Surface Water Hydrology
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Lecture 47
Nash's Conceptual Model

Today's lecture is focused on this conceptual model and we will see how to develop the instantaneous hydrograph using Nash's conceptual model that will use the concept of hydrograph analysis as well as flood routing.

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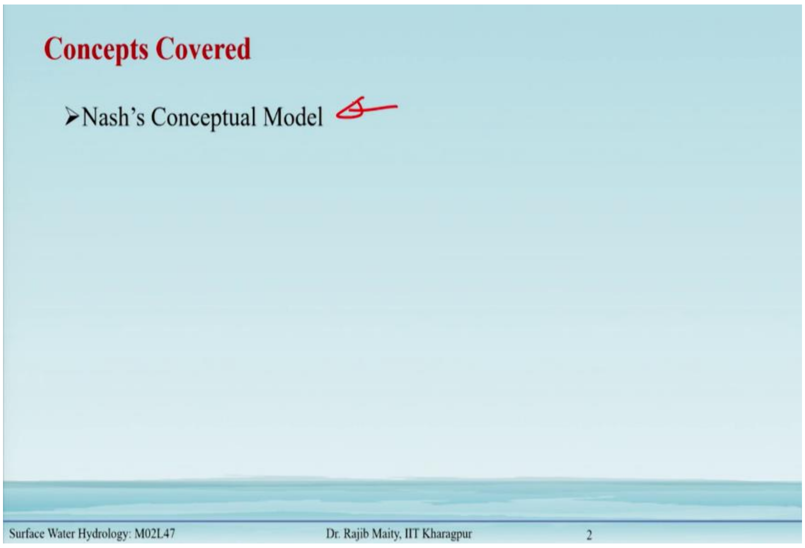

Under this Nash's conceptual model is a thing that is under the concept covered in today's class.

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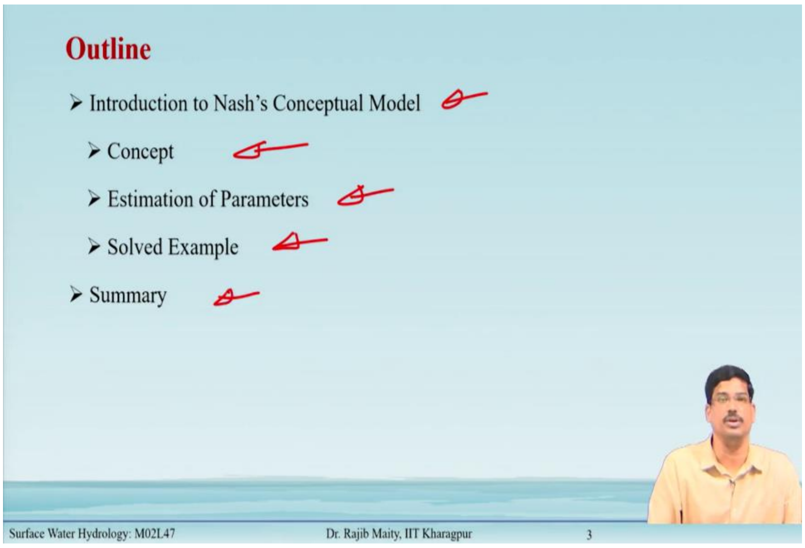

The outline goes like this, that introduction to Nash's conceptual model. There are two parameters specifically in n and K that we will discuss. We will also take up some example problems to discuss methodology further. Finally, we will go to the summary.

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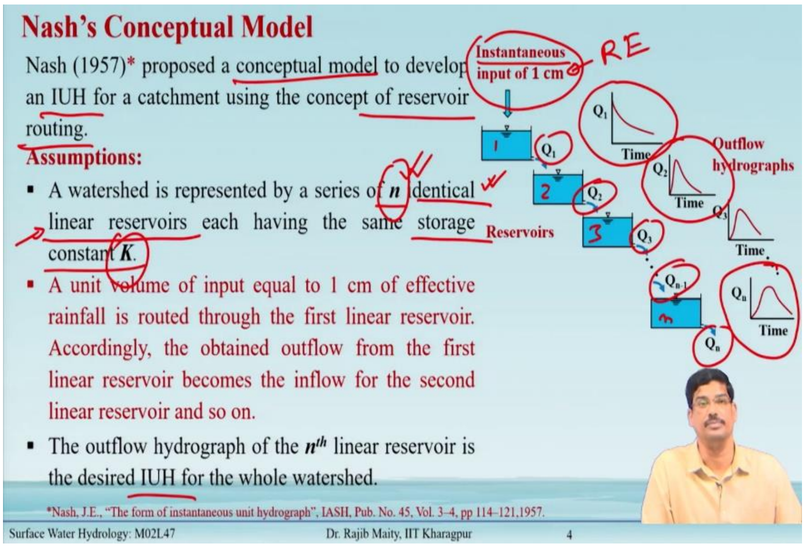

Nash's Conceptual Model

Nash (1957) proposed a conceptual model to develop an IUH for a catchment using the concept of reservoir routing.

Assumptions:

- A watershed is represented by a series of ***n*** identical linear reservoirs each having the same storage constant ***K*** as shown in fig.1.
- A unit volume of input equal to 1 cm of effective rainfall is routed through the first linear reservoir. Accordingly, the obtained outflow from the first linear reservoir becomes the inflow for the second linear reservoir, and so on.
- The outflow hydrograph of the n^{th} linear reservoir is the desired IUH for the whole watershed.

