**Soil Dynamics**
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**Lecture 18**
**Wave Propagation in an Elastic Rod of Finite Length**

Hello everyone, today we will continue our last class on wave propagation.

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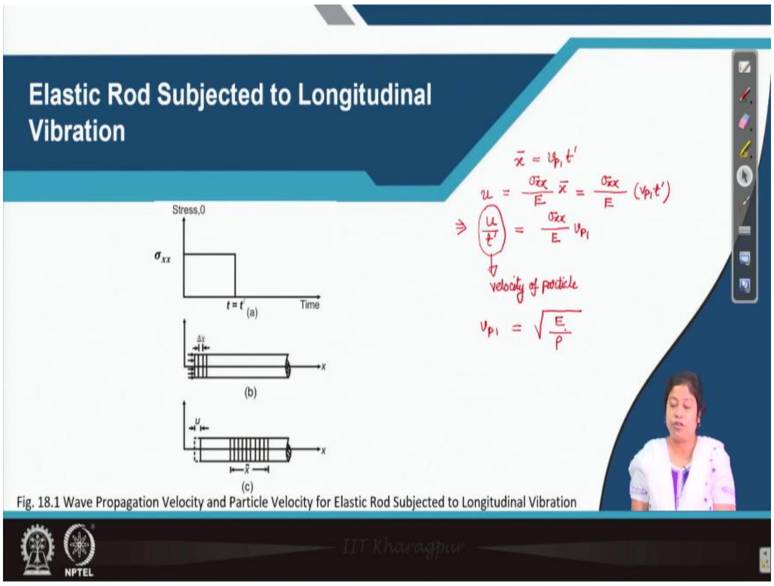


First, we will see what we have discussed last class. So, last class we have studied wave propagation in and infinite length elastic rod. So, they are first we have studied what will be the equation of motion when the longitudinal wave is propagating in the rod, thereafter, we have studied what will be the equation of motion when torsional wave is propagating in the rod.

Today, first I had like to say something about the longitudinal wave propagating through the in rod of infinite length in this diagram, what we can see in this diagram, we can see that u is the here you can see u is the this displacement or I can see completion of the rod because of the propagation of longitudinal wave in this rod and x bar is the length of that element which is under consideration.

Then what will be the expression for x bar, I am writing here itself we can write x bar is equal to vp 1 times t dashed, just give me one minute here, yes, vp 1 v times t dashed likewise, we can write that displacement u also in this case. So, how much is the displacement is in this case it is sigma x x divided by E which is elastic modulus times x bar.

Now, in place of x bar what I can write I can write the vp 1 times t dashed, so, let us do that sigma x x divided by E times vp 1 times t dashed. Now, from this we can calculate the velocity of the particles by calculating u divided by t dashed which is equal to sigma x x divided by E times vp 1. So, here are these term u divided by t dashed is nothing but the velocity of a particle.

So, from this what we can see, velocity of the particle depends upon sigma x x whereas, the velocity of the longitudinal wave depends upon only the material properties, which are elastic modulus and the density of the material. That means, vp 1 we have already seen is equal to E divided by rho, where E is elastic modulus and rho is the density of the material.

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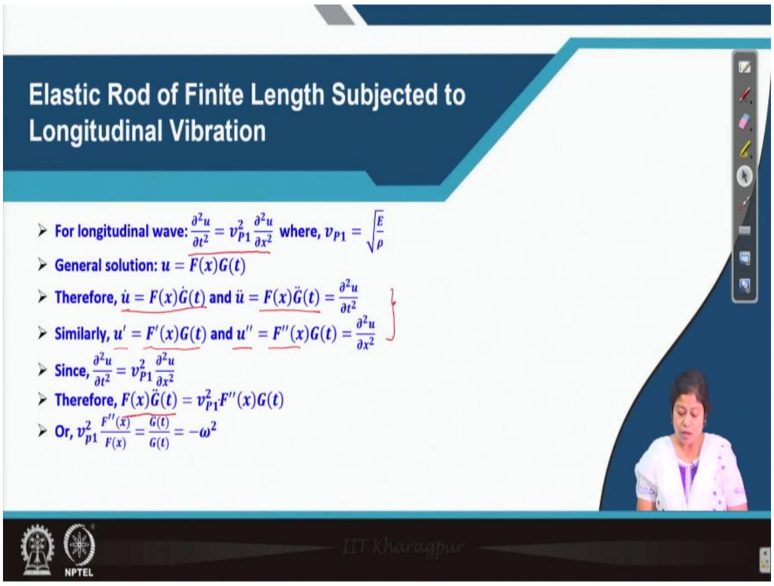


Now, let us see when elastic rod of finite length is subjected to longitudinal vibration. So, we already have the equation for wave propagation in one direction one dimensional or 1d direction. So, that equation you can see here del 2u of del t2 is equal to vp 1 square times del 2u divided by del x2. So, here vp 1 is the velocity of the body wave or we can call a longitudinal wave, we can also call it as P wave.

So, now, here if what is u, u is the displacement. So, now, we can consider that u is equal to the product of function x and function G, F is a function of space that is x and G is a function of time. Then, what is u dot, u dot is the first derivative of u with respect to the time t therefore, we can write for u dot is equal to a fixed times G dot t because, as I already said u dot is the first derivative of u with respect to time not with respect to space then u2 dot will be a fixed times G2 dot t which is del 2u of del t2.

