Rock Mechanics and Tunneling
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Lecture 52
Methods to determine stresses around openings: Kirsch equation

Hello everyone, I welcome all of you to the first lecture of module 11. So, in module 11, we will discuss about the analysis of stresses in tunnel or opening. So, we will discuss about the Kirsch equations in lecture one.

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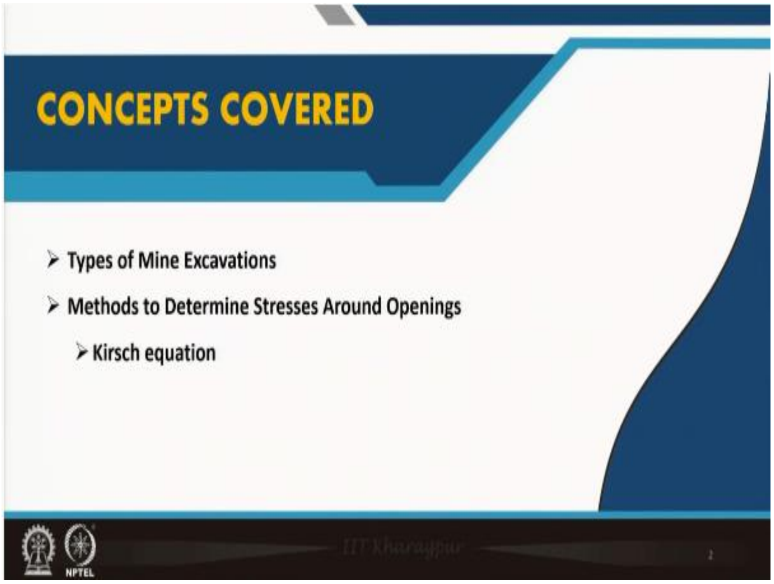

So, types of Mine Excavation first we will discuss and then our focus will be on these methods to determine stresses around openings and as I mentioned Kirsch equation.

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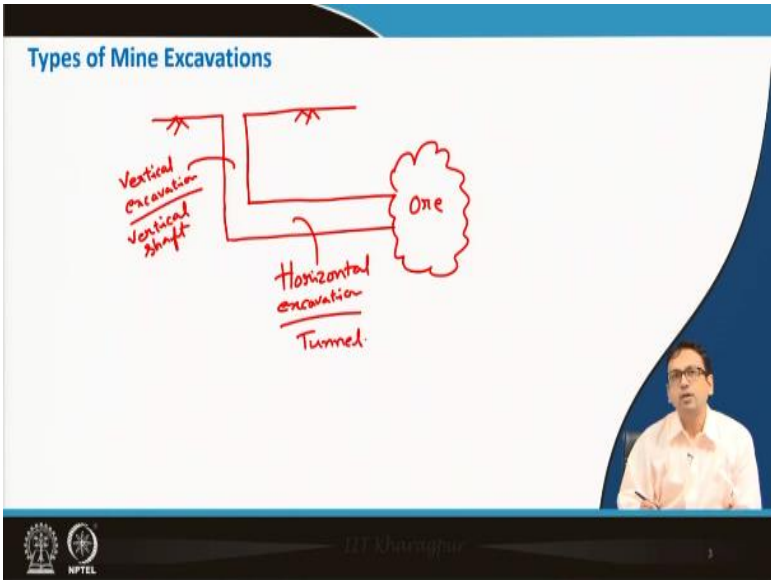

Types of mine excavation

Not only mine excavations, it can be used in other like metro rail project. We use circular tunnel mostly. Different other shapes of tunnel may use but here we mainly focus on analysis of stresses related to the circular tunnels. I have discussed about the other shapes also but mainly we will focus over here on the circular excavation, circular openings actually or tunnels.

We know that if we consider a mine. If suppose the ground level then what will happen? We have to go for a maybe vertical shaft and then maybe the tunnel you may have to construct horizontal excavation to may be reach here. The ore is over here. So, it is the vertical shaft and this is nothing but the horizontal excavation or simply the tunnel.

