Theory of Computation
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Lecture - 11
Single Degree of Freedom System (SDOF) - Part 9

Hello friends, welcome to the class Soil Dynamics. So, far we have discussed single degree of freedom system under undamped condition or damped by viscous damping. Now, our today's topic is single degree of freedom system with coulomb damping.

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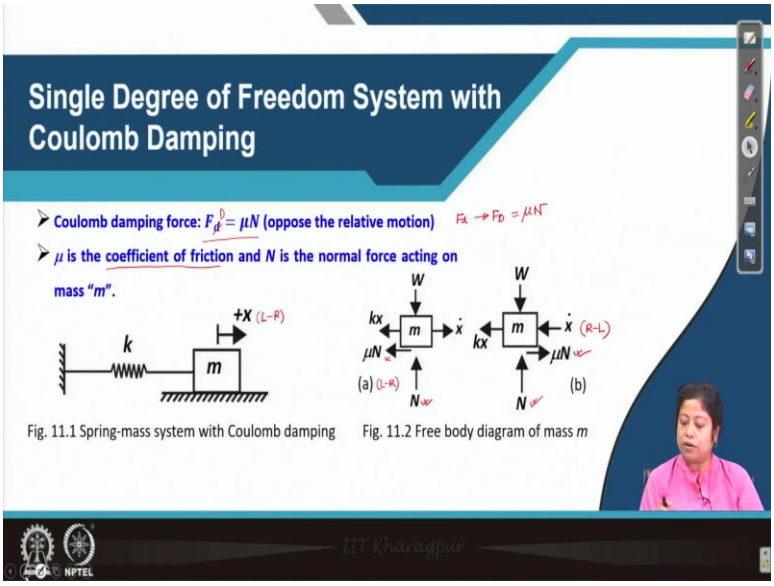

So, first of all how coulomb damping works? We can see this spring mass system with the coulomb damping. So, here we have a mass m which is attached to the spring having stiffness k. Now, when the mass m will be allowed to move in positive x direction, in this case positive x reduction means left to right. I can write it here, left to right.

Then what will be happened? There will be a friction between the mass and the surface on which it is resting. And these friction opposes the motion of the mass m. Then how the freebody diagram looks? Here we can see that freebody diagram. In this case, you can see the mass is moving, the velocity of the mass is shown which is left to right. So, figure a shows left to right, figure b shows you can see the direction of the velocity.

So, this is right to left. Now, when the masses moving from left to right that means, we can see in this figure it is moving from left to right, then what is happened? The surface on which these mass resting offers a resistance in the opposite direction of motion. In this case, the direction of motion is left to right. So, resistance from right to left you can see here.

Now, for the case, when motion is operating from right to left, then the surface on which the mass was resting offers a resistance opposition to the motion that means, in this case resistance will act from left to right. You can see in this figure it is occurring from left to right. And what is the magnitude of the damping force because of the coulomb damping? The magnitude of the damping force you can see, F d is equal to mu N.

You can write instead of F d here, you can make it capital D for damping. So, in place of a small d I am writing it F capital D. So, then what will be happen? The damping force because of the coulomb damping is capital F D and the magnitude of that forces mu times capital N. What is mu? In this case, mu is the coefficient of friction you can see here and what is N? N is the normal force acting on the mass m.

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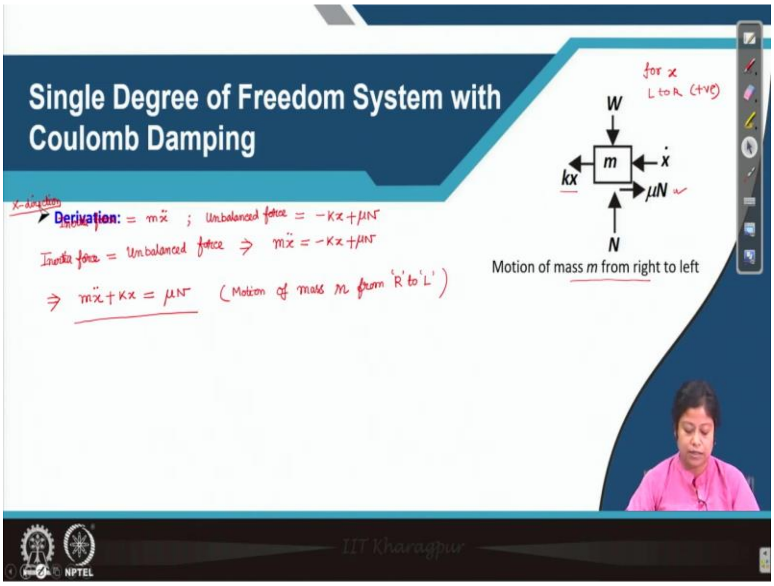