Similarly, we can calculate the first derivative and second derivative of u with respect to space that means x. So, u dash t is equal to F dash x times Gt and u double dash is equal to F2 dash x times Gt. Now, we have these equations that del 2u by del t2 is equal to vp 1 square times del 2u by del x2. So, here if we will write what we get from previous two lines, then what we will write, we will get Fx on the left-hand side we have a Fx times G2 dot t which is equal to vp 1 square times Fx, sorry, F double dash x times Gt.

Then next step we can write vp 1 square times F 2 dash x divided by Fx is equal to G2 dot p divided by Gt. Now, we have already learned when we have a function, we can express one function by harmonic motion, then its acceleration is proportional to the displacement and also acceleration is proportional to the frequency of motion and direction of acceleration is opposite to the direction of displacement.

So, in this case, if we express Gt by harmonic motion, then we can write it same way. So, that is a reason what we have done here, we have taken Gt as…

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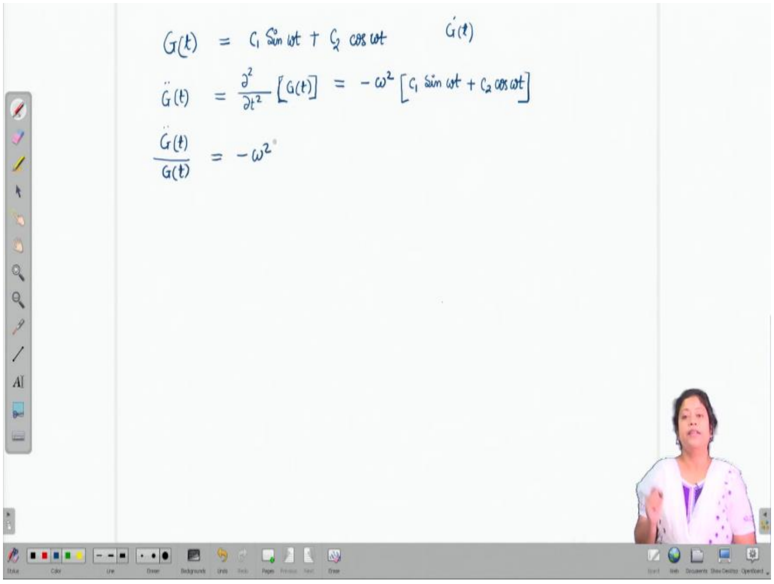

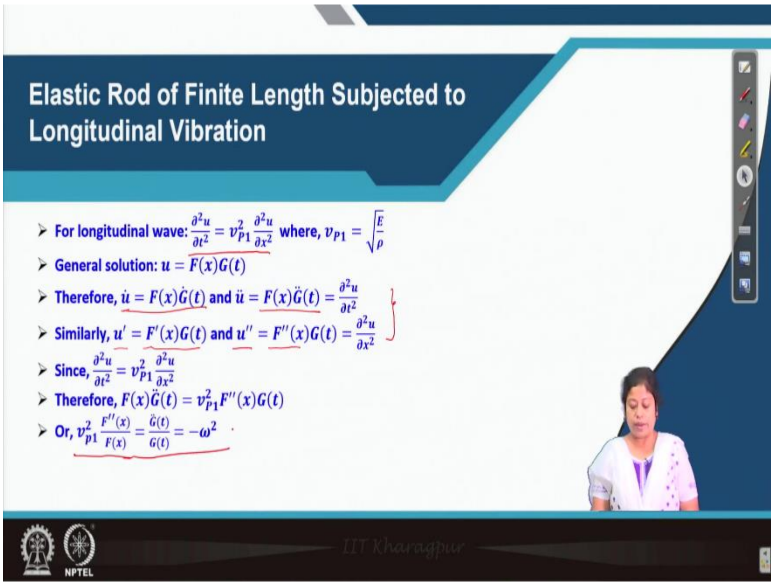


I can write here, we can take Gt as... Let us take C1 sine omega t plus C2 cosine omega t, then what we can write G2 t that means I am taking the second derivatives of Gt which will be equal to, here you can see I have stepped one, I have jumped one step, that is G dot t, you can calculate first G dot t and then also you can calculate G2 dot t, or you can jump one step and directly write this also.

So now from here, if we will write G2 dot t divided by Gt, what we are getting that is equal to minus omega square. So, the same thing I have written here that G2 dot t divided by Gt is equal to minus omega square. Now, eventually, G2 dot t by Gt is equal to vp 1 square times F 2 dashed x by Fx, that is the reason this entire thing is equal to minus omega square.

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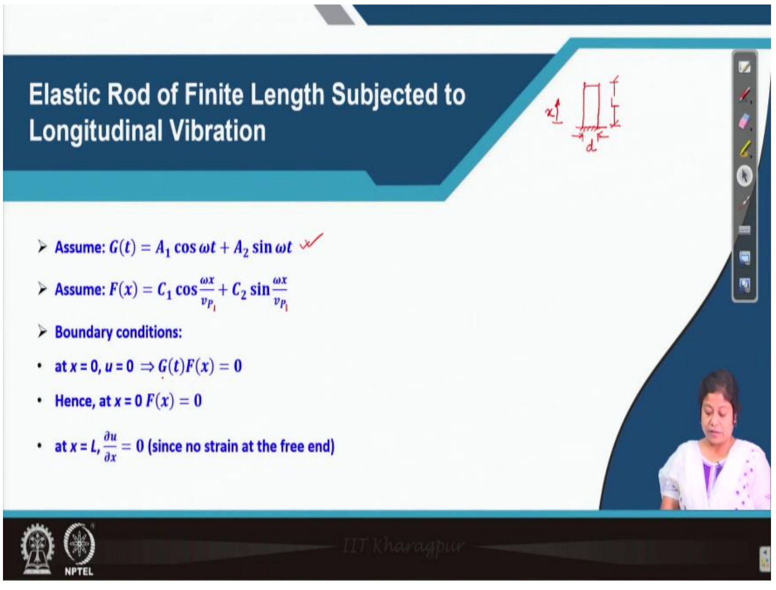

