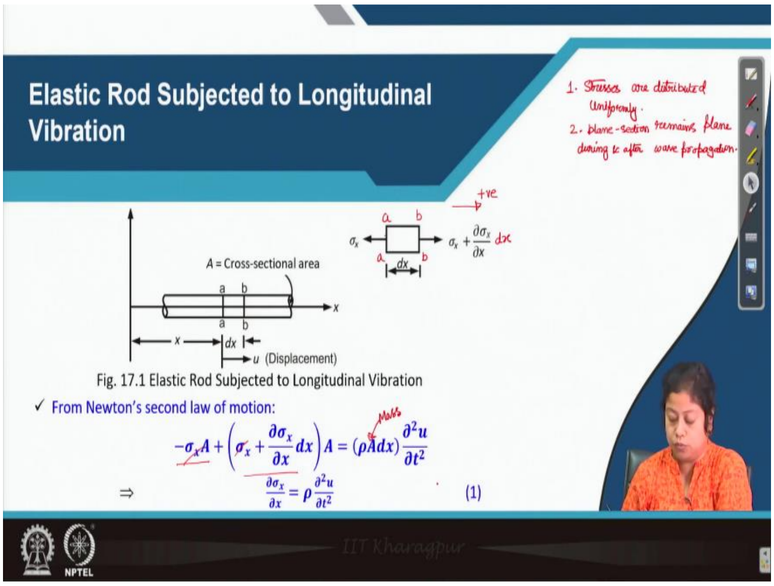
Soil Dynamics
Professor Paramita Bhattacharya
Department of Civil Engineering
Indian Institute of Technology, Kharagpur
Lecture 17
Wave Propagation in An Elastic Rod

Hello friends, today we will start a new topic which is Wave Propagation in an Elastic Rod.

(Refer Slide Time: 00:32)



So, here you can see we have taken one elastic rod, it is length is in we assume that it has in finite length. So, when longitudinal wave will propagate in this rod through these rod, then what will be happened, stress will be induced. So, if I will take only the portion a, b having length dx, how does it look, you can see here this is the a, b element.

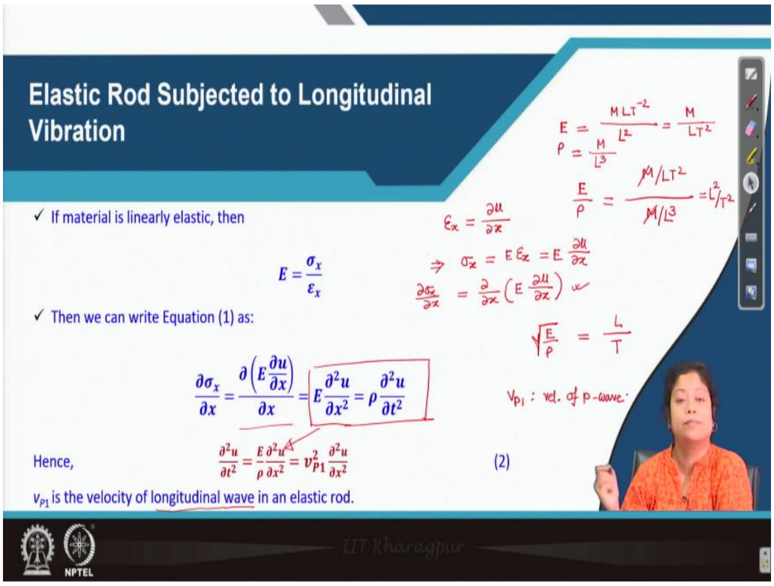
So, wave is propagating through these elements also and it has an, it has a length of dx. So, under longitudinal vibration, you can see on left hand side of these elements that means, on this area stresses is sigma x at a distance dx that means, on the face b, b the stress is normal stresses sigma x plus del sigma x by the del x of dx there is a dx here.

Now, in this case we have assumed that the stresses are distributed uniformly this is one thing and the second thing which we have assumed here is the cross sectional area of the rod which is plane remains plane even after wave propagation that means, plane section remains plane. Now, with this if we will write the equation Newton's if you will use the Newton's second law of motion, what will be the equation since our positive direction.

Then, we can write the force acting on the plane A, A is minus sigma x times a likewise the force acting on the plane B, B is sigma x plus del sigma x of del x times dx this entire thing is multiplied by the area A, A is the cross sectional area A A and B B and the this is the left hand side so, the right hand side will be inertia force which is mass, mass means in this case rho is the density of the element of this rod, A is the cross sectional area and dx is the length of this small element.

So, these represent mass so mass times acceleration. Then from this next step we can see del sigma x of del x is equal to rho times del 2u of del t2.

(Refer Slide Time: 05:06)



Now, if the betrayal of the rod is linearly elastic, then we can write E which is that elastic modulus is equal to sigma x divided by epsilon x. What is epsilon x here, epsilon x is longitudinal strain. So, epsilon x longitudinal strain. So, we can write it also as del u of del x, then from these two what we can write, we can write sigma x is equal to E times of epsilon x or E times of del u of del x.