Now, let us consider the motion is occurring from right to left first and the left to right. So, when motion is occurring from right to left, what will be happened? Damping force acts from left to write in this case you can see it. Then if we write the force equilibrium equation, then what will be that? Inertia force in this case I am writing here, inertia force in this case is how much, obviously, mass times acceleration.

Now, you can find out the unbalanced force, we are talking in x direction, so, better I write it here also in x direction. So, unbalanced force is how much? That is minus k x plus mu N.

Why it is plus? Mu N and minus k x, minus k x is the spring force whereas, damping forces plus mu N because we have considered lift to right motion when the direction of x is left to right that is positive so, for x left to right we will consider positive direction.

In this case, you can see spring force is acting from right to left that on the body or on the mass m that is the reason it is with negative sign. Then in equilibrium what we can write? This inertia force is equal to the unbalanced force then we can write m x two dot is equal to minus k x plus mu N.

So, once again I can rewrite it test in x two dot plus k x is equal to mu N. So, in this case motion is occurring from right to left, better I should write it in full sentence here also, motion of mass m from right to lift. This is the equation of motion in that time.

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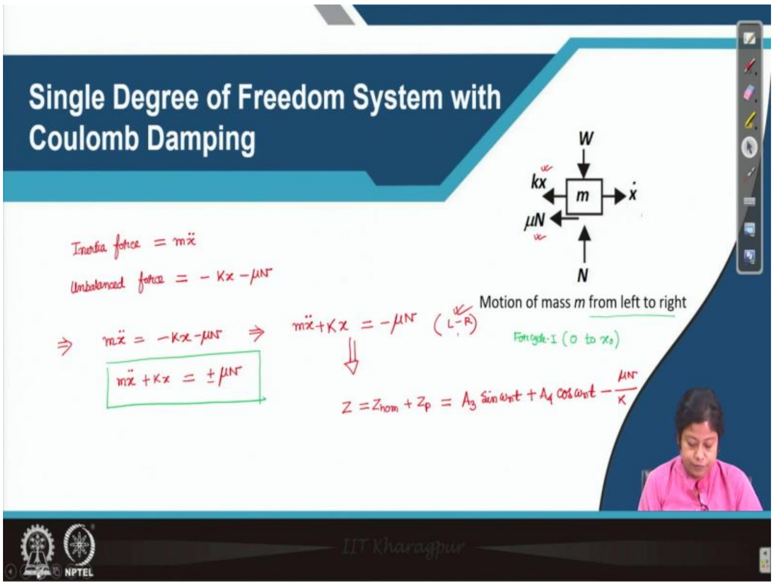

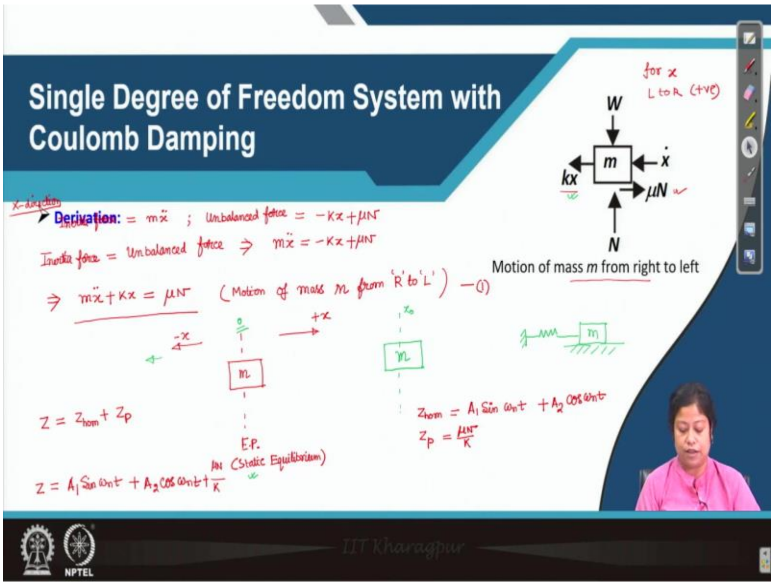

Now, next case you can see when the mass is moving from left to right, in this case mass is moving from left to right then what will be the equation of motion in this case? So, in this case inertia forces m times x two dot. What about the unbalanced force? In this case, unbalanced forces because of the spring force and because of the damping force offered by the coulomb damping then it is minus k x minus mu times N because why minus because we have considered, when direction is left to right we will consider it as positive direction.