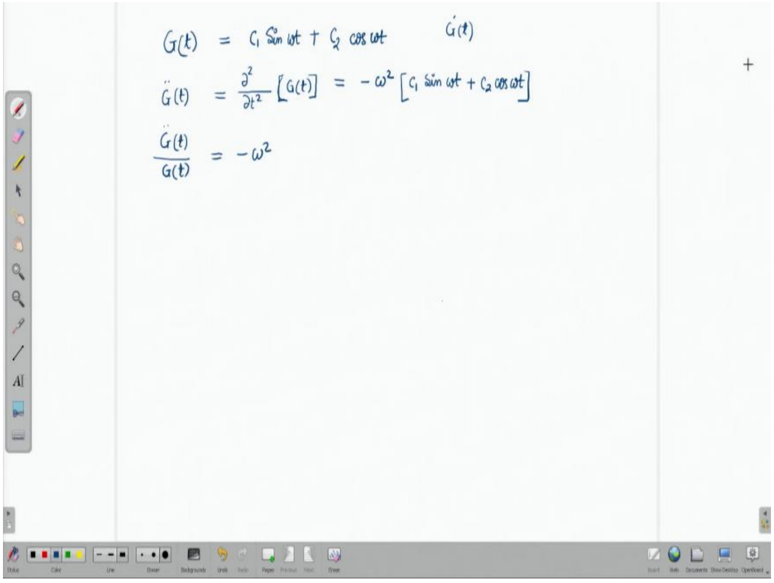


Now, I am just making one change here, here I have taken C1, C2. Now, in this slide, you can say that two constant terms associated with the cosine omega t and sine omega t are taken as A1 and A2 respectively. So, with these only, we will continue our discussion. So, let us assume Gt here is equal to A1 cosine omega t plus A2 sine omega t.

We need to assume also something for Fx, here we have taken Fx is equal to C1 times cosine omega x divided by it will be vp 1 and plus C2 times sine omega x divided by vp 1, please make it vp 1. Then we need to set boundary condition. Let us take a simple case. When a since we are dealing, we will deal with soil. So let us take a soil sample is resting on traxial pedestal, that means its bottom is fixed.

We are not applying anything at the top of this sample. And the length of the sample is A and its diameter is D. Small d. So here, what boundary condition we can set, when x, let us take x is measured in this direction. So, when x is equal to 0, that means at the bottom end, what we can say displacement is 0. So, first boundary condition, I can write that x is equal to 0, u is equal to 0, then u means Gt times Fx that is also equal to 0.

So, at x is equal to 0, u is equal to 0. So, what we can write we can write Gt times Fx is equal to 0. Now under any circumstances, Gt cannot be 0, then what will be 0, Fx. So let us write at x is equal to then o, Fx is 0. When Fx is 0, what we can write that we will see.

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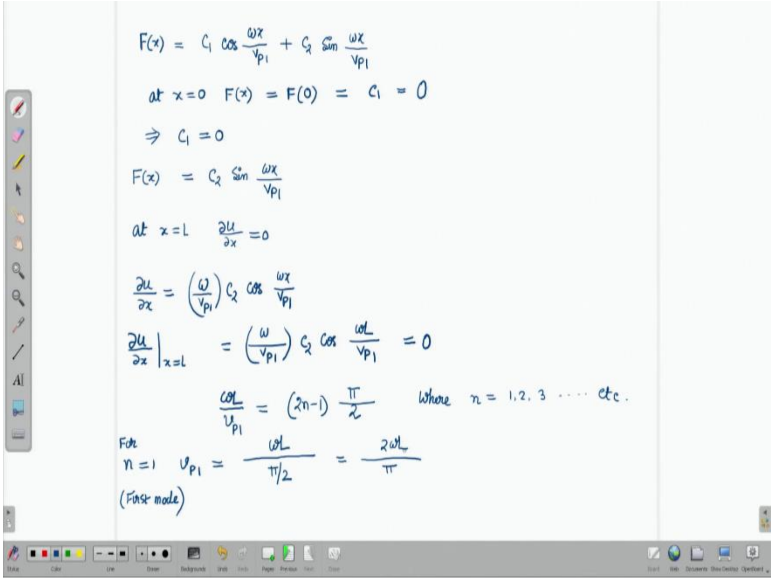