Now, in the previous equation, there was a term called del sigma x of del x. So, in place of del sigma x of del x we can write del del x of E times del u by del x like this. So, let us do this in the next step. This we have done here which is shown in the previous tip. So, now, we can simplify these del sigma x by del x as E times of del 2u by del x2, you can see that in this expression and that should be equal to the right-hand side of the equation, which is rho times del 2u by del t2.

Then, here you can see that next part where I have written del 2u by del t2 is equal to E divided by rho times del 2u by del x2. So, from these we are getting this expression. Now, finally, these E by rho is expressed by vp 1 square why so. So, let us see what the unit of E is and what is the unit of rho. E is elastic modulus. So, it has a unit of stress, stress means, force divided by area is is the unit for stress, force means mass times acceleration.

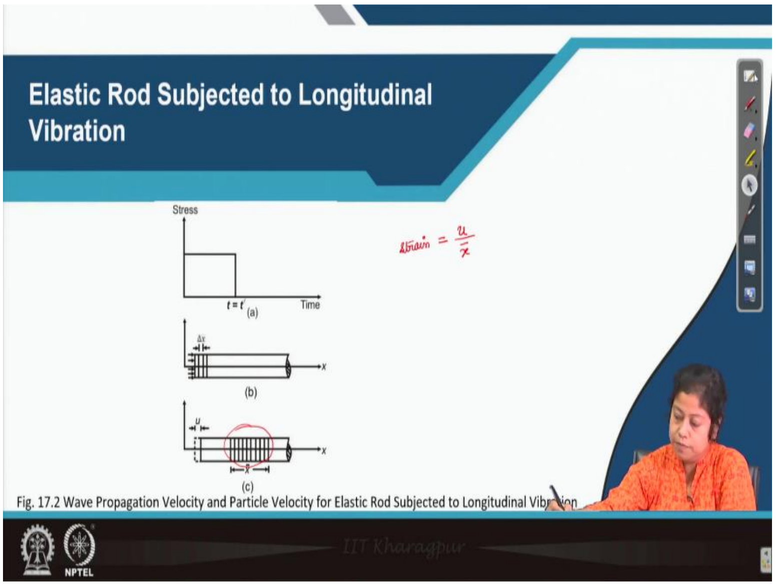
So, I am writing here mass times acceleration, divided by cross sectional area or area. So, L2 that means, what we are getting is mass divided by L2 square, L is length. T is time, M is mass. Likewise, now, here I can draw I can write rho as mass divided by rho is density. So, mass divided by volume that means, M divided by L cube then, what will be E by rho, E by rho will be I can write it as M by LT square divided by M by L cube.

So, what I am getting here I am getting here L square divided by T square that is E by rho. Then, if I will take square root of E by rho, I am writing here E by rho square root of that is L by T. So, the dimension of square root of E by rho is L by T which is the dimension for the velocity that is a reason here E by rho is expressed by a term vp 1 square, where vp 1 is the velocity of wave, 1 stands for one dimensional problem.

So, basically when we are considering wave is propagating through an elastic rod in one direction, which is here x direction that means, we are interested for one dimensional problem, that is the reason I have written vp 1 for the velocity of the wave considering only one direction of propagation of wave. Now, what is exactly vp 1, vp 1 is exactly the velocity of the elastic wave here you can see it is related to the elastic modulus.

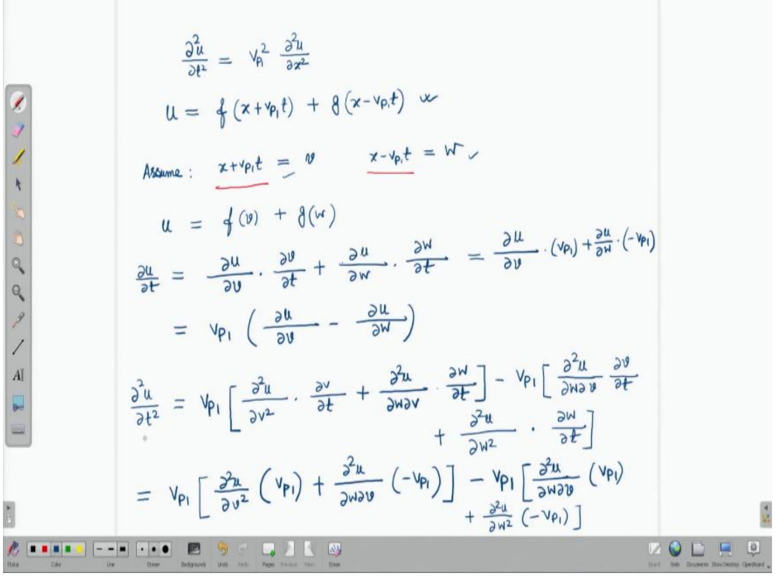
So, we can call vp 1 as the velocity of elastic or longitudinal wave in an elastic rod. So, I am just writing one more line for vp 1, we can call it also as velocity of elastic wave or we can call it as velocity of p wave also which is very well known term.