So, then in equilibrium condition what we can write? Inertia force is equal to the unbalanced force which is in this case minus k x minus mu N then we can write the equation of motion the final form is m x two dot plus k x is equal to minus u N. So, this time motion of mass m is occurring from left to right. So, what we have seen?

Depending upon the direction of motion of the mass m, the equation of motion will change. So, there are two equations of motion in this case, we can write both together as a m x two dot plus k x is equal to plus minus mu N. When it is plus? When motion is occurring from right to left. And when it is minus? When motion is occurring from left to write, but, there is a question.

So, question is that in the previous case, when motion was occurring right to left that time, you may think, we can understand the direction of mu N which is the direction of the damping force because of coulomb damping, but what about why k x is in this direction that means, spring is trying to push sorry trying to pull the mass m. In this case you can see spring is exerting pulling force to the mass m, why so?

Actually, when we are seeing right to left basically what is happen that I would like to explain now. This is the mass in equilibrium static equilibrium position. So, this is your equilibrium position of mass N, static equilibrium I am writing. Now, what is happen when it is moving from right it is it can move in both direction positive x and negative x also.

Now, in positive x direction, let us take it reach to the extreme position let us say for first case, extreme position is x 0. Now, what is happen, this is the extreme position in the positive x direction when mass m is subjected to our motion. Now, what is happen? From this extreme position, now, the mass will try to go back to its equilibrium position here. And then it will cross this point and will move towards this direction.

As long as these mass m is this equilibrium position, let us take O to x 0, what is happen? If you think about the spring, which is attached to the mass, how it was attached, I am just showing here. Now, when mass is this is in equilibrium position. Now when mass is at x 0, what is happened to this spring? This spring is stretched. And that condition will be there as long as the mask is placed paste in between 0 to x 0.

So, x 0 is the extreme position however 02 to x 0 at any distance from or O to x 0 we can say at any position what will be happen? The spring will be stretched, if the spring is stretched, then what will be happen to the mass m that will be pulled by the spring that is the reason in this case when mass is moving even from right to left. Right to left means x 0 to this position, that time also the spring is exerting pull out force to the mass m and we can see that direction of k x which is the spring force is opposite to positive direction.

I hope for this case, there is no confusion because in this case mass is moving from left to right that means O to x 0 position for first cycle then I can write it here for cycle 1. This is the position. So, obviously, spring force will act in negative x direction. So, in this way, we can derive the equation of motion for the mass spring system with coulomb damping. Now, we need to find out the general solution for this problem.

Now, for this type of problem, let me go back to right to left case. So, this is our right to left case. Now, in this case, we can see this equation is no more homogeneous equation I can give a name to this equation is this equation 1. So, this equation one is not homogeneous equation there is a term non0 term on the right hand side. So, the solution for equation 1 will be solution of homogeneous equation plus solution of a particular solution in this case.

Now, what is the solution of homogeneous equation for this case, that we already have learned. So, set homogeneous in this case is can be expressed as A 1 sin omega n t plus A 2 cosine omega n t. And what will be the particular solution for these case? For this case, particular solution will be you can see the equation that will be mu N divided by k, then, we can write that solution set is equal to A 1 sine omega n t plus A 2 cosine omega n t then what is left for us?

In this case, it is plus let me write it close to this otherwise, yes mu N by k. This is for the case when motion of mass is occurring from right to left. Now, come to the case when motion of mass is occurring from left to right that means, these case. Then for these case, z will be the same way z homogeneous plus z particular and that means z for homogeneous equation plus particular solution, then we can write it here as A 3.

A 3 sin omega n t plus A 4 cosine omega n t minus mu N divided by k, this is for the case when motion of mass from left to right. Now, we have two equations and two solutions depending upon the direction of motion of mass from 0 to x 0 position.