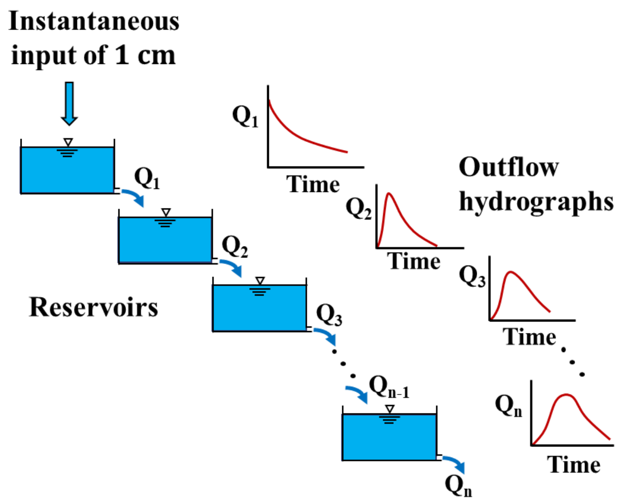

Figure 1: Nash Model: Cascade of Linear Reservoirs

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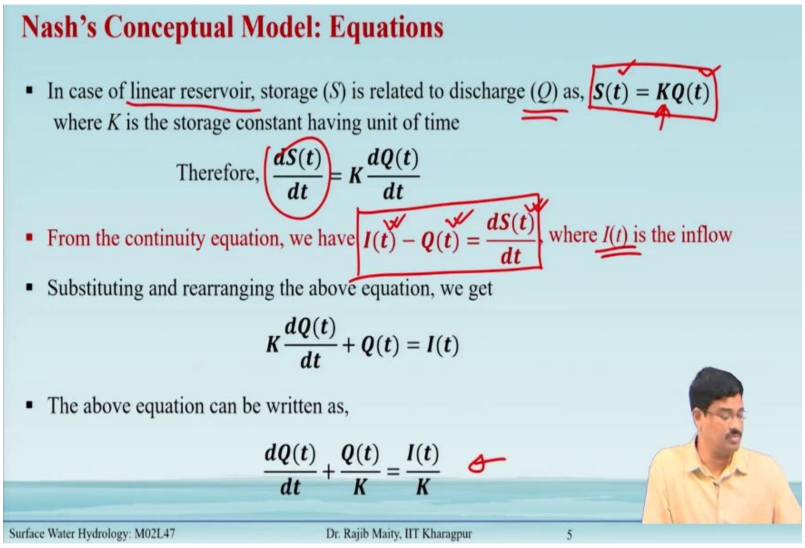

Nash's Conceptual Model: Equations

In the case of the linear reservoir, storage (S) is related to discharge (Q) as, $\boldsymbol{S(t)=KQ(t)}$

where K is the storage constant having a unit of time

Therefore,

$$\frac{d\boldsymbol{S(t)}}{dt} = \boldsymbol{K}\frac{d\boldsymbol{Q(t)}}{dt}$$

From the continuity equation, we have

$I(t) - Q(t) = \frac{dS(t)}{dt}$ where I(t) is the inflow

Substituting and rearranging the above equation, we get

$$K\frac{dQ(t)}{dt} + Q(t) = I(t)$$

The above equation can be written as,

$$\frac{dQ(t)}{dt} + \frac{Q(t)}{K} = \frac{I(t)}{K}$$

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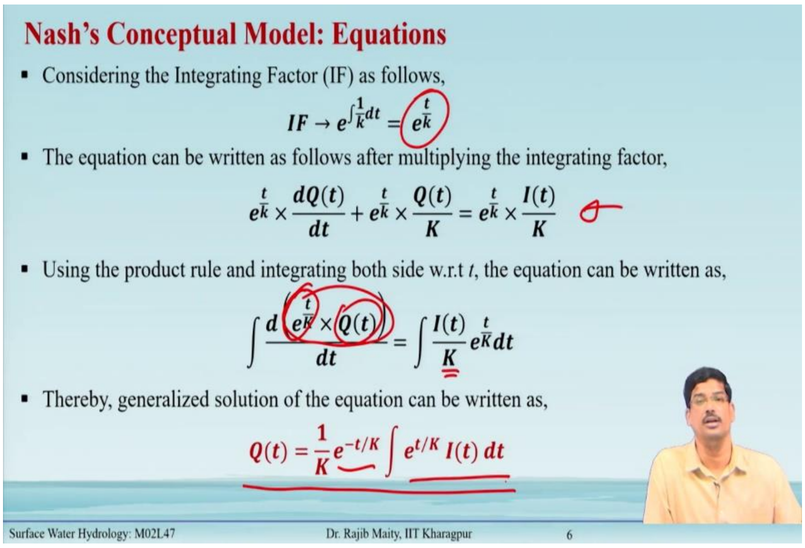

Considering the Integrating Factor (IF) as follows,

$$IF \rightarrow e^{\int \frac{1}{K}dt} = e^{\frac{t}{k}}$$

The equation can be written as follows after multiplying the integrating factor,

$$e^{\frac{t}{k}} \times \frac{dQ(t)}{dt} + e^{\frac{t}{k}} \times \frac{Q(t)}{K} = e^{\frac{t}{k}} \times \frac{I(t)}{K}$$

Using the product rule and integrating both sides w.r.t t, the equation can be written as,

$$\int \frac{d\left(e^{\frac{t}{K}} \times Q(t)\right)}{dt} = \int \frac{I(t)}{K} e^{\frac{t}{K}} dt$$