(Refer Slide Time: 12:02)



Now, let us see the consequence of the propagation of longitudinal wave in an elastic rod. At some time t is equal to t dashed, what we can see the displacement of the rod in positive x direction and that amount of displacement here is u. So, u is displacement at time t is equal to t dashed. Then, what is happened for this zone, these zone actually is compressed. Now, how much strain is developed in this zone that we can see. So, how much strain is developed here strain is equal to u divided by x bar, this is the amount of the strain.

(Refer Slide Time: 13:35)



Now, we can go to the board, del 2u by delta t2 that is equal to vp 1 square times del 2u of del x2. Now, here what we can do we can assume solution for u, u is equal to a function of x function of x not anything else. So, function of x plus vp, I can take it vp 1 times t plus another function g which is a function of x minus vp 1 times t. Now, we can assume we can assume that x plus vp 1 times t is equal to v and x minus vp 1 times t is equal to w small w.

Then we need to now find out del 2u of del t2. So, for that first we will find out del u of del t, which is equal to. So, we are differentiating this expression with respect to t. So, del u of del t that t is equal to what we can write, before that I can write one more step here. So, after assuming this thing x v is equal to x plus vp 1 times t and w is equal to x minus vp 1 times t what is the expression for u, we can write u is equal to function of v plus function of small w. So, function of v is represented by a f and function of small w is represented by g.

Now, we can find out del u of del t, which is equal to del sorry, I can write it as del u of del v, because u is a function of v times del v del t plus del w of sorry, del u of del w will be del u of del w as u is function of w also times del w del t. Now, we know the value for del v of del t from this expression, and we can find out the value of del w of del t from this expression. So, what is del v of del t, del v of del t is vp 1 and del w of del t is minus vp 1.

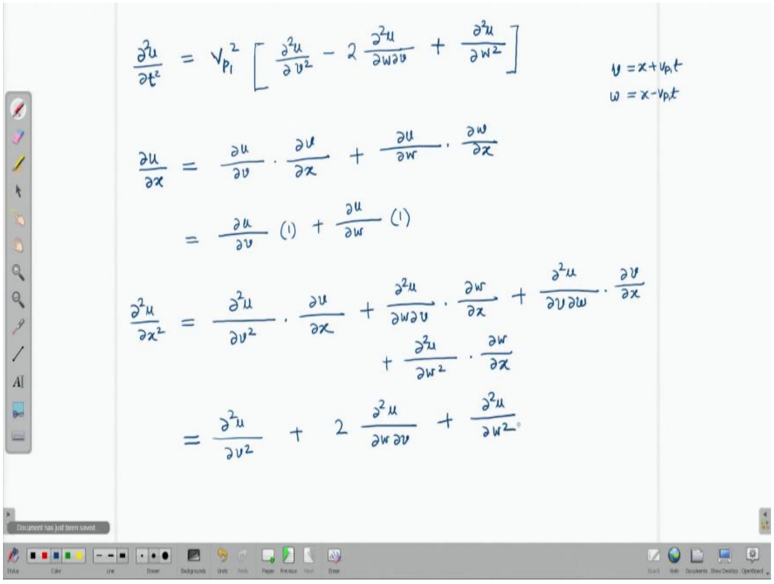
So, then I can write it as del u of del v times vp 1 plus del u of del w times minus vp 1. Now, we need to find out the second derivative for this expression. So, just let me write this line also minus del u, del w. So, now, I am finding out the second derivative of u with respect to t. So, del 2u of del t2, which is equal to what I can write here, vp 1 times first I am writing the first part del u del v.

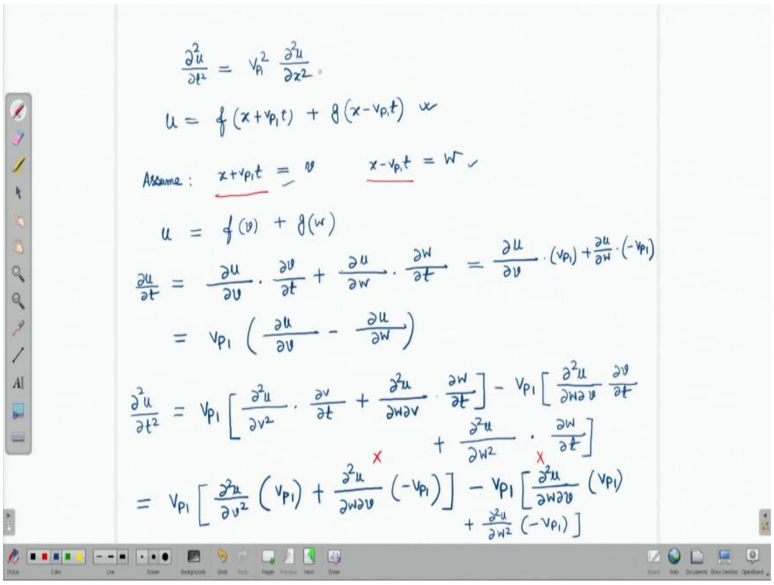
So, if I will differentiate del u of del v with respect to t what I can write here that I am writing. So, del 2u of del v2 times del v del t plus del 2u of del w del v times del w del t, this is for the first part for the second part I can write del 2u of del 2, give me one minute time, just a minute for the first part I have written these thing. So, for the first part basically. So, I have the differentiated E with respect to v and w.