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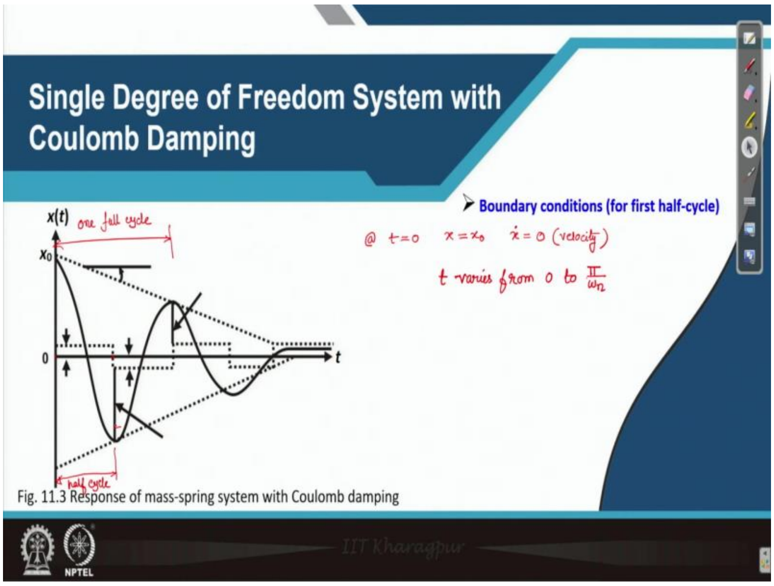
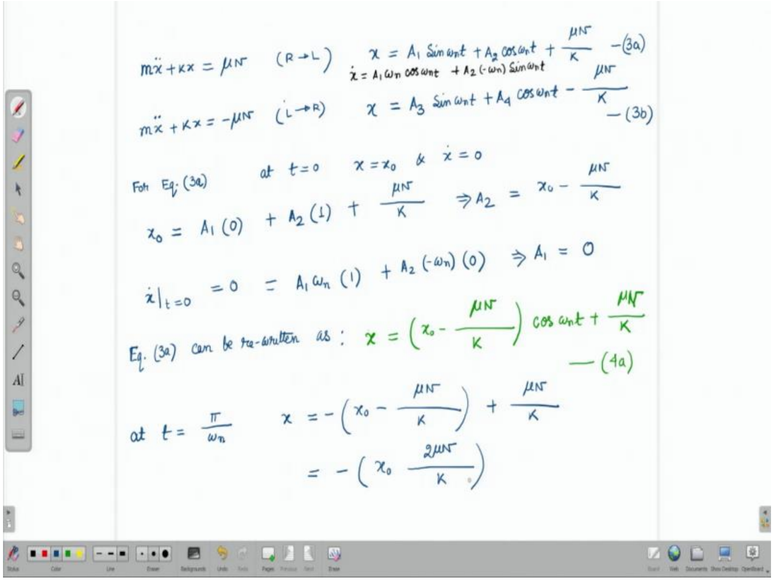
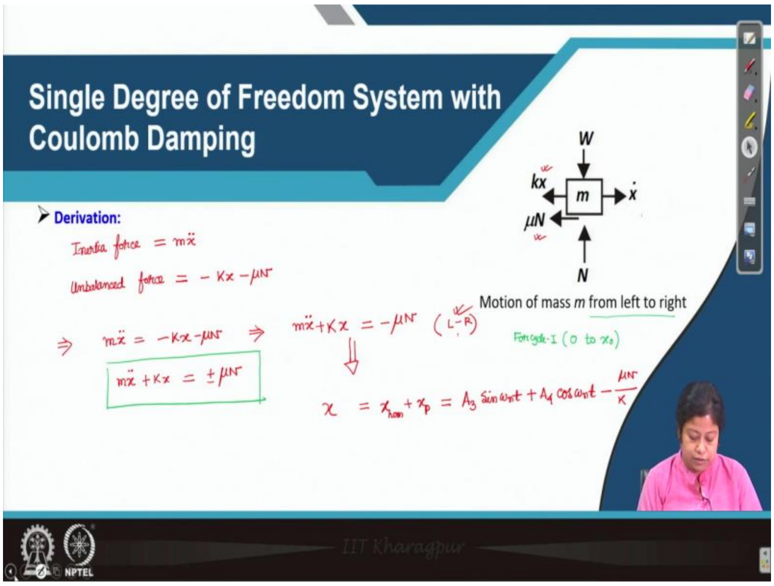