Thereby, the generalized solution of the equation can be written as,

$$Q(t) = \frac{1}{K} e^{-t/K} \int e^{t/K} \, I(t) \, dt$$

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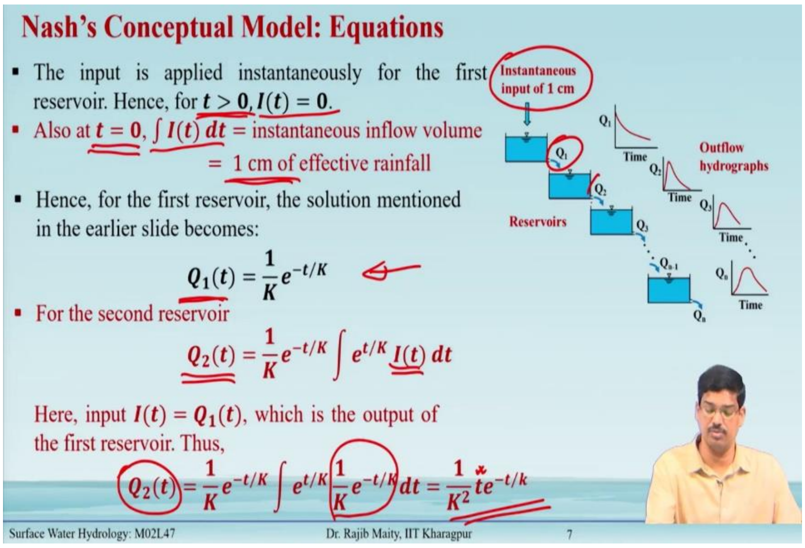

The input is applied instantaneously to the first reservoir. Hence, for $t > 0,\ I(t) = 0$.

Also at t=0, $\int I(t)\ dt$ = instantaneous inflow volume = 1 cm of effective rainfall$t = 0 \int I(t)\, dt =$ = 1 cm

Hence, for the first reservoir (shown in fig.1), the solution mentioned in the earlier slide becomes:

$$Q_1(t) = \frac{1}{K} e^{-t/K}$$

For the second reservoir

$$Q_2(t) = \frac{1}{K} e^{-t/K} \int e^{t/K} \, I(t) \, dt$$

Here, input $I(t)$=Q_1 (t), which is the output of the first reservoir. Thus,

$$Q_2(t) = \frac{1}{K} e^{-t/K} \int e^{t/K} \frac{1}{K} e^{-t/K} dt = \frac{1}{K^2} te^{-t/k}$$

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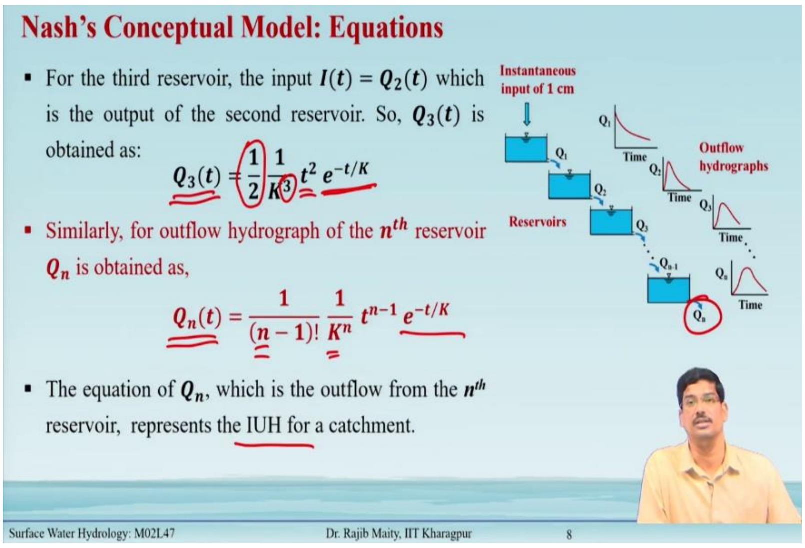

For the third reservoir, the input $\boldsymbol{I(t)}$=$\boldsymbol{Q_2}$ $\boldsymbol{(t)}$ is the output of the second reservoir. So, $\boldsymbol{Q_3}$ $\boldsymbol{(t)}$ is obtained as:

$$\boldsymbol{Q_3(t) = \frac{1}{2}\,\frac{1}{K^3}\,t^2\,e^{-t/K}}$$

Similarly, for outflow hydrograph of the $\boldsymbol{n^{th}}$ reservoir $\boldsymbol{Q_n}$ is obtained as,

$$\boldsymbol{Q_n(t) = \frac{1}{(n-1)!}\,\frac{1}{K^n}\,t^{n-1}\,e^{-t/K}}$$

The equation of $\boldsymbol{Q_n}$, which is the outflow from the nth reservoir, represents the IUH for a catchment.

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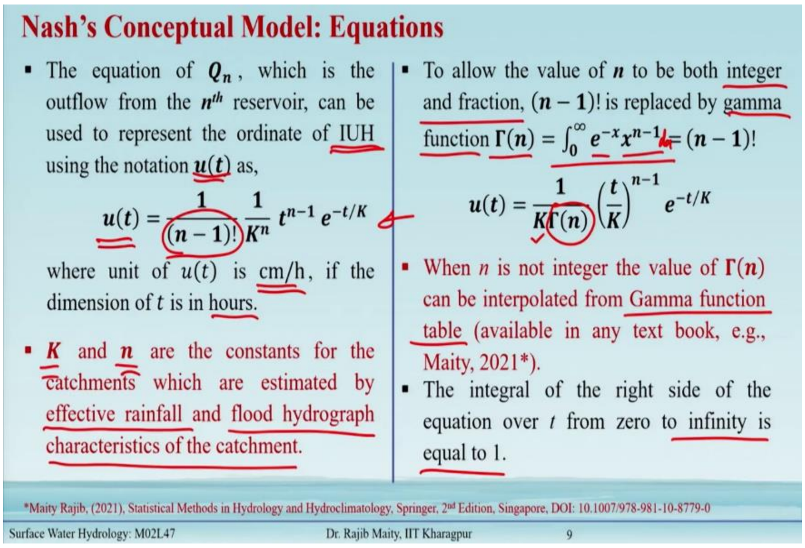

The equation of $\boldsymbol{Q_n}$, which is the outflow from the nth reservoir, can be used to represent the ordinate of IUH using the notation $\boldsymbol{u(t)}$ as,

$$\boldsymbol{u(t) = \frac{1}{(n-1)!} \frac{1}{K^n} t^{n-1} e^{-t/K}}$$

where the unit of $u(t)$ in cm/h, if the dimension of t is in hours.