So, the same thing I will do here now v and w. So, these will be del 2u of del w del v times del w sorry, times del w del I am sorry del v del t del v del t then the second part is del 2u of del w2 times del w del t. Now what we can write here again I need to use I need to write the value of del v del t which is del v del t means vp 1 and del w of del t means minus vp 1. So, I am writing that once again.

So, here what I can write del 2u of del v 2 times vp 1 plus here I can write del 2u of del w del v times minus vp 1. Now, for this remaining term here what I can write del 2u divided by del w del v times vp 1 plus del 2u of del w2 sorry, I can write in the next line also plus del 2u by del w2 times minus vp one. So, this is that expression for del 2u by del t2.

(Refer Slide Time: 21:59)





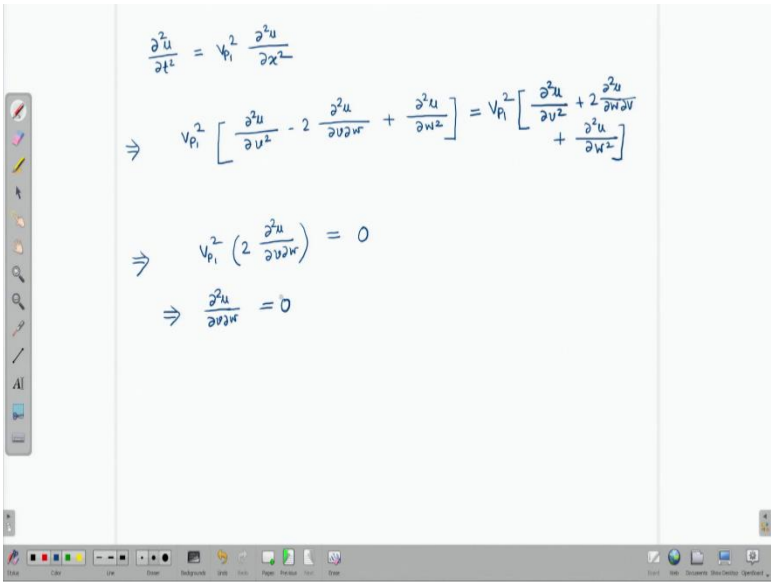
I can write one more step. So, del 2u of del t2 that is equal to what I can right now vp 1 square times del 2u by del v 2 minus you can see here two times of del 2u yes, 2 times minus 2 times a del 2u by del w del v then plus del 2u of del w2. So, I can right now here this thing vp 1 square inside that bracket, I am writing del 2u of del v2, then I can write minus 2 times of del 2u divide by del w del v or I can write it del 2u of del v del w both are same plus del 2u of del w2.

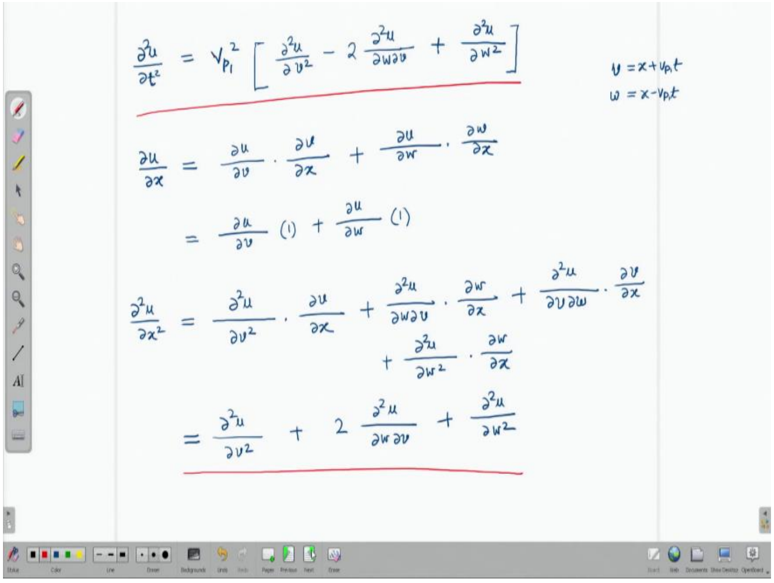
So, this is the expression for del 2u of del t2, which is the left hand side of the governing equation. Now, what we need to find out we need to find out the right-hand side that means this part vp 1 square times del 2u by del x2. So, this is very simple. I am just erasing this underlined portion here. So, first thing I need to find out del u of the del x.