mẍ + kx = μN (R→L) x = A1 sin ωnt + A2 cos ωnt + μN/k —(3a)
ẋ = A1 ωn cos ωnt + A2 (−ωn) sin ωnt
mẍ + kx = −μN (L→R) x = A3 sin ωnt + A4 cos ωnt − μN/k —(3b)
For Eq. (3a) at t = 0 x = x0 & ẋ = 0
x0 = A1 (0) + A2 (1) + μN/k ⇒ A2 = x0 − μN/k
ẋ|t=0 = 0 = A1 ωn (1) + A2 (−ωn) (0) ⇒ A1 = 0
Eq. (3a) can be re-written as : x = (x0 − μN/k) cos ωnt + μN/k —(4a)
at t = π/ωn x = −(x0 − μN/k) + μN/k
= −(x0 2μN/k)


Single Degree of Freedom System with Coulomb Damping
Derivation:
Inertia force = mẍ
Unbalanced force = −kx − μN
⇒ mẍ = −kx − μN ⇒ mẍ + kx = −μN (L−R)
mẍ + kx = ±μN
For cycle-I (0 to x0)
x = xhom + xp = A3 sin ωnt + A4 cos ωnt − μN/k
W
kx
m
ẋ
μN
N
Motion of mass m from left to right
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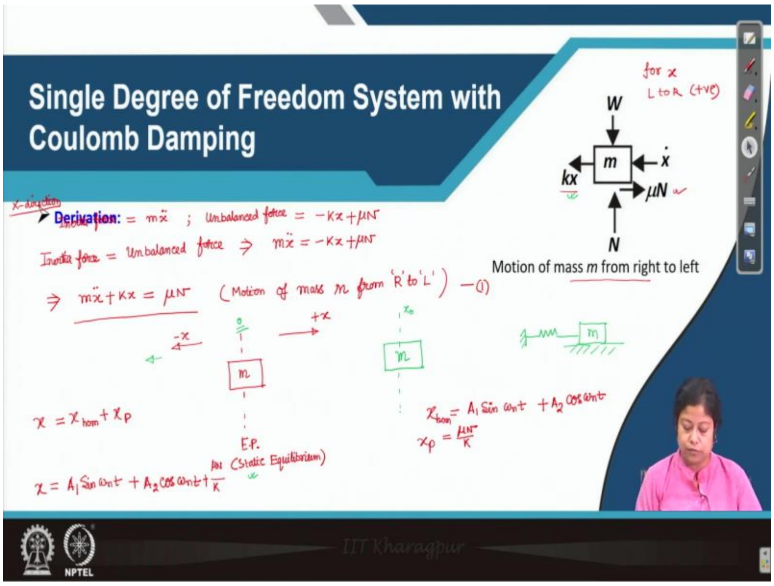
Single Degree of Freedom System with Coulomb Damping
for x L to R (+ve)
x-direction
Derivation: = mẍ ; Unbalanced force = −kx + μN
Inertia force = Unbalanced force ⇒ mẍ = −kx + μN
⇒ mẍ + kx = μN (Motion of mass m from 'R' to 'L') —(1)
−x
+x
x0
m
E.P. (Static Equilibrium)
x = xhom + xp
x = A1 sin ωnt + A2 cos ωnt + μN/k
xhom = A1 sin ωnt + A2 cos ωnt
xp = μN/k
W
kx
m
ẋ
μN
N
Motion of mass m from right to left
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So, here you can see the situation. So, for first half cycle x 0 to 0, and then 0 to minus x 0 in the other direction. So, before coming here, now, we need to find out the solution, let us go to the whiteboard. So, we have two equations of motions. I am writing here once again first one is m x two dot plus k x is equal to mu N for right to left and solution is A 1 sine omega n t plus A 2 cosine omega n t plus mu N divided by k.

The second equation is for motion left to right and that is m x two dot plus k x is equal to minus mu N. So, in this case, our general solution is z that is A 3 times sin omega n t plus A 4 times cosine omega n t mu N divided by k. Now, let us see this slide. So, for the first half cycle, first half cycle means from where to where? Up to this because full cycle means from here to here, this is full cycle or I can write one full cycle. This is half cycle.

Now, for first half cycle one boundary condition we can see it here you can see at t equals to 0 that means at this point, what is the value of x? Yes, here it is x not z please correct. I have made a mistake here. This is x homogeneous. That means solution for homogeneous equation plus particular solution, same thing for the previous case also. Two, three places mistakenly I have written z which should be x for this case.