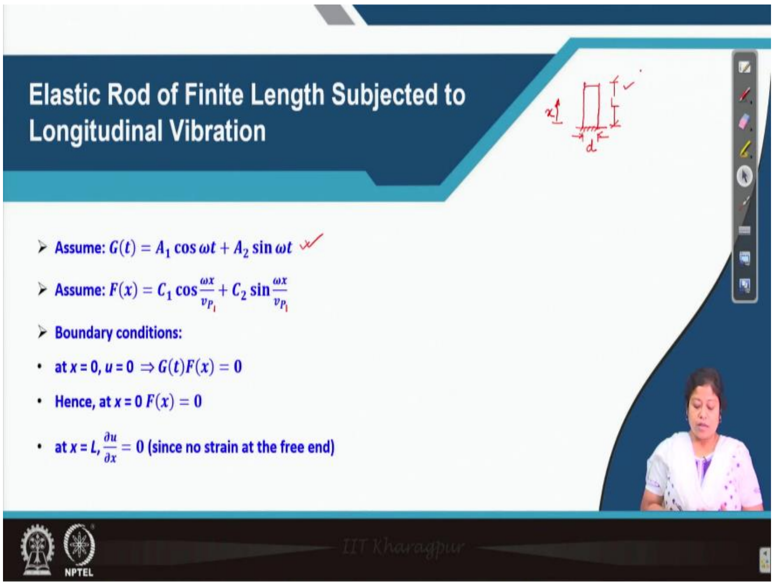


So first, this, I am going to a new page. So here we have taken Fx is equal to C1 times cosine of omega x by vp 1 plus C2 times sine of omega x by vp 2 sorry vp 1, 1 stands for one dimension that it. Then when x is equal to 0 at x is equal to 0 Fx that is F0 is equal to C1 and that we already defined that u is 0, so, Fx is 0 here. So, from this we can say C1 is 0, then we can write Fx is equal to C2 times sine of omega x divided by vp 1.

Now, see the second condition, second condition says that at x is equal to L that means at the top of these sample del u by del x is equal to 0, since no strain at the free end. So, let us write this condition also at x is equal to L del u of del x is equal to 0. So, first we need to find out what del u of del x that is nothing but omega divided by vp 1 times C2 times cosine of omega x divided by vp 1.

Now here we will write in place of x, we will write L. So, del u of del x at x is equal to L is omega divided by vp 1 times C2 times cosine of omega L by vp 1 and that is equal to 0. If this is equal to 0, what does it mean? It means that the angle that is omega L divided by vp 1 is either pi by 2 or 2n minus 1 times pi by 2 where n is varying from one to any integer. So, let us write that then omega L divided by vp 1 I can write it vp 1 is equal to 2n minus 1 times pi by 2, where n is equal to 1, 2, 3 etcetera, it can be any integer.

So, with this now, what we can calculate, we from this we can calculate what will be vp 1. So, here vp 1 will be how much? Let us take first mode of vibration that means n is equal to 1. So, four m is equal to 1 that is first mod of vibration, vp 1 is equal to omega L divided by pi by 2. So, in this way, we can calculate the velocity of the longitudinal wave or P wave for first mode of vibration and also from these. So, this is our final. So, in this way we can calculate vp 1. I can write one more step here just like this. Now come to the next.

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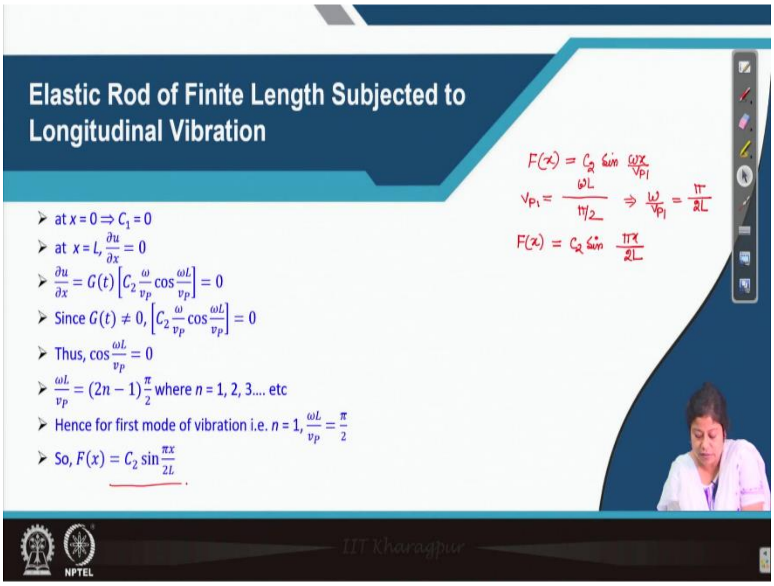


What I said that is explained here you can go through this. So, finally, we can write Fx using the expression C2 times in place of omega L, what we have I am just writing here we have considered actually Fx is equal to C2 times sine of omega x divided by vp 1 and already we have established relationship between omega and vp 1.

What was that relationship we have seen that vp 1 is equal to our omega L divided by pi by 2. So, from these what we can write in place of omega divided by vp 1 we can write pi divided by 2 L. So, Fx is equal to C2 times sine of pi x divided by 2L when that is written here.