$\boldsymbol{K}$ and $\boldsymbol{n}$ are the constants for the catchments which are estimated by effective rainfall and flood hydrograph characteristics of the catchment.

To allow the value of n to be both integer and fraction, $(\boldsymbol{n-1})!$ is replaced by the gamma function

$\boldsymbol{\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} = (n-1)!}$

$$\boldsymbol{u(t) = \frac{1}{K\Gamma(n)} \left(\frac{t}{K}\right)^{n-1} e^{-t/K}}$$

When n is not an integer the value of $\boldsymbol{\Gamma(n)}$ can be interpolated from the Gamma function table (available in any textbook, e.g., Maity, 2021).

The integral of the right side of the equation over t from zero to infinity is equal to 1.

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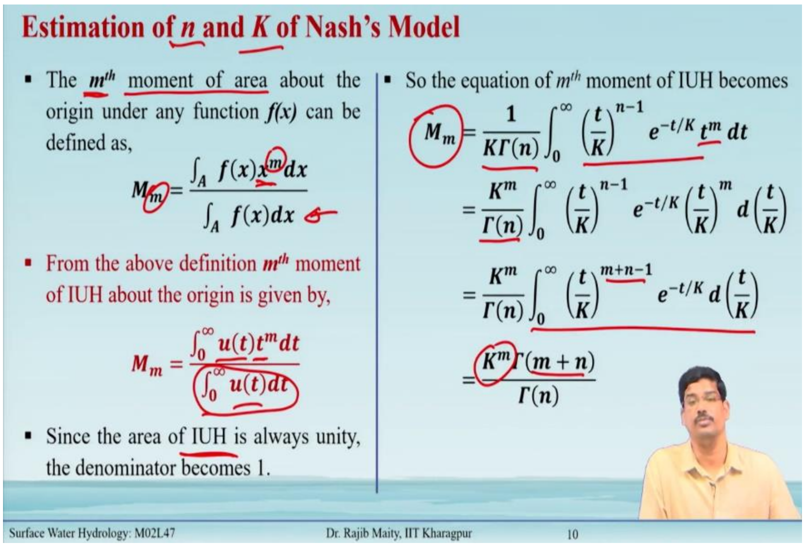

Estimation of *n* and *K* of Nash's Model

The m^{th} moment of area about the origin under any function f(x) can be defined as,

$$M_m = \frac{\int_A f(x)x^m dx}{\int_A f(x)dx}$$

From the above definition m^{th} moment of IUH about the origin is given by,

$$M_m = \frac{\int_0^\infty u(t)t^m dt}{\int_0^\infty u(t)dt}$$

Since the area of IUH is always unity, the denominator becomes 1.

So, the equation of m^{th} moment of IUH becomes

$$M_m = \frac{1}{K\Gamma(n)}\int_0^\infty \left(\frac{t}{K}\right)^{n-1} e^{-t/K} t^m \, dt$$

$$= \frac{K^m}{\Gamma(n)}\int_0^\infty \left(\frac{t}{K}\right)^{n-1} e^{-t/K} \left(\frac{t}{K}\right)^m d\left(\frac{t}{K}\right)$$

$$= \frac{K^m}{\Gamma(n)}\int_0^\infty \left(\frac{t}{K}\right)^{m+n-1} e^{-t/K} d\left(\frac{t}{K}\right)$$

$$= \frac{K^m \Gamma(m+n)}{\Gamma(n)}$$

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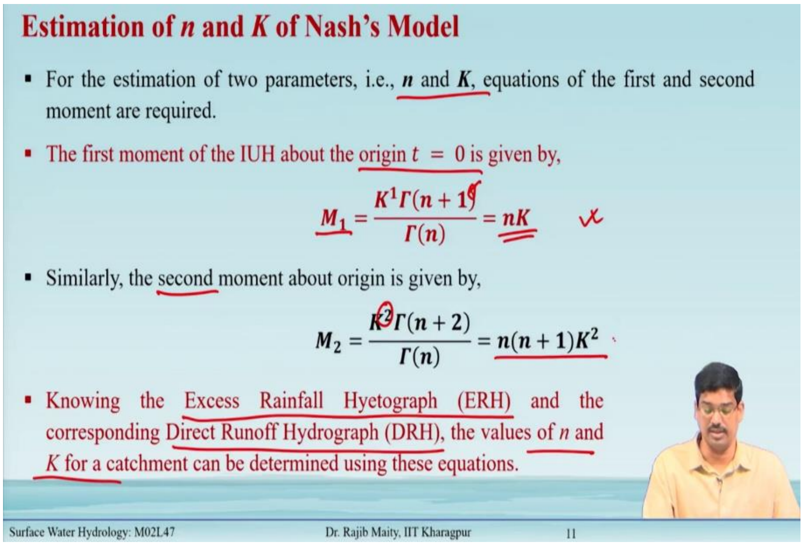

For the estimation of two parameters, i.e., n and K, equations of the first and second moment are required.

The first moment of the IUH about the origin $t = 0$ is given by,

$$M_1 = \frac{K^1 \Gamma(n+1)}{\Gamma(n)} = nK$$

Similarly, the second moment about the origin is given by,

$$M_2 = \frac{K^2 \Gamma(n+2)}{\Gamma(n)} = n(n+1)K^2$$

Knowing the Excess Rainfall Hyetograph (ERH) and the corresponding Direct Runoff Hydrograph (DRH), the values of n and K for a catchment can be determined using these equations.