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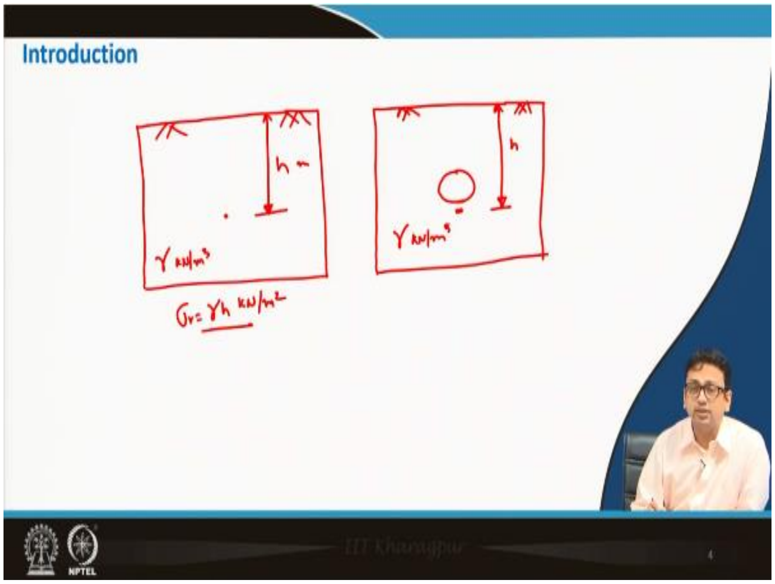

Now, we have to understand that our main topic of discussion is the stresses around the opening. We are discussing this because the initial maybe the rock domain or soil domain whatever you consider, maybe since we are mainly focusing on rocks, so rock mechanics so suppose this is the rock domain, now at this location suppose this is at a depth of h and the unit weight is by γ kN/m^3. So, obviously here the stress will be γh kN/m^2.

Now, what is happening suppose in the same domain, again let me draw another diagram, now this was the point, now suppose over here a tunnel is excavated and same way this is h and the unit weight is supposed γ kN/m^3, what will happen at this point. Stresses will change definitely right, so that is what, means here you will have see in this location you will have maybe some σ_x, σ_y at this point but here you see because of the now that there some excavation is created so that is why the state of stress at this point is going to definitely change.

So, how much change, what is going to happen, how we can represent those changing state of stresses that we should actually learn or understand, so that is what we have to understand in this part of our lecture. So, this was just a simple introduction, means why we are doing it that I think should be clear to you.

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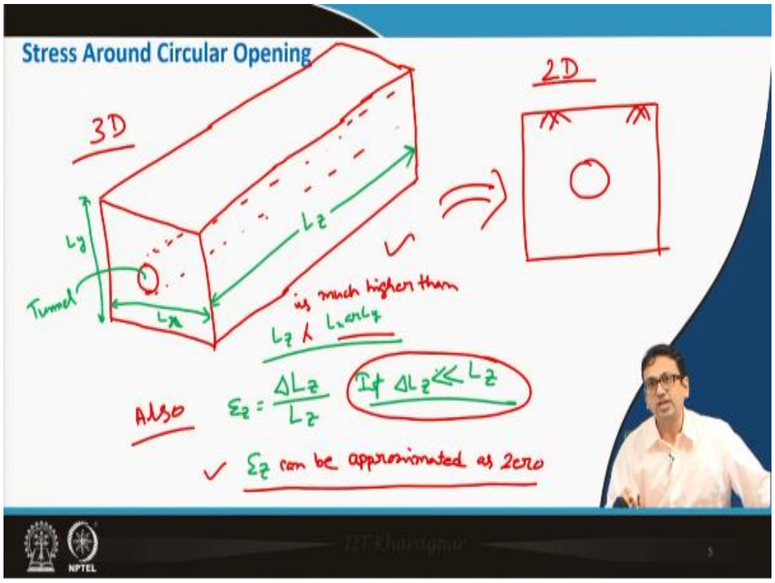

Now, the main thing of discussion is the stresses around circular opening. Now, before going into this discussion what we can think about, see in reality our domain will be 3D. Suppose, it is the three-dimensional domain. In addition, this is the tunnel, this is the rock domain, now suppose here you are creating the tunnel, so this is going like this. Now, what we can understand from here.