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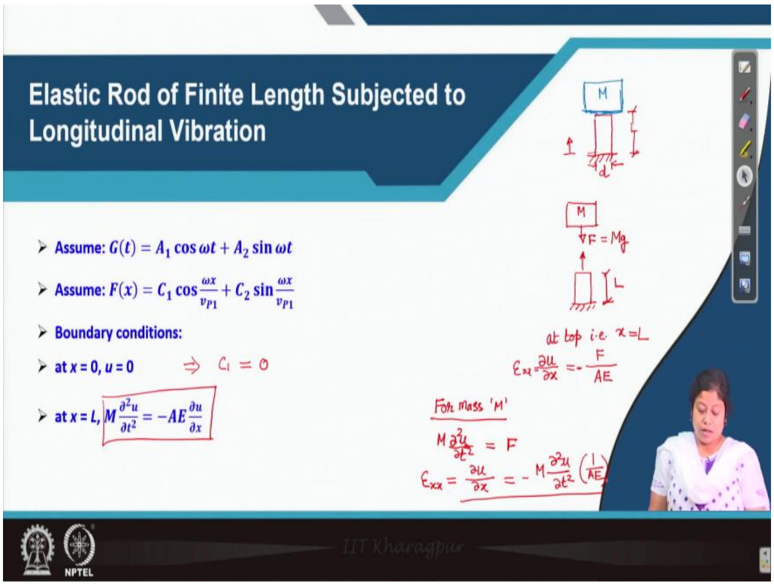


Now, another boundary condition is possible, let us take the same soil sample resting on a pedestal fixed at the pedestal. Now, at the top what we can do let us take we have placed a mass in so, sorry. So, here a mass is placed at the top of this soil sample and mass is N capital N and the length of the soil sample and its diameter remains same L and d respectively. I can write I can show here L is the length and d is the diameter.

Now, what boundary condition we can write for these keys when x is equal to 0 that time u is 0 because x0 means at the bottom it is fixed so, displace there is no displacement. So, from this we can write C1 is 0. So, from the first boundary condition we can write C1 is equal to 0. Now, second boundary condition what is happening at the top end.

So, at the top end a mass of magnitude capital M is placed. Now, if I will draw the Freebody diagram for this mass what it will be let us take these the weight of the mass we can write F is equal to capital MG also here just, then what is happened to the soil sample, the soil sample, I am just showing I am just interested for the free top end.

So, at the top end it is also subjected to the same force capital F, the only difference is the direction of the force, if the for the mass M the direction of the forces downward for the soil the direction of the same force F under which it is subjected is in upward direction. So, for such case what we can write about the strain at top that is x is equal to L, how much is the strain, strain is del u of del x or I can write epsilon xx that is equal to F divided by AE.

In this case, what we have from the mass M if I will write the equation of motion what I can write for mass M for mass M what I can write, I can write mass times acceleration. So, mass times acceleration which is inertia force is equal to the unbalanced force we consider F positive in downward direction.

So, here F is positive if we consider F is positive in downward direction then for the other case, we get we need to consider minus F here and eventually when we deal with soil compressive forces positive and tensile forces negative. So, in that way the sign convention matches perfectly. So, now, what we will do here in place of F we can write mass times acceleration.

So, I can write epsilon xx or del u of del x is equal to minus M times del 2u of del t2 times 1 upon AE. So, from this we can write this expression that here I have multiplied both sides by minus AE, if I will multiply both sides by minus AE, what we can write mass times acceleration is equal to minus AE times del u of del x that is our actual strain for this case. So, now using this boundary condition what we will see.

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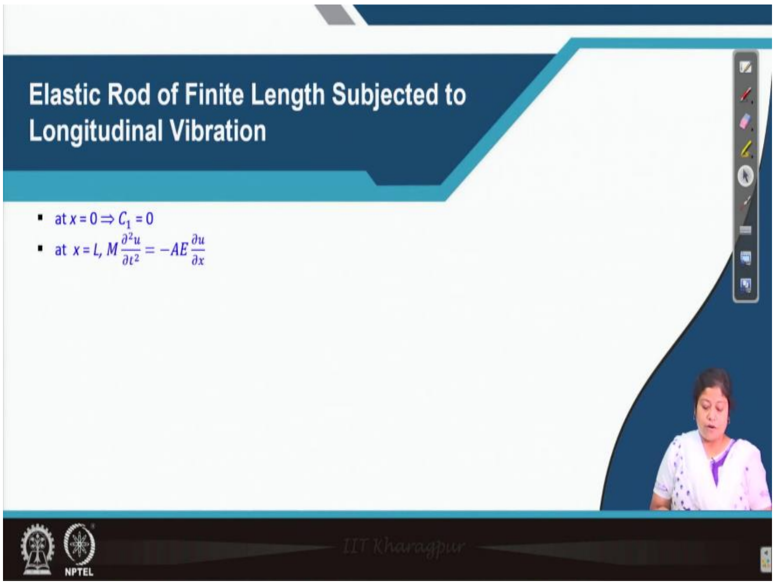


So, alright. So, now, after setting the two boundary conditions what we can write?