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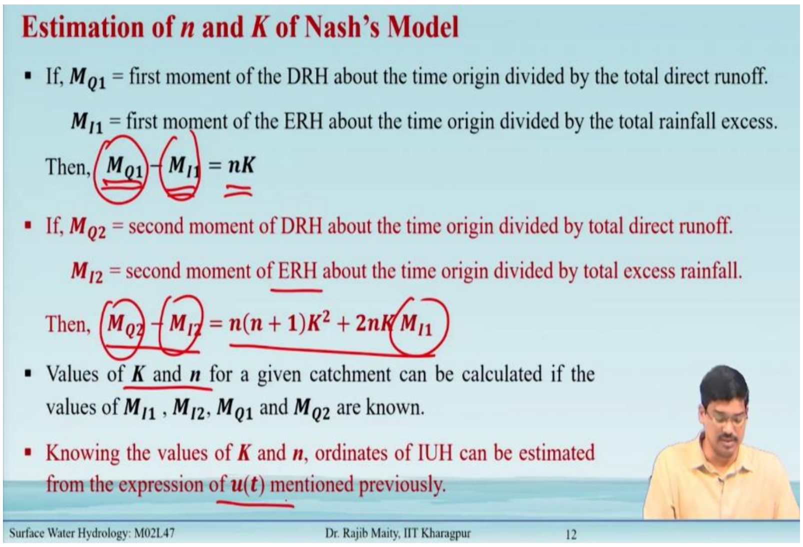

If, $\boldsymbol{M_{Q1}}$ = the first moment of the DRH about the time origin divided by the total direct runoff.

$\boldsymbol{M_{I1}}$ = the first moment of the ERH about the time origin divided by the total rainfall excess.

Then, $\boldsymbol{M_{Q1} - M_{I1} = nK}$

If, $\boldsymbol{M_{Q2}}$ = the second moment of DRH about the time origin divided by total direct runoff.

$\boldsymbol{M_{I2}}$ = the second moment of ERH about the time origin divided by total excess rainfall.

Then, $\boldsymbol{M_{Q2} - M_{I2} = n(n+1)K^2 + 2nK\, M_{I1}}$

Values of $\boldsymbol{K}$ and $\boldsymbol{n}$ for a given catchment can be calculated if the values of $\boldsymbol{M_{I1}}$, $\boldsymbol{M_{I2}}$, $\boldsymbol{M_{Q1}}$ and $\boldsymbol{M_{Q2}}$ are known.

Knowing the values of $\boldsymbol{K}$ and $\boldsymbol{n}$, ordinates of IUH can be estimated from the expression of $\boldsymbol{u(t)}$ mentioned previously.

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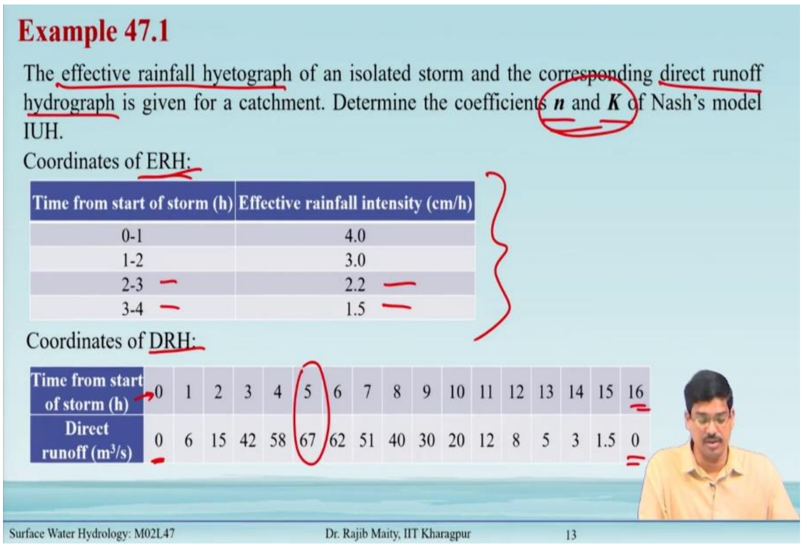

Example 47.1

The effective rainfall hyetograph of an isolated storm and the corresponding direct runoff hydrograph are given for a catchment. Determine the coefficients ***n*** and ***K*** of Nash's model IUH.

Coordinates of ERH:

Time from start of storm (h)	Effective rainfall intensity (cm/h)
0-1	4.0
1-2	3.0
2-3	2.2
3-4	1.5

Coordinates of DRH:

Time from start of storm (h)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Direct runoff (m^3/s)	0	6	15	42	58	67	62	51	40	30	20	12	8	5	3	1.5	0

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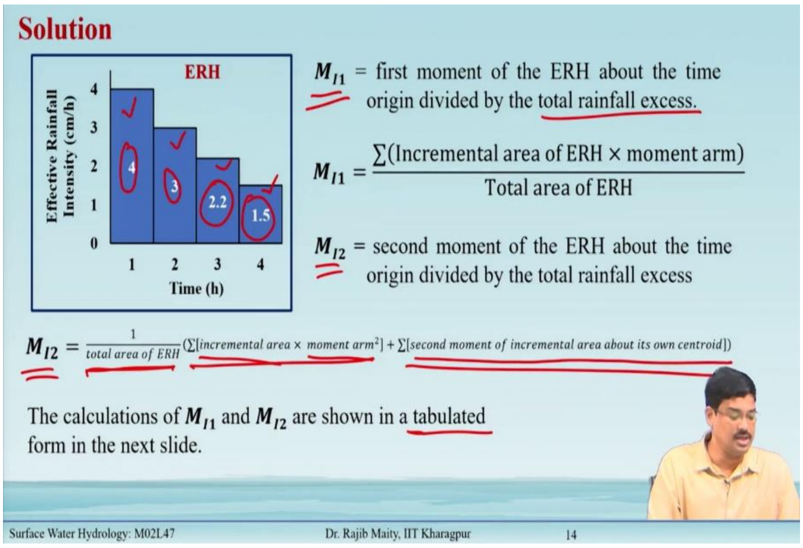