So, del u of del x means what we can right here. Del u of del v times del, just let me correct del u of del v times, del v of del x plus del u of del w time del w of del x. Now, if you remember what is v, v is x plus vp 1 times t whereas, w is x minus vp 1 times t. So, del v of del x means 1, del w of del x means also 1. So, I can write here then del u of del v times 1 plus del u of del w times 1. Now, I can take the second derivative of u with respect to x.

So, del 2u of del x2 that is equal to the del 2u of del v2. Then what I can write del v of del x plus del 2u of del w del v times del w del x plus del 2u of del v del w times del v del x plus del 2u of del w2 times del w del x. So, what we can right here it means del 2u of del x2 is equal del 2u of del v2 plus 2 times of del 2u of del w del v plus del 2u of del w2.

(Refer Slide Time: 26:49)



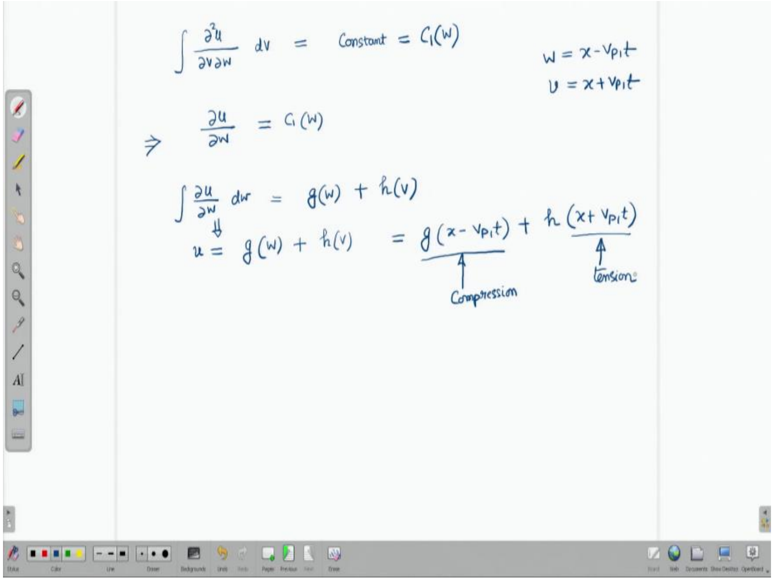


Now, what I can write in the next line, we have this governing equation del 2u of del t2 is equal to vp 1 square times del 2u of del x2. Now here I can write the expression for del 2u of del t2 which is already known I can just show. So, this is the expression for left hand side and this expression for the right-hand side. So, I can write it then the next page.

So here what I will write is vp 1 square times del 2u of del v2 minus 2 times del 2u of del v del w plus del 2u of del w2, this is that left hand side, right hand side is first we need to write vp 1 square then the expression for del 2u of del x2 that means this expression. So, it is del 2u of del v2 plus 2 times of del 2u of del v del w or I can write del w del v both are equal same. Now, third term is del 2u of del w2.

So, now from these expressions what we can see, we can see vp 1 square times 2 times of del 2u of del v del w is equal to 0. Now here. So, here what is vp 1 square sorry. What is vo 1? Vp 1 is the velocity of the longitudinal wave or P wave and that cannot be 0. So, the second term in this product that means del 2u of del v del w is 0 here. So, what I can write them del 2u of del v del w is equal to 0.

(Refer Slide Time: 29:31)



Now, if we will integrate the expression which we get in the previous page, let us integrate it for function v, then what we will get? We will get a constant term and that constant term is a function of w but not v, if it is a function of v, then when we are differentiating it that time del 2u by del v del w should not be 0.

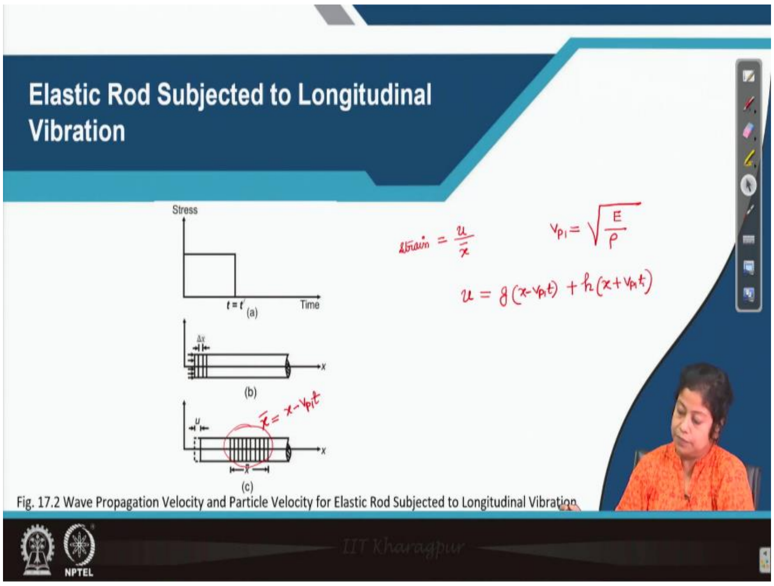
Since it is 0 that means, when we are integrating the constant term is coming appearing that constant term is not a function of v but a function of w only. Now, again we will integrate it. So, basically what we are getting here del u by del w which is c1 which is a constant term, maybe a function of the w. Now, if we will integrate it once again with respect to dw, then what we will get we will get another term.