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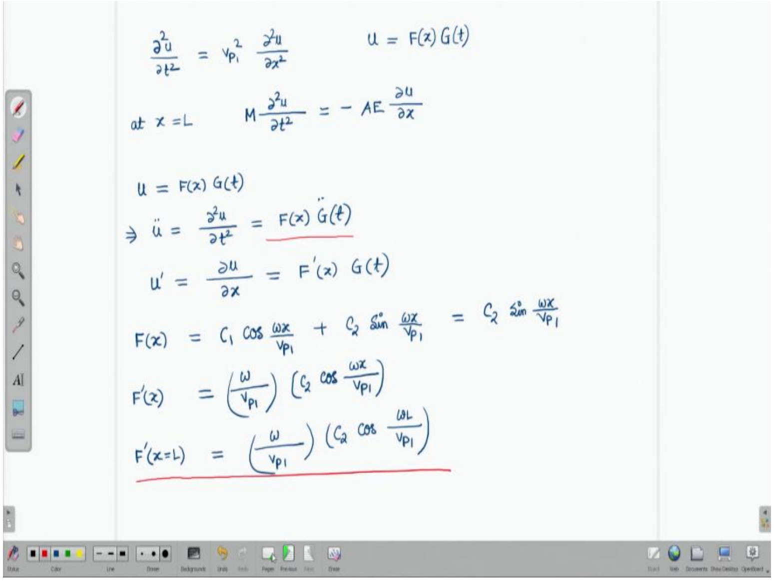

So, now we have this equation del 2u of del t2 that is equal to vp 1 square times del 2u of thdel x2. Now, in this equation what we have assumed u is equal to function of space which is Fx times function of time, which is Gt capital G represents the function of time and capital F represents the function of space.

What is the second boundary condition here at x is equal to L that means, at the top end or other end we have these boundary condition that mass times acceleration, which is inertia force is equal to minus AE times del u of del x. Del u of del x means longitudinal strain here. Now, since we have assumed that u is equal to Fx times Gt. So, from this we can find out del 2u of del t2 which is the acceleration. So, I am once again writing it here.

So, what is acceleration then u2 dot which is del 2u of del t2 is equal to Fx times G2 dot and what is the expression for u dashed or we can write it as del u of del x it is F dashed x times G. Now, already we have assumed the expression for F dash sorry Fx what is that Fx is equal to C1 cosine of omega x by vp 1 plus C2 times sine of omega x by vp 1.

So, here what is F dash x, F dash x is. Now, in the expression of Fx we have already seen C1 is equal to 0 from the first boundary condition. So, I can just write it as C2 times sine of omega x divided by vp1 then F dash x is equal to C2 times better I write it first the omega divided by vp 1. So, F dash x is equal to omega divided by vp 1 times C2 times cosine of omega x divided by vp 1.

So, now, I need to write F dashed at x is equal to L which is omega divided by vp 1 times C2 times cosine of omega L divided by vp 1. Now, we can use the expression of F dash L and that expression of this one. So, I can write it here itself. So, for these what I can do just go to the next page a new page.

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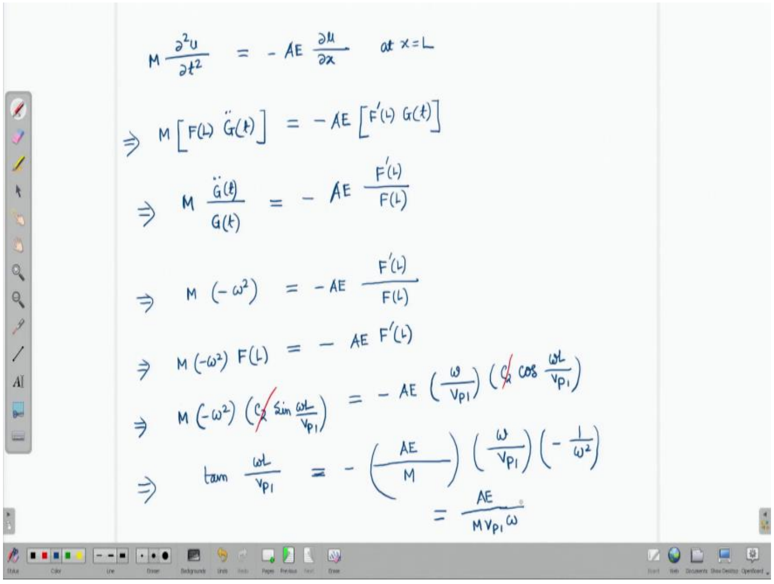


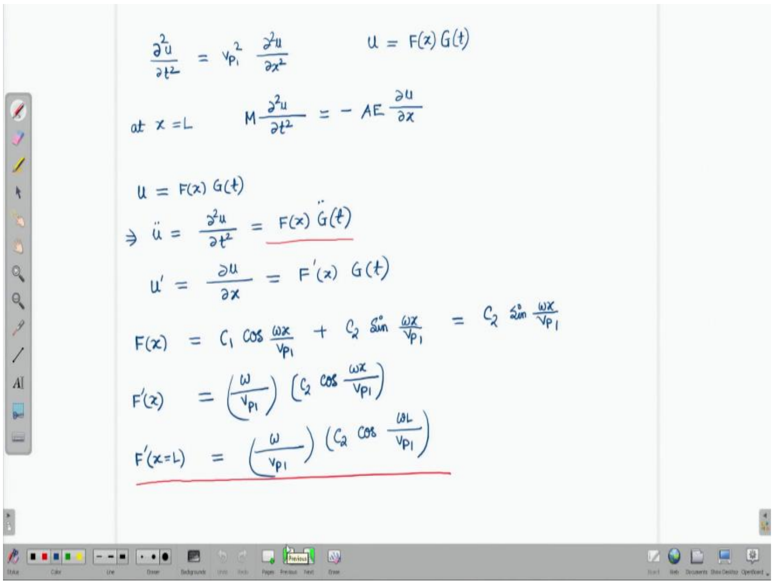


So, now, I can first write M times acceleration is equal to minus AE times del u of del x at x is equal to L. So, in the next line what we can write mass times acceleration, acceleration means del 2u of del t2 at x is equal to L, what I can write if L times G2 dot which is equal to minus AE times F dashed L times Gt or what I can write here mass times G2 dot divided by G is equal to minus AE times F dashed L divided by FL.