What we can observe over here is that if this length this one is suppose L_z suppose and these are suppose x, y directions, now what we can observe over here is that L_z, suppose this is maybe let me write down this one also, suppose it is L_x and this one is suppose L_y. So, now we can clearly understand that L_z is quite greater than quite, quite greater than other dimension like L_x or L_y. Now, if we try to find out the strain in this z direction, suppose this one it will be nothing but if we consider like change in length in this direction this and this now, if this ΔLz is very less than Lz. So, we can say the strain this z will be close to or it may be approximated as maybe 0.

So, if we consider that in z direction, this strain is close to 0 then what we can say, that we can analyze this 3d problem. Now, Lz is much higher than maybe L_x and L_y. Then and also if we find this condition is satisfied, then what we can assume that εz is can be approximated at 0, so what we can do basically in that case we can assume or analyze this problem as a plane strain problem.

So, what is plane strain problem? we have discussed earlier that under this type of situation like not only for tunnel even some strip footing, there also we can reduce the 3d problem to a 2d problem. With the reasonable accuracy, we will get the outcome because the condition. The strain in that out of plane direction will be close to 0 considering that we can analyze this problem as a this 2d problem, so this 2d problem is nothing but the or called as the plane strain problem. So, now why I have shown this because now we will carry forward our remaining discussion considering this tunnel as a plane strain problem.

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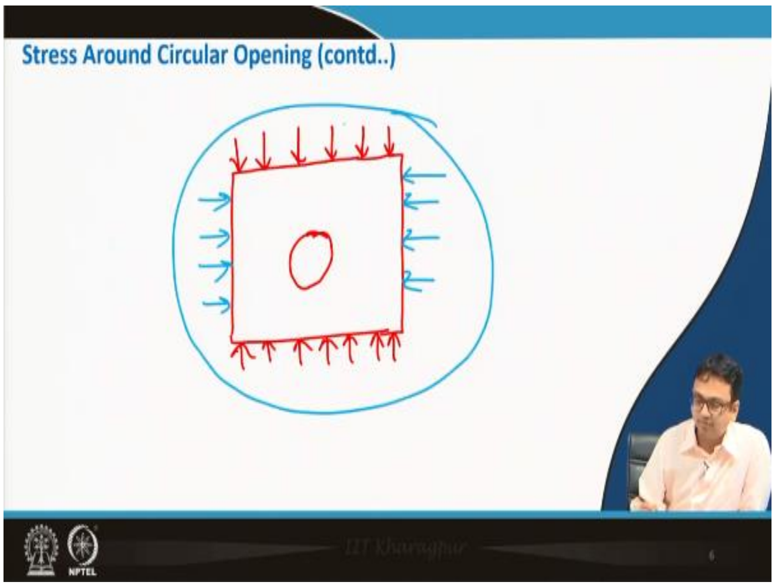

Now, once we have understood, so just what I can say that this is my suppose the domain, as I have drawn, so now basically at a particular, means this is suppose my tunnel and surrounding area is this, so now what will happen obviously this type of vertical this loading or vertical stress will be there as well as this time vertical, vertical as well as horizontal. So, let me use maybe other color for horizontal, so this kind of situation will develop in and around area of the tunnel.