So, let us write it then when we are getting another term, how we can write that another term we can write it as some function of like this. So, from what this left-hand side is u, so, what we can see u is equal to g which is a function of w plus h which is a function of v. So, in this way and already we have assumed that w is. So, I can write it here x minus vp 1t likewise for v it is x plus vp 1t.

So, u can be expressed now, x minus vp 1t plus h which is x minus sorry x plus vp 1t. Now, here what we can see basically this function g which is a function of x minus vp 1t, what exactly is this x minus vp 1 times t, x minus vp 1 times t indicates reduction in the value of x or the length that means, that material or rod is compressed here.

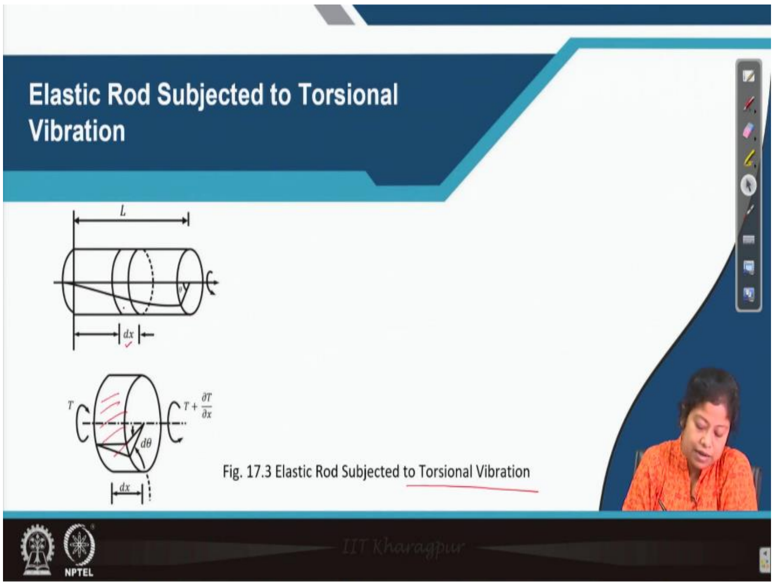
So, these function represent compression whereas, this one indicates elongation of the rod. So, it is a tension related to the tension then what we can see u which is the displacement is a combination of completion and tension or I can say completion and elongation.

(Refer Slide Time: 33:46)



Now, if I will go back to this figure here what we have seen is compression. So, this x x bar is your x minus vp 1 times t, what is vp 1 here, I hope we all remember it is square root of E by rho, rho is the density E the elastic modulus. So, now here what we can do is that we can express the elongation. So, in this way we can express for the total displacement u is sum of compression and elongation. So, I am writing here, this is gx what we get here at xg is compression. So, x minus vp 1t plus h which is a function of x plus vp 1t.

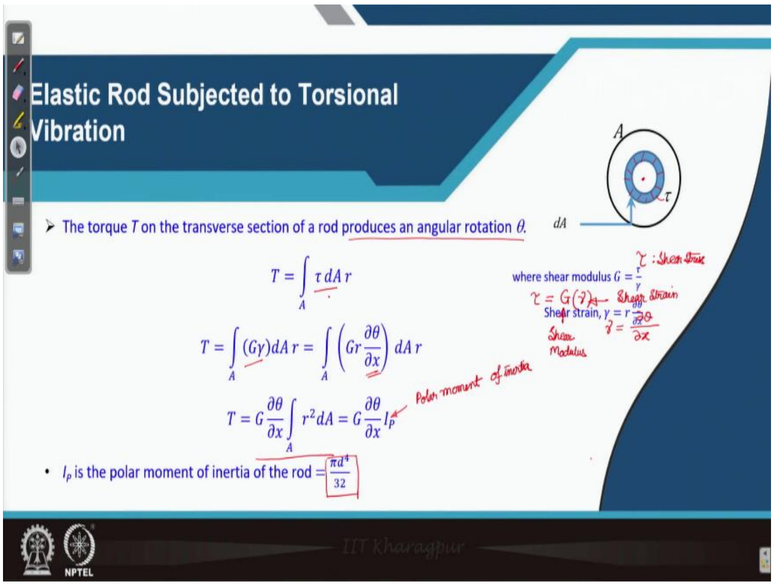
(Refer Slide Time: 35:25)