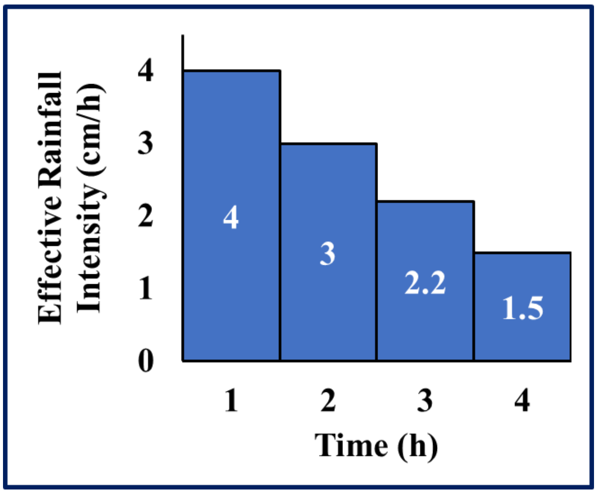

Figure 2: Excess Rainfall Hyetograph of example 47.1

M_{I1} = the first moment of the ERH about the time origin divided by the total rainfall excess.

$$M_{I1} = \frac{\sum(\text{Incremental area of ERH} \times \text{moment arm})}{\text{Total area of ERH}}$$

M_{I2} = **the** second moment of the ERH about the time origin divided by the total rainfall excess

$M_{I2} = 1/\ (total\ area\ of\ ERH)\ (\sum\ [incremental\ area\ \times\ moment\ arm^2]\ +\sum\ [second\ moment\ of\ incremental\ area\ about\ its\ own\ centroid])$

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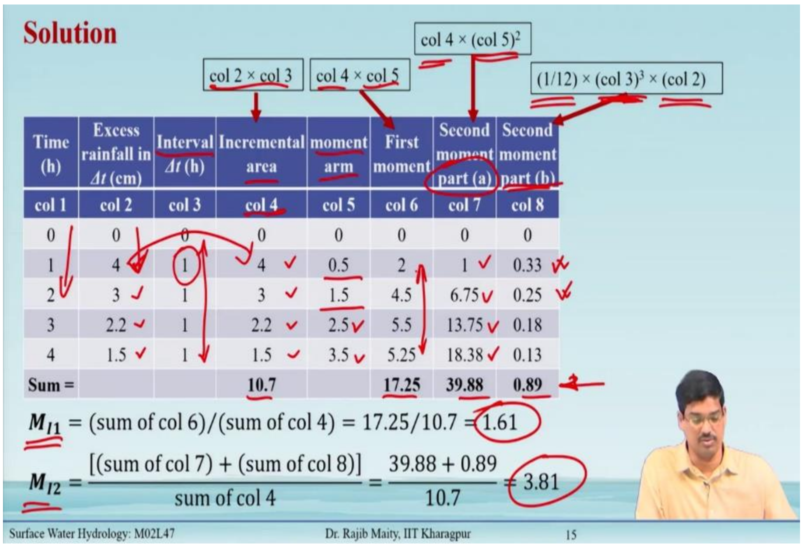

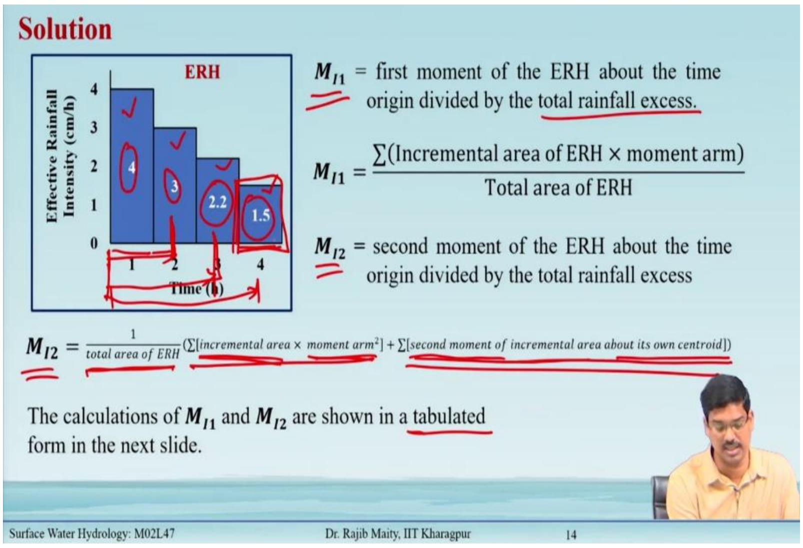

Time (h)	Excess rainfall in Δt (cm)	Interval Δt (h)	Incremental area	moment arm	First moment	Second moment part (a)	Second moment part (b)
col 1	col 2	col 3	col 4	col 5	col 6	col 7	col 8
0	0	0	0	0	0	0	0
1	4	1	4	0.5	2	1	0.33
2	3	1	3	1.5	4.5	6.75	0.25
3	2.2	1	2.2	2.5	5.5	13.75	0.18
4	1.5	1	1.5	3.5	5.25	18.38	0.13
Sum =			**10.7**		**17.25**	**39.88**	**0.89**