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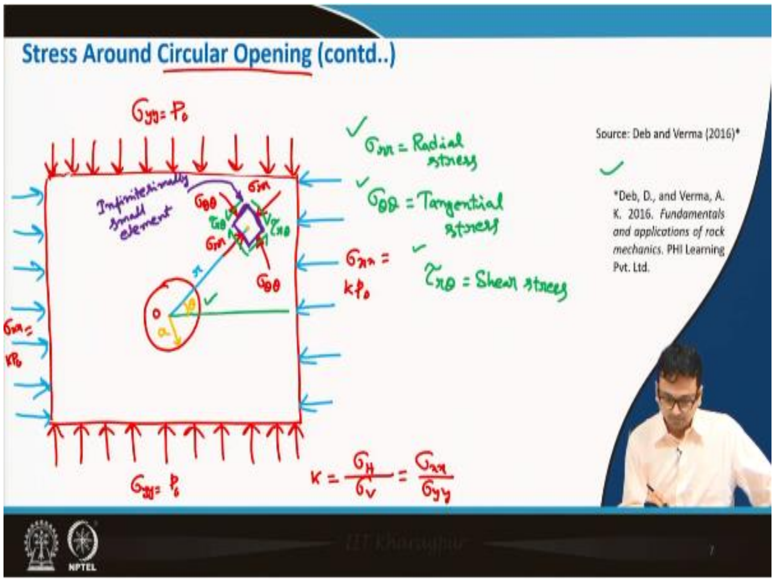

So, now regarding this actually the next part of the Kirsch equation will come after this only, now what will be considered over here, just see it, suppose the tunnel is located somewhere maybe here and now this is the surrounding area.

Now, here the vertical loading is acting, if now this one if suppose, this one is suppose P_0 so here it will be then kp_0. We can indicate the stresses as σ_{yy} and σ_{xx}.

So, that means here this k is equal to nothing but the σ_H/σ_V that is nothing but the horizontal stress by vertical stress or simply we can write it as σ_{xx} by σ_{yy}. Now, the purpose of drawing this one is we have to understand suppose the o is the origin, means of this tunnel, this is a circular tunnel that you have to understand like circular opening rather to say.

So, now if this is the horizontal line, now at a distance suppose r from the tunnel, let me also give the consider the radius of this tunnel also, so let us consider that as maybe this one suppose a and at a distance r, this suppose this some point is located and we want to find out the state of stress at that location.

Now, this line is making an angle suppose θ with the horizontal, now if we try to understand or draw an infinite small element. Therefore, this kind of element so this you see one thing obviously that this point is defined by r and θ in polar coordinate. I hope you have understood.

Now, if this is an infinite small element, let me write it also maybe somewhere here. Now, here what are the stresses will act, obviously you can understand clearly that here σ_{rr} which is the radial stress, like here also σ_{rr}, now here what will have σ_{θ} element.

Now, what I was telling σ_{rr} and also $\sigma_{\theta\theta}$ which are nothing but the tangential stresses along with that what will act over there, obviously the shear stresses like and this side so what is this basically this one is $\tau_{r\theta}$. So, this is at a distance r from the center of the tunnel and this angle is θ as you can see from here, so this point the state of stress, means the I mean to say that I mean the state of stress can be represented by these stress components like $\sigma_{\theta\theta}$, σ_{rr} and $\tau_{r\theta}$.

So, let me keep it little this side because I need to little more things over there, so σ_{rr} which is the radial stress, $\sigma_{\theta\theta}$ is the tangential stress, and $\tau_{r\theta}$ is the shear stress. So, in order to present the state of stress at this point as it is shown over here, basically I need to know the what is σ_{rr}, what is $\sigma_{\theta\theta}$ and what is $\tau_{r\theta}$, these three things we need to know, so these things as you know that we are also, this is one of your reference book as I have mentioned earlier also, so you can follow this book also for better in order to see it in more detail but anyway I am trying to explain it in detail only, but still you can refer this book also for further understanding also.