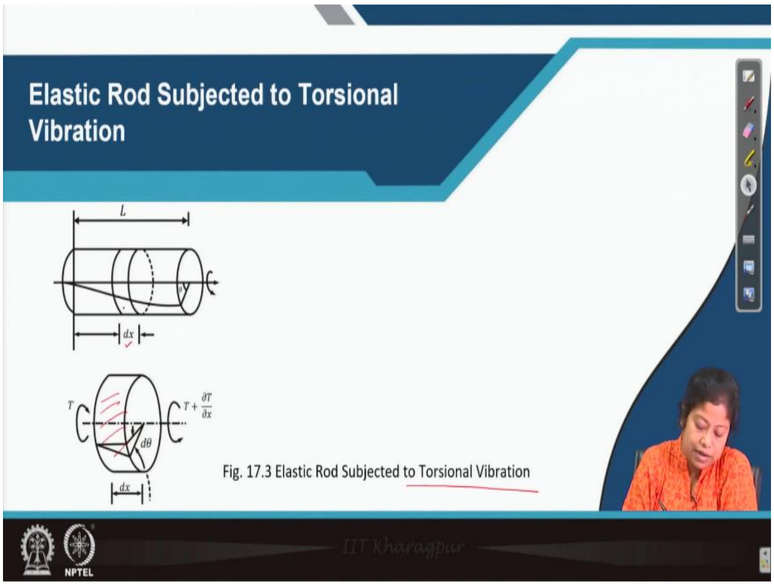


Now, another now, see the next one. Now, what will be happening when the same elastic rod is subjected to torsional vibration. So, if we will apply torque at the ends of this elastic rod, then what will be happening it will be subjected to torsional vibration and that time what how it will react, what will be the displacement or what will be the response better word is to is here response.

So, what will be the response that time. So, let us see that. So, the finger which is shown here, here we can see an elastic rod which is subjected to torsional vibration. Now, we have chosen a small element of length dx you can see here this is this small element.

(Refer Slide Time: 36:37)





So, first we will see how we can express torque T here, this is the cross-sectional area of the small element dx. Now, if we will divide this cross sectional area by a number of annular portion like this, then how much shear stresses develop that we need to know. So, that is tau it is shear stress then total force will be tau times da because dA is that area of this annular portion. Now, because of these top T the rod produces an angular rotation theta, if you go back to the previous slide here you can see the rod is subjected to an angular rotation of theta.

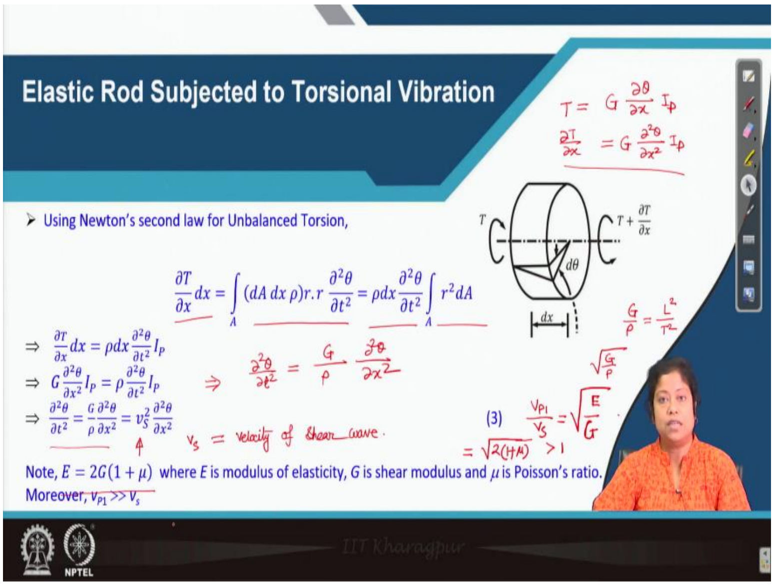
Then what we can write here then that torque is equal to these shear stress times r r is the radial distance from the centre. Then, we know tau, which is the shear stress can be written as G times gamma here G is shear modulus and what is gamma this is shear strain, then we can write in place of tau here we can write g times gamma times the area of this annular portion which is dA times the distance from centre which is r here.

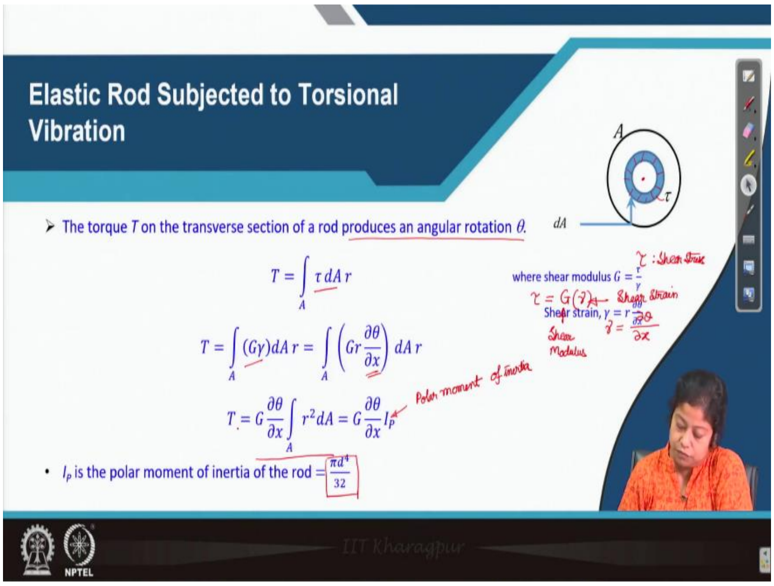
Now, shear strain gamma can also be written as del theta of del x for the small element of having length dx, then here what we can do we can replace the shear strain del theta of del x you can see in this expression. So, in the next step, we can take out all the terms which is not a function of the area A. So, G neither G nor del theta of del x that means, the shear strain is function of the area. So, I can take it out of this integration to find out the total torque.