So, here it is x, x, same thing here also it is x, x for homogeneous equation, x particular solution, here also I have corrected now. Now come to the boundary condition. So, what we can see here at t is equal to 0, what is the value of x? x is x 0. And if we will draw a slope here, that means x dot that will be how much, x dot will be 0 here, and x dot is nothing but the velocity. So, now, we have two boundary conditions at t is equal to 0.

One boundary condition says x is equal to x 0 and the other boundary condition says velocity is 0 at t is equal to 0. Then what will be the then, what we can do here, give me name for this one, let us take this is 3 a e and this is 3 b. So, then we can write for equation or expression 3 a at t is equal to 0, x is equal to x 0 and x dot is 0. So, first, t is equal to 0, x is x 0. So, I am writing equation 3 a once again, x 0 is equal to A 1 times this is 0 because t is 0 plus A 2 times 1 plus mu N by k.

So, from this we can get A 2 is equal to x 0 minus mu N by k. Likewise, if we will find out x dot at t is equal to 0 which is 0 here. Now, what is the expression for x dot? The expression for x dot is A 1 times omega n times cosine omega n t, I can write here also A 1 times omega n times cosine omega n t plus A 2 times minus omega n times sin omega n t. This is the expression for velocity.

Now at t is equal to 0, velocity is 0. So, we can write then A 1 omega n times 1 plus A 2 times minus omega n times 0, because sin omega n t is 0 at t is equal to 0, then from this we can get A 1 is 0. That means now equation 3 a can be rewritten as x is equal A 1 is 0, so, first term, it will not be there, A 2 is how much, x 0 minus mu N by k. So I am writing x 0 minus mu N by k, let me write it better way mu N by k times cosine of omega N t plus mu N by k. So, this is our new equation 4 a.

Now, what about equation 3 b? What will be the boundary condition for equation 3 b? So, when I am talking about first half cycle, that means t from where to where? t is driving from 0 to pi by omega N this is pi by omega N. This is the domain for the first half cycle. Now, for the second half cycle when at the beginning of second half cycle that means, when t is 0 for the second half cycle, what will be the value of displacement this one.

So, let us see what is the value of displacement. So, here I can write at t is equal to pi by omega N, what will be x? So, cosine omega n t at pi t is equal to pi by omega N is nothing but cosine pi, cosine pi means minus 1 plus mu N by k so, it is coming minus x 0 minus 2 mu N by k.

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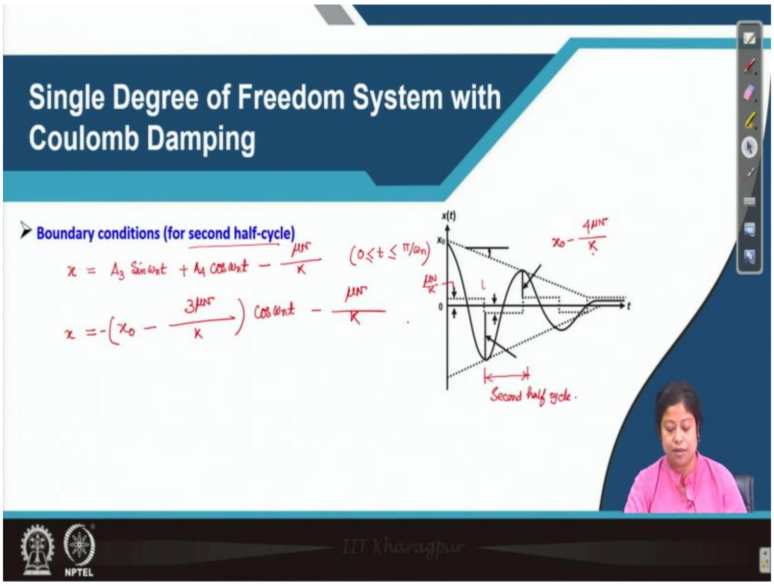