M_{I1}= (sum of col 6) / (sum of col 4) =17.25⁄10.7= 1.61

$$M_{I2} = \frac{[(\text{sum of col 7}) + (\text{sum of col 8})]}{\text{sum of col 4}} = \frac{39.88 + 0.89}{10.7} = 3.81$$

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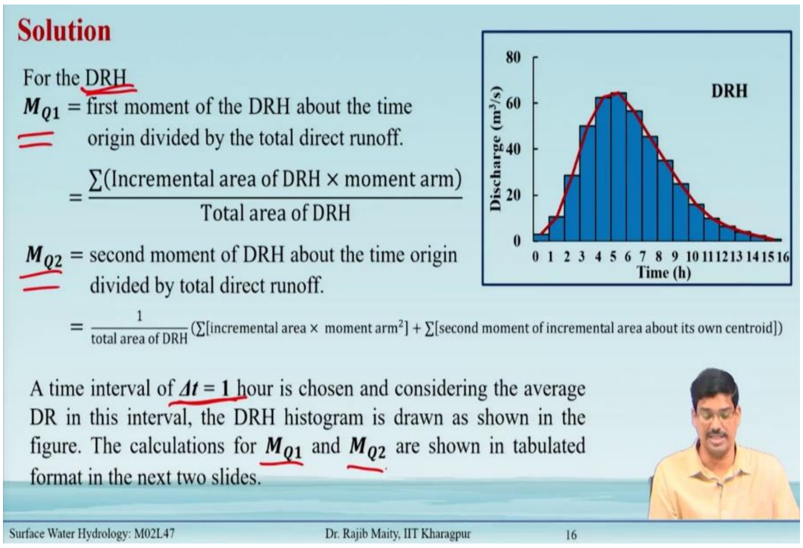

For the DRH

M_{Q1} = the first moment of the DRH about the time origin divided by the total direct runoff

$$= \frac{\sum(\text{Incremental area of DRH} \times \text{moment arm})}{\text{Total area of DRH}}$$

M_{Q2} = the second moment of DRH about the time origin divided by total direct runoff.

=1/ (total area of DRH) (∑ [incremental area× moment arm^2] +∑ [second moment of incremental area about its own centroid])

A time interval of Δt = 1 hour is chosen and considering the average DR in this interval, the DRH histogram is drawn as shown in the figure. The calculations for M_{Q1} and M_{Q2} are shown in the tabulated format in the next two slides.

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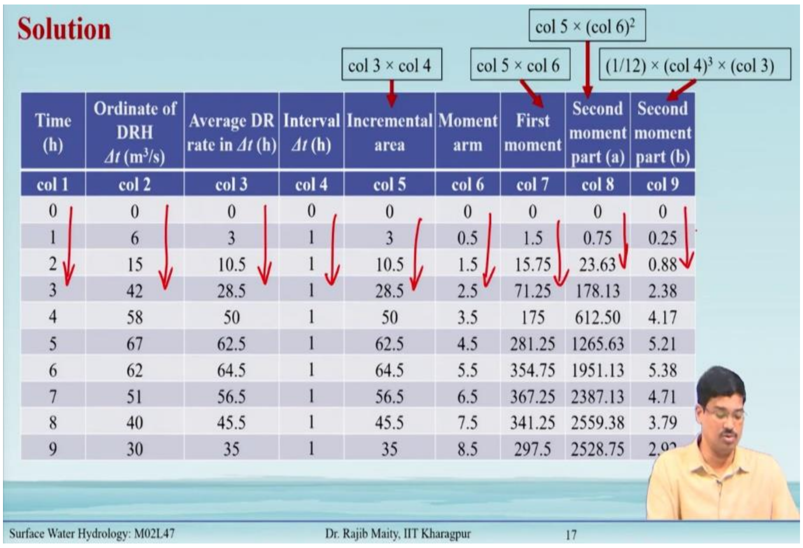

Time (h)	Ordinate of DRH Δt (m^3/s)	Average DR rate in Δt (h)	Interval Δt (h)	Incremental area	Moment arm	First moment	Second moment part (a)	Second moment part (b)
col 1	col 2	col 3	col 4	col 5	col 6	col 7	col 8	col 9
0	0	0	0	0	0	0	0	0
1	6	3	1	3	0.5	1.5	0.75	0.25
2	15	10.5	1	10.5	1.5	15.75	23.63	0.88
3	42	28.5	1	28.5	2.5	71.25	178.13	2.38
4	58	50	1	50	3.5	175	612.50	4.17
5	67	62.5	1	62.5	4.5	281.25	1265.63	5.21
6	62	64.5	1	64.5	5.5	354.75	1951.13	5.38
7	51	56.5	1	56.5	6.5	367.25	2387.13	4.71
8	40	45.5	1	45.5	7.5	341.25	2559.38	3.79
9	30	35	1	35	8.5	297.5	2528.75	2.92

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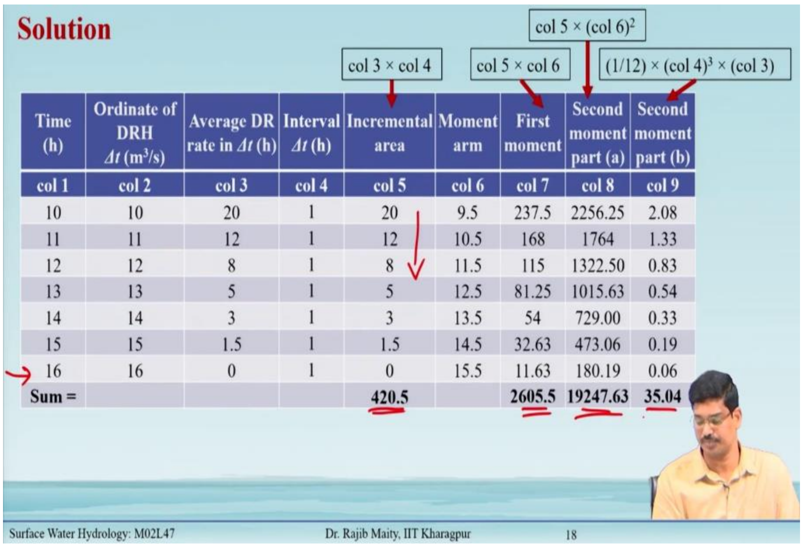