Now here, what is the value of G2 dot divided by G, we know g is a harmonic function. So, already we have seen for harmonic function or harmonic motion, what we can write acceleration is proportional to or acceleration is equal to minus omega square times the equation of the motion, which is Gt here. So, I can actually write here M times minus omega square which is equal to minus AE times F dashed L divided by FL or I can write it also as M times minus omega square times function of L is equal to minus AE times F dashed L.

Now, here I can write the value of Fl. So, M times minus omega square, what is the value of a Fl. So, here, FL means I can write here itself, that is C2 times sine omega L divided by vp 1. So, I can write it in the next page, then C2 times sine omega L divided by vp 1 in place of Fl and that is equal to minus AE times F dash del, what is f dash del omega divided by vp 1 times C2 times cosine of omega L divided by vp 1. Now, from these you can see C2 which is a nonzero quantity present in both sides.

So, I can cancel it out like this and I can rewrite the in this expression as what I can write I can write then omega divided by vp 1 omega no I cannot write I can write omega divided by vp 1 which is better I should write first the left hand side the right hand side. So, now, what I can write tangent of omega L divided by vp 1 that is equal to minus AE divided by M this M is used here times, omega divided by vp 1 times minus 1 divided by omega square.

So, simplifying it we can write AE divided by M times vp 1 times omega. So, this is the value of tangent of omega L divided by vp 1.

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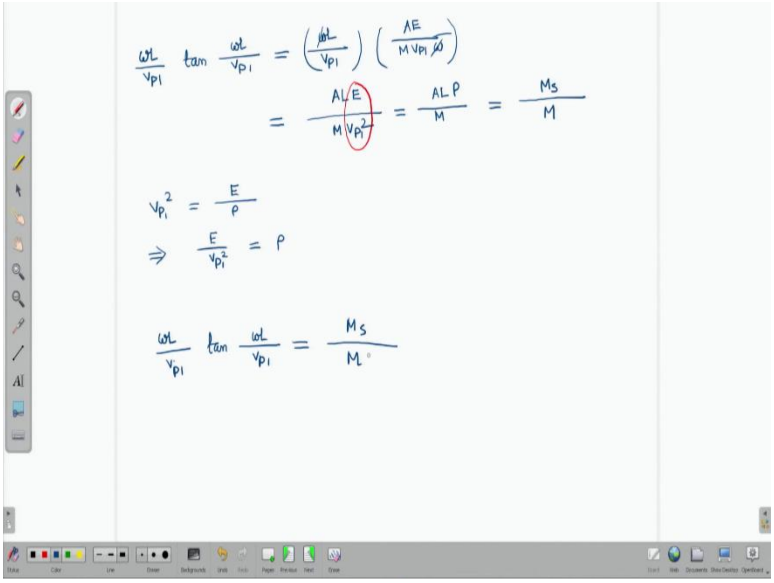

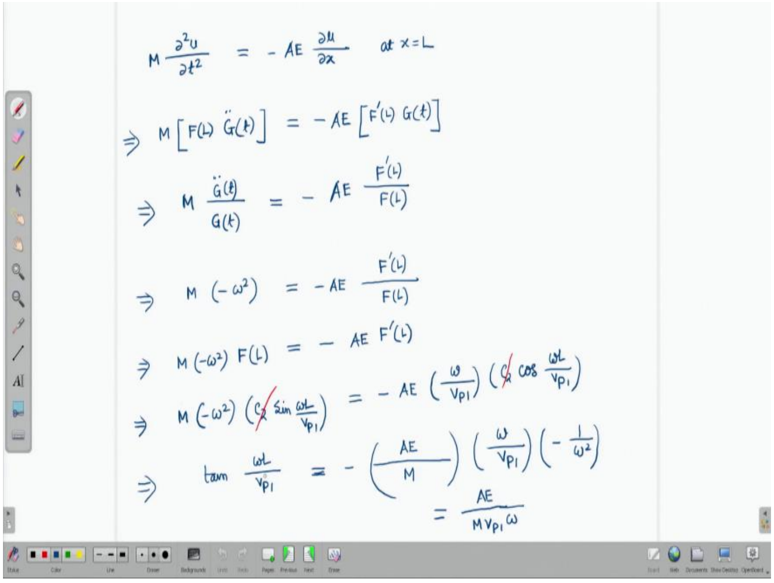


Now, what we can write here. Here if I will multiply omega L by vp 1 on both sides what we will get? Let us see, omega L divided by vp 1 times tangent of omega L divided by vp 1 which is equal to omega L divided by vp 1, times what we have on the right-hand side earlier AE divided by M times vp 1 times omega. So, I can write here AE divided by M times vp 1 times omega. So, finally, what we are getting here finally, I am getting AE or I can AE L I can write L before writing AE.

So, A L E divided by M times vp 1 square. So, what is AL here, AL is the volume of the rod or the soil sample when we are taking soil. So, AL is volume and what else we can what is vp 1 square, we have already seen vp 1 square means E divided by rho where E is modulus of elasticity and rho is the density of the material. So, from these what I can write, I can then write E divided by vp 1 square is equal to rho. So, you can see here that term E divided by vp squared is present.