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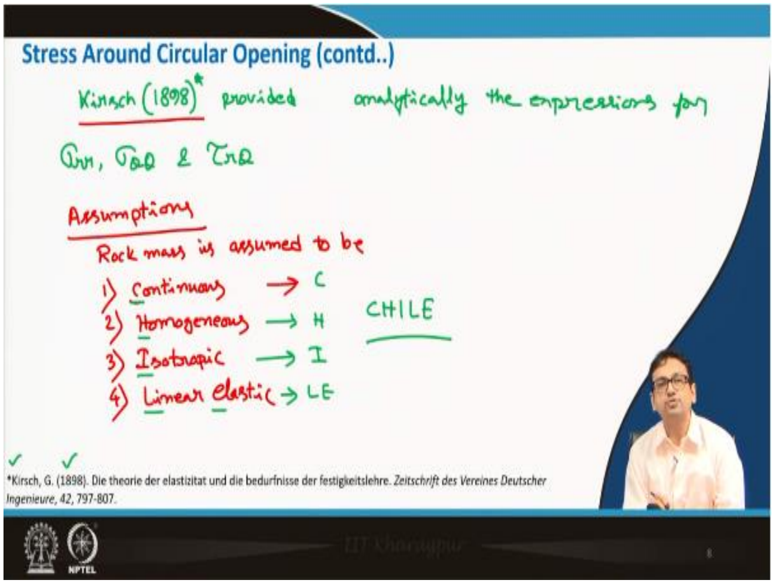

So, anyway this is the domain, now this is the infinite small element and in order to present the state of stress at a point, we need three stress component. Now, the reference the Kirsch, G. in the year 1898, so you see means about you can understand 123 years ago, you can understand that he has provided some equation based on some assumptions, the equations for the $\sigma_{\theta\theta}$, σ_{rr} and $\tau_{r\theta}$, so that we will learn now.

So, some of the things relate to this, means the assumptions what he has taken that also you should learn, so, I was telling about the Kirsch (1898), so this one is only the reference is given. Provided analytically the expressions for this $\sigma_{\theta\theta}$, σ_{rr} and $\tau_{r\theta}$. Now, it is based on some assumptions so the assumptions related to these expressions what I will show you in my next slide, so assumptions are the rock mass is assumed to be, so rock mass is assumed to be.

Number one, like continuous, number two homogeneous, number three isotropic and number four linearly elastic, means as you know this first letter is earlier as we have probably discussed about the C, here first letter is H, here first letter is I and here from here LE, so together CHILE material, based considering the rock mass as continuous, homogeneous, isotropic and linear elastic material, Kirsch has provided the analytical solution for $\sigma_{\theta\theta}$, σ_{rr} and $\tau_{r\theta}$.

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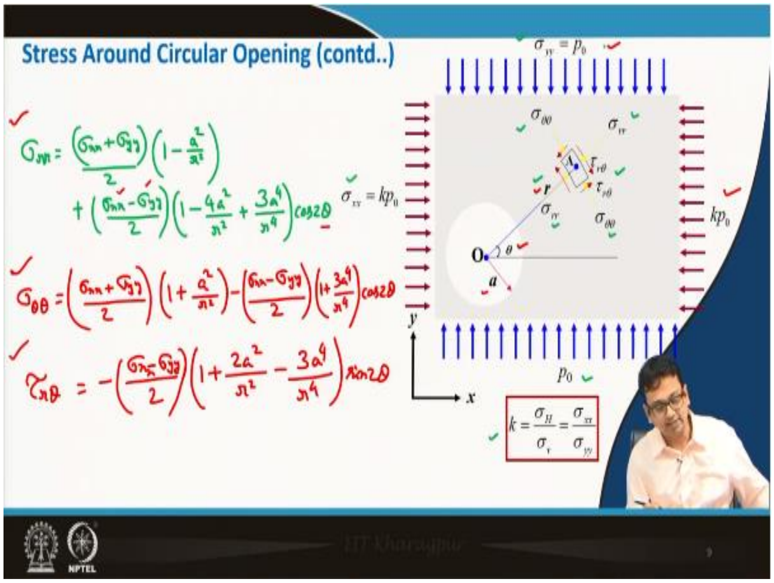

An Analytical formulation for a long circular tunnel in an ideal rock mass was developed by Kirsch (1898). The equation are expressed as

$$\sigma_{rr} = \frac{\left(\sigma_{xx}+\sigma_{yy}\right)}{2}\left(1-\frac{a^2}{r^2}\right)-\frac{\left(\sigma_{xx}-\sigma_{yy}\right)}{2}\left(1-\frac{4a^2}{r^2}+\frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{\left(\sigma_{xx}+\sigma_{yy}\right)}{2}\left(1+\frac{a^2}{r^2}\right)+\frac{\left(\sigma_{xx}-\sigma_{yy}\right)}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = \frac{\left(\sigma_{xx}-\sigma_{yy}\right)}{2}\left(1+\frac{2a^2}{r^2}-\frac{3a^4}{r^4}\right)\sin 2\theta$$

So, these are the three expressions have provided for determining this stresses

where $$k=\frac{\sigma_H}{\sigma_v}=\frac{\sigma_{xx}}{\sigma_{yy}}$$

Now, you see now what we can do σ_{xx} is nothing but here kP_0 and σ_{yy} is P_0.

$$\sigma_{rr} = \frac{p_0}{2}\left\{(1+k)\left(1-\frac{a^2}{r^2}\right)-(1-k)\left(1-\frac{4a^2}{r^2}+\frac{3a^4}{r^4}\right)\cos 2\theta\right\}$$

$$\sigma_{\theta\theta} = \frac{p_0}{2}\left\{(1+k)\left(1+\frac{a^2}{r^2}\right)+(1-k)\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right\}$$

$$\tau_{r\theta} = \frac{p_0}{2}\left\{(1-k)\left(1+\frac{2a^2}{r^2}-\frac{3a^4}{r^4}\right)\sin 2\theta\right\}$$

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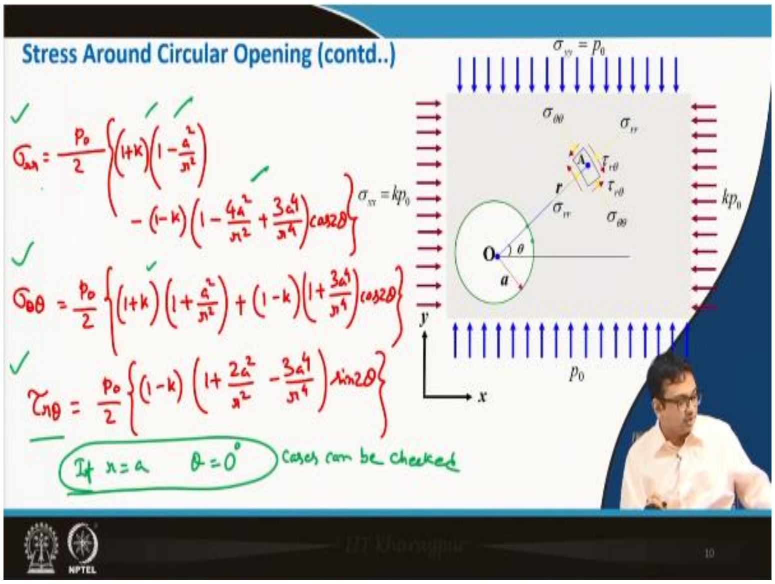

So, you can also cross check like if θ is 0°. means these are fine cos 0 is 1 no problem, but here sin 0 will become 0, so your this shear stress will vanish, so these are the things like you can cross check from here. So, these are the few cases can be checked from here, check from these equations, but anyway nevertheless these three equations are what you need actually for solving different problems.

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