For Second half cycle

$$x = -\left(x_0 - \frac{2\mu N}{K}\right) \quad \text{at } t=0$$

$$-\left(x_0 - \frac{2\mu N}{K}\right) = A_3(0) + A_4(1) - \frac{\mu N}{K}$$

$$\Rightarrow \quad A_4 = -\left(x_0 - \frac{3\mu N}{K}\right)$$

for $\dot{x} = 0$ at $t=0$ $\quad A_3 = 0$

$$x = -\left(x_0 - \frac{3\mu N}{K}\right)\cos\omega_n t - \frac{\mu N}{K} \qquad (4b)$$

at $t = \pi/\omega_n$ (for Second half cycle)

$$x = (-)(-1)\left(x_0 - \frac{3\mu N}{K}\right) - \frac{\mu N}{K}$$

$$= x_0 - \frac{4}{\quad}$$

Single Degree of Freedom System with Coulomb Damping

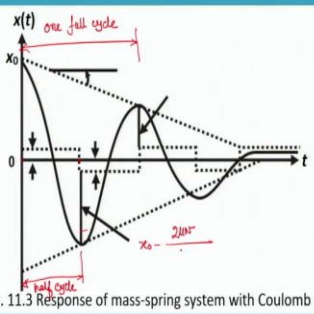

Fig. 11.3 Response of mass-spring system with Coulomb damping

- **Boundary conditions (for first half-cycle)**

@ $t=0$ $\quad x = x_0 \quad \dot{x} = 0$ (velocity)

t varies from 0 to $\frac{\pi}{\omega_n}$

$$m\ddot{x} + Kx = \mu N \quad (R \to L) \qquad x = A_1 \sin\omega_n t + A_2\cos\omega_n t + \frac{\mu N}{K} \quad -(3a)$$

$$\dot{x} = A_1\omega_n \cos\omega_n t + A_2(-\omega_n)\sin\omega_n t$$

$$m\ddot{x} + Kx = -\mu N \quad (L \to R) \qquad x = A_3 \sin\omega_n t + A_4\cos\omega_n t - \frac{\mu N}{K} \quad -(3b)$$

For Eq. (3a) $\quad$ at $t=0$ $\quad x = x_0$ & $\dot{x} = 0$

$$x_0 = A_1(0) + A_2(1) + \frac{\mu N}{K} \quad \Rightarrow A_2 = x_0 - \frac{\mu N}{K}$$

$$\dot{x}\big|_{t=0} = 0 = A_1\omega_n(1) + A_2(-\omega_n)(0) \quad \Rightarrow A_1 = 0$$

Eq. (3a) can be re-written as: $x = \left(x_0 - \frac{\mu N}{K}\right)\cos\omega_n t + \frac{\mu N}{K}$ — (4a)

at $t = \frac{\pi}{\omega_n}$ $\quad x = -\left(x_0 - \frac{\mu N}{K}\right) + \frac{\mu N}{K}$

$$= -\left(x_0 - \frac{2\mu N}{K}\right)$$

So, these boundary condition now we can use for the second half cycle that means, here itself let me write this value here also x 0 minus 2 mu n by k. Now, that means, for second half cycle what we can write at for second half cycle our equation was x is equal to A 3 sin omega n t plus A 4 times cosine omega n t then minus mu N by k.

This is the general solution where t varies from once again 0 to pi by omega N, so, when I am writing t here, same thing t varies from 0 to pi by omega N that means, from here to here, this is second half cycle. So, here x at t is equal to 0, I need to write the expression. So, let us go back to the whiteboard for a second half cycle x is equal to minus x 0 minus 2 pi N by key at t is equal to 0. Then what we can write?

Minus x 0 minus 2 by N by k time that is equal to A 3 times 0 plus A 4 times 1 which gives us A 4 is equal to how much, 3 mu N by k. So, the and for x dot equals to 0 at t is equal to 0 we can get A 3 will be 0 just like the previous case that means first half cycle. So, finally, what we can see? The general solution for the second half cycle becomes A 3 which is 0 here. So, I am not writing A 3 sin omega n t here, I am directly writing A 4 cosine omega n t.

So, the value of A 4 is a minus of x 0 minus 3 mu N by k times cosine omega n t plus mu N by k, it will not be plus it will be minus. So, this is the general solution, you can see here the first one, now, the second one that means, second half cycle general solution is this one, so, we can give it the name 4 b, then once again see here at the end of the second half cycle that means, at this point what will be the value of our x?

That means, at t is equal to pi by omega N for second half cycle, x will be minus, in this case cosine omega n t at t is equal to pi by omega N is minus 1. So, there are two minus times minus 1 times x 0, 3 mu N by k minus mu N by k. So, finally, what we are getting is x 0 minus 4 mu N by k here. So, this is nothing but let me write this is x 0 minus 4 mu N divided by k and these distance is mu N by k this one.