Time (h)	Ordinate of DRH Δt (m³/s)	Average DR rate in Δt (h)	Interval Δt (h)	Incremental area	Moment arm	First moment	Second moment part (a)	Second moment part (b)
col 1	**col 2**	**col 3**	**col 4**	**col 5**	**col 6**	**col 7**	**col 8**	**col 9**
10	10	20	1	20	9.5	237.5	2256.25	2.08
11	11	12	1	12	10.5	168	1764	1.33
12	12	8	1	8	11.5	115	1322.50	0.83
13	13	5	1	5	12.5	81.25	1015.63	0.54
14	14	3	1	3	13.5	54	729.00	0.33
15	15	1.5	1	1.5	14.5	32.63	473.06	0.19
16	16	0	1	0	15.5	11.63	180.19	0.06
Sum =				**420.5**		**2605.5**	**19247.63**	**35.04**

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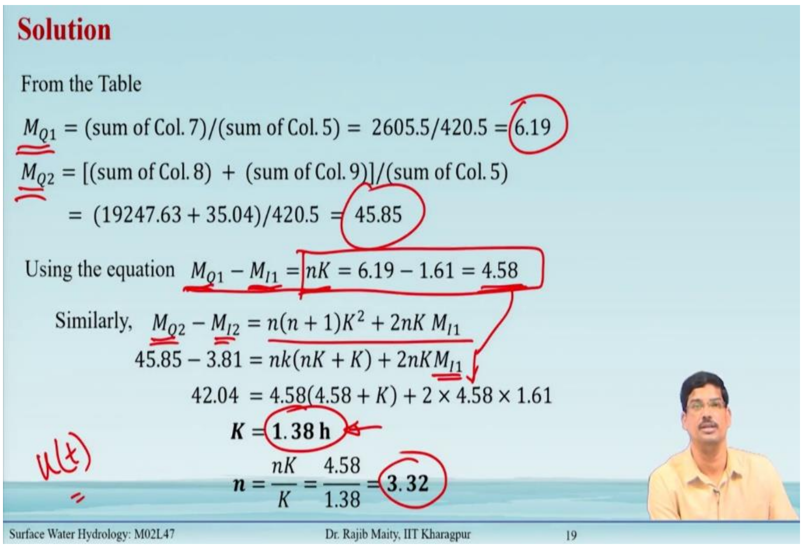

From the Table

M_{Q1}= (sum of Col. 7)/ (sum of Col. 5) = 2605.5/420.5=6.19

M_{Q2}= [(sum of Col. 8) + (sum of Col. 9)]/ (sum of Col. 5)

= (19247.63+35.04)/420.5 = 45.85

Using the equation $MQ_1 - M_{I1} = nK$ = 6.19−1.61 = 4.58

Similarly, $M_{Q2} - M_{I2} = n(n+1)\,K^2 + 2nK\,M_{I1}$

$$45.85 - 3.81 = nk(nK+K) + 2nKM_{I1}$$

$$42.04 = 4.58(4.58+K) + 2\times4.58\times1.61$$

$$\boldsymbol{K = 1.38\ \mathrm{h}}$$

$$\boldsymbol{n} = nK/K = 4.58/1.38 = \mathbf{3.32}$$

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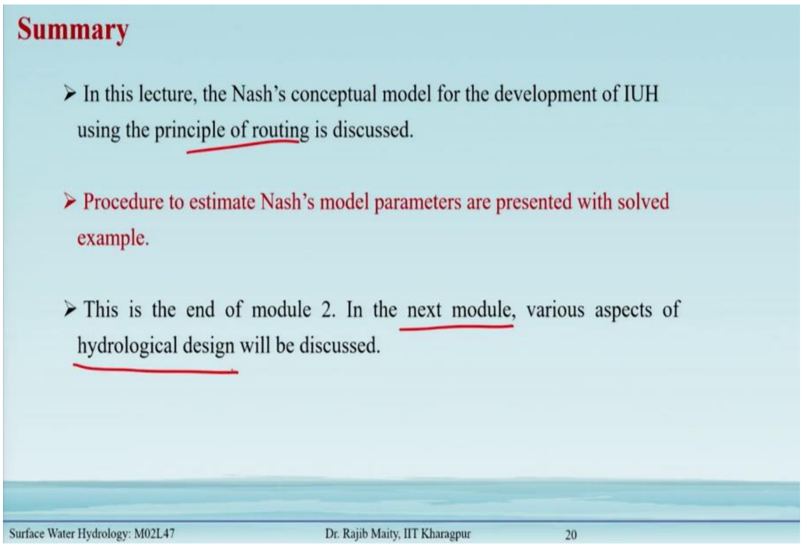

Summary

In summary, we learned the following points from this lecture:

- ➤ In this lecture, Nash's conceptual model for the development of IUH using the principle of routing is discussed.
- ➤ Procedure to estimate Nash's model parameters are presented with a solved example.

- This is the end of module 2. In the next module, various aspects of hydrological design will be discussed.

References

- Maity R., (2021), Statistical Methods in Hydrology and Hydroclimatology, Springer, 2nd Edition, Singapore, DOI: 10.1007/978-981-10-8779-0
- Nash, J.E., "The form of instantaneous unit hydrograph", IASH, Pub. No. 45, Vol. 3–4, pp 114–121,1957.