrival of Rayleigh wave or R-wave causes significant ground displacement in comparison to the body waves that we have already seen. During the propagation of body waves along hemispherical wavefront the energy should be distributed, so, I can write it should be distributed, how that let us see, distributed energy over an area increases with the square of the radial distance.

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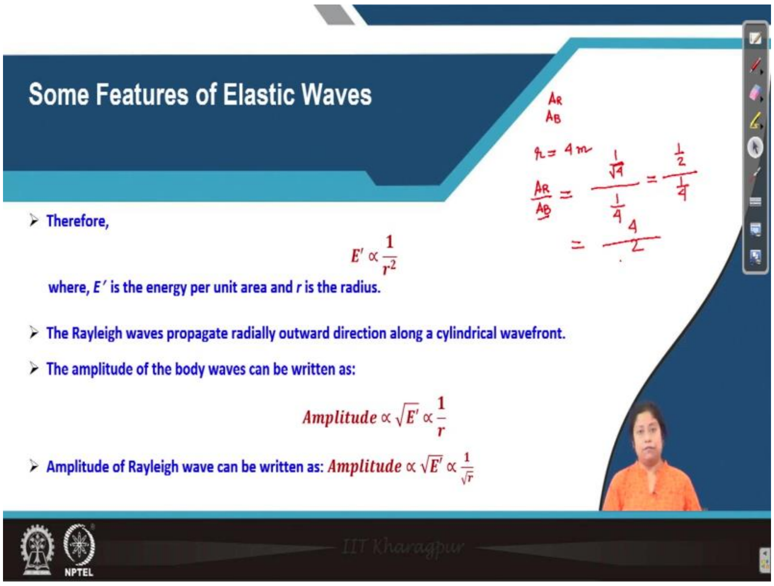

So, we can write energy over an area that is E dashed is proportional to 1 by r square, because energy is distributed over an area. So, it is as r increases, E dashed which is the energy per unit area decreases then what will be happened for the Rayleigh wave? The Rayleigh waves propagate radially outward direction along the cylindrical wavefront and that we have already actually mentioned somewhere.

Now, what about the amplitude. Amplitude for the body waves is proportional to the energy per unit area, but not directly proportional square root of is E dashed. So, that means amplitude is proportional to 1 by r, what does it mean? If the distance increases, radial distance increases, then the amplitude decreases. What happens it for the Raleigh wave? For Raleigh wave, we can see amplitude is proportional to 1 divided by square root of r.

So, one thing we can understand from here is that, at same distance radial distance are where attenuation of energy will be high or we can measure the higher amplitude let us take amplitude for Raleigh wave is A R and amplitude for any of these body wave is A B. Now, if we will take suppose, R is some 4 meter distance, then what will be this ratio of A R divided by A B? You can see here, what will be for the body wave, body wave it will be sorry for body wave it will be 1 by 4 approximately.

What about Raleigh wave? Square root of 4. So, finally what we are getting is it is half and it is 1 by 4, so, 4 by 2, so A R to A B is equal to 4 by 2 that means, the amplitude at same radial distance amplitude of Rayleigh wave is more in comparison to the amplitude of the body waves, where amplitude of Rayleigh wave is 4 unit, body wave is 2 unit that is the difference.

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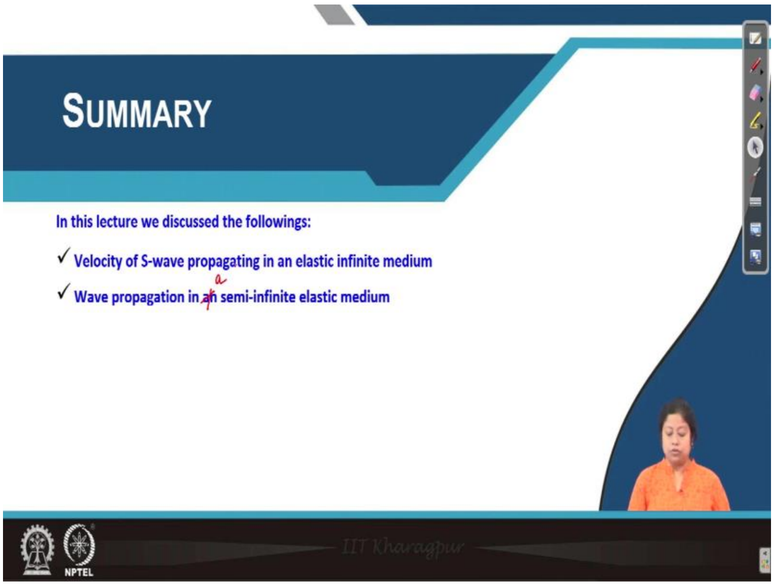

So, come to the summary of today's lecture. Today, we first studied the velocity of S-wave when wave is propagating in an elastic infinite medium and we have seen that the velocity of S-wave remains the same in case of wave propagating in an elastic infinite medium when we compare it to the velocity of the S-wave propagating in a elastic rod. Now, after that we have also started the wave propagation in an sorry it is not an, it is a in a semi-infinite elastic medium, that we have also studied in today's class.

So, with this I would like to conclude today's class. These are the references. Thank you.