So, in this way finally, we are getting the general solution which is see it once again here minus x 0 minus 3 mu n by k times cosine omega n t minus mu N by key. So, that means, what we have seen today? At the end of the first full cycle the amplitude reduced by an amount 4 mu N divided by k because at the starting amplitude was x 0 and at the end of the first cycle, it is x 0 minus 4 mu N divided by k which indicates that amplitude is reduced by an amount 4 mu N divided by k.

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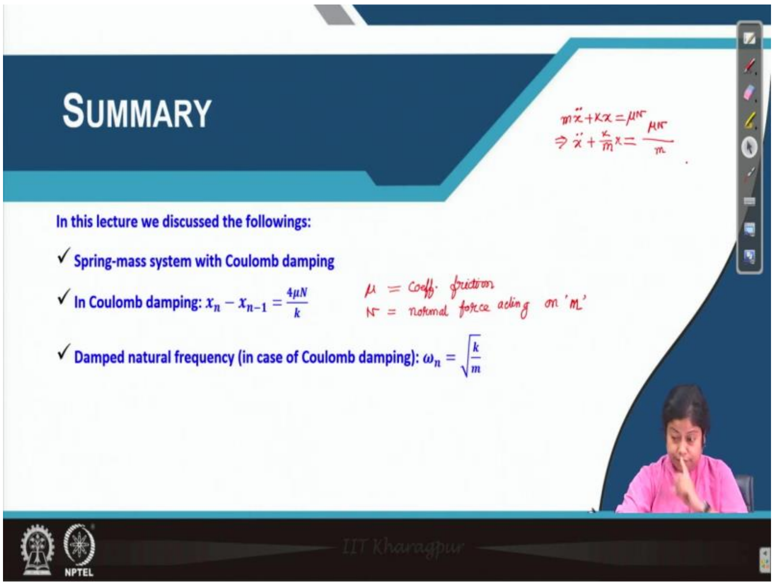

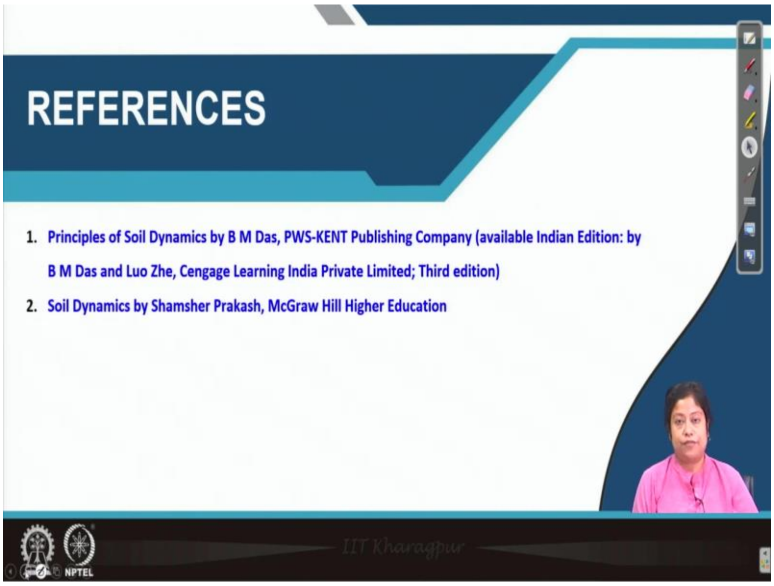

So, come to then the summary of today's class, we learned spring mass system with coulomb damping, how we will form the equation of motions and from that how we will say the boundary conditions and from that what will be the solution. Then what we have seen, the difference between two consecutive amplitude in case of coulomb damping is 4 mu N divided by k where mu is coefficient of friction, N is normal force acting on mass m, this is small m and k is the spring stiffness.

Another interesting thing which we have studied in these case is that in case of coulomb damping the damped natural frequency remains same as the undamped natural frequency that means, omega N is equal to square root of k by m in this case that is the reason actually I have used I have taken the general solution as A 1 omega m t plus A 2 cosine omega n t. If you see the equation of motion, equation of motion for one case it was like this.

So, that means basically it can be written like this. So, with these now I can conclude today's class. These are the references which I have used for this lecture. Thank you.