So, I can then write this expression as aAL divided by M times rho. Now, AL which is the volume of the rod or sample times rho gives the mass of that sample or mass of the rod. So, I can for soil sample it is mass of the soil divided by M which is mass of the which is the mass kept at the top of this soil sample. So, in this way finally, what I can write omega L divided by vp 1 times tangent of omega L divided by vp 1 is equal to Ms divided by M.

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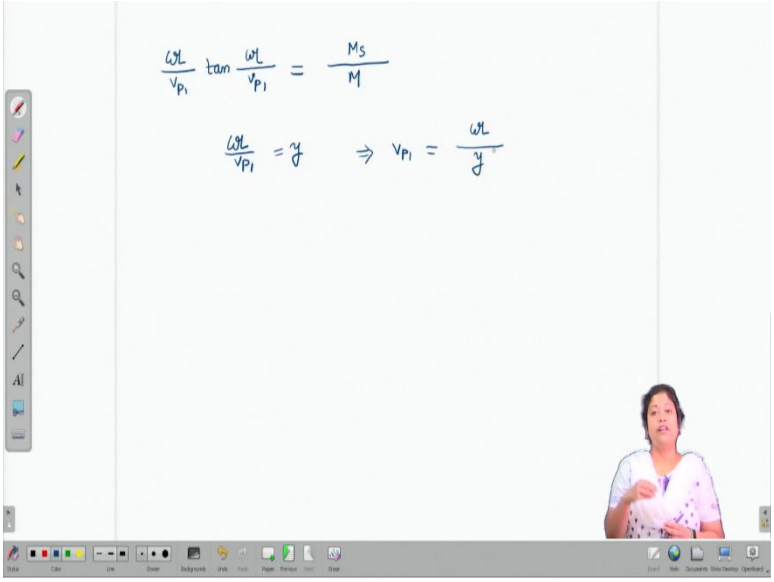


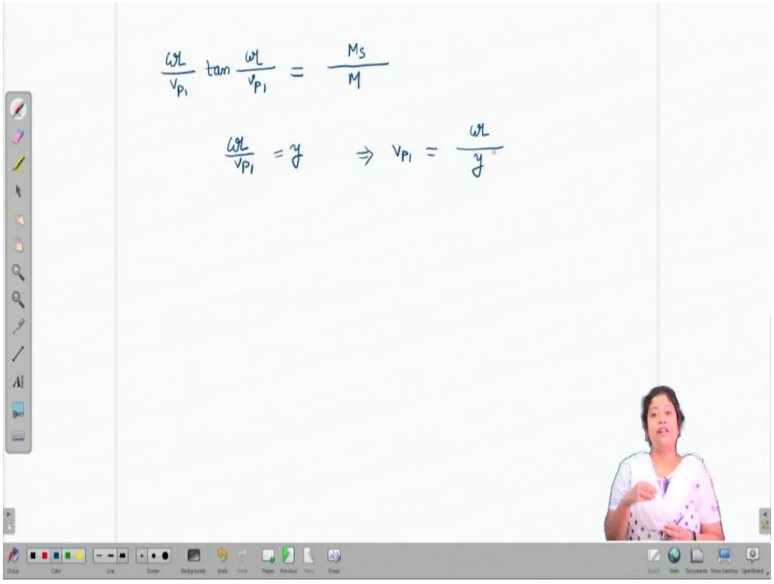


We get I am writing here omega L divided by vp 1 times tangent of omega L divided by vp 1 is equal to M is divided by M, where M is the mass of the soil sample and capital M is the mass which is placed at the top of the soil sample. Now, we know how to solve this type of equation. So, we can solve this type of equation and find out something for omega L by vp 1 let us take that something is y then from that we can express vp 1 as omega L divided by y.

So, in this way, we can find out the velocity of the P wave propagating through one propagating through soil element.

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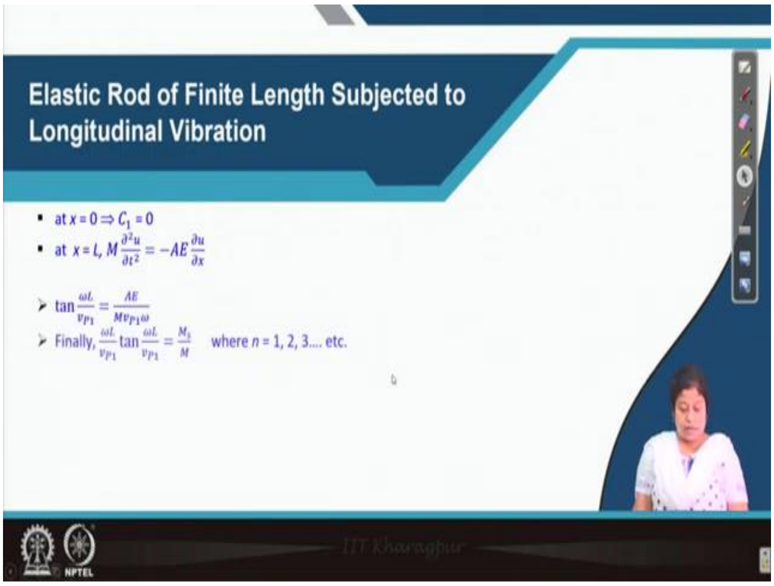


In this case, what we have considered is that we have considered one directional wave propagation.