So, it is written this way and what is integration of r square dA over the area A it is nothing but the polar moment of inertia which is Ip polar moment of inertia and for rod having diameter d small d, what is the value of Ip or what is the expression of Ip that is pi times d to the power 4 divided by 32. So, we can use this expression when it is needed. So, in this way we can express total torque T is equal to G times del theta by del x times Ip.

I think I told here only the shear stress it is not just the shear stress because the tau is the shear stress here and dA is the area of this angular element. So, basically tau times dA is your shear force.

(Refer Slide Time: 41:27)





Now, we need to use the Newton's second law of motion for the unbalanced torsion in this case, unbalanced torsion if you see that is del t of del x times dx. So, this is your unbalanced torsion and that can be represented already we have the expression for t which is we can see just the previous slide for T this is our expression G times del theta of the del x times Ip.

So, del t of del x in place of that now, we will write what we can write is G times del 2 theta del x2 times Ip for the left-hand side and the right-hand side is equal to mass moment of inertia. So, here this is mass, so, mass moment of inertia the expression for that is this one this is the final expression once again here we can see integration of r square dA which is polar moment of inertia.

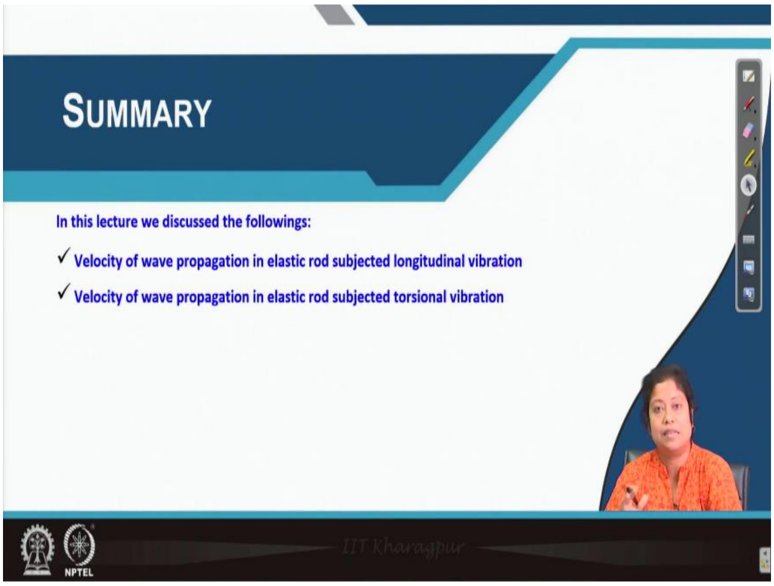
So, we can write this expression as del t of del x times dx is equal to rho times of the row times of dx times del 2 theta by delta t2 times Ip. Now, in place of del t by del x we can write this. So, writing that what we are getting that is written here that G times del 2 theta by del x2 times Ip is equal to rho times del 2 theta by del t2 times Ip. Now, Ip cannot be 0 and it appears on right hand side as well as left hand side.

So, we can divide Ip from both sides and final expression is this one that del 2 theta by del t2 is equal to. So, what we are doing basically, if I will right here G by rho times del 2 theta by the del x2. So, that is written here. Now, if you see that dimension of G by rho G has a dimension of stress and rho has a dimension of mass per unit volume that means mass divided by L cube.

So, finally, we will get in this case also L square by T square like the previous case for a by rho and then we can see G by rho square root of G by rho represent some velocity and that is because of the shearing of the road. So, that velocity and shear wave is that time shear wave is generated in the rod. So, vs is the velocity of the shear wave you can I can write here vs is velocity of shear wave and this is a well-known relationship between elastic modulus and shear modulus where mu is that Poisson's ratio.

So, from this what we can see if someone will ask us to find out the ratio of vp 1 by vs that is nothing but square root of E by G, E can be expressed by this way. So, finally, what we are getting is square root of 2 times 1 plus mu. So, this is always greater than 1, then finally, what we can see is vp 1 is always greater than vs that means, the velocity of the P wave through elastic rod is greater than the velocity of the shear wave in the same elastic rod.

(Refer Slide Time: 47:03)



So, today's lecture we have discussed the velocity of wave propagation in elastic rod, when it is subjected to longitudinal vibration, the velocity of the wave propagation when it is in the elastic rod when it is subjected to torsional vibration. With this, I am concluding today's class.

(Refer Slide Time: 47:37)

These are the references which I have used for this lecture. Thank you for attending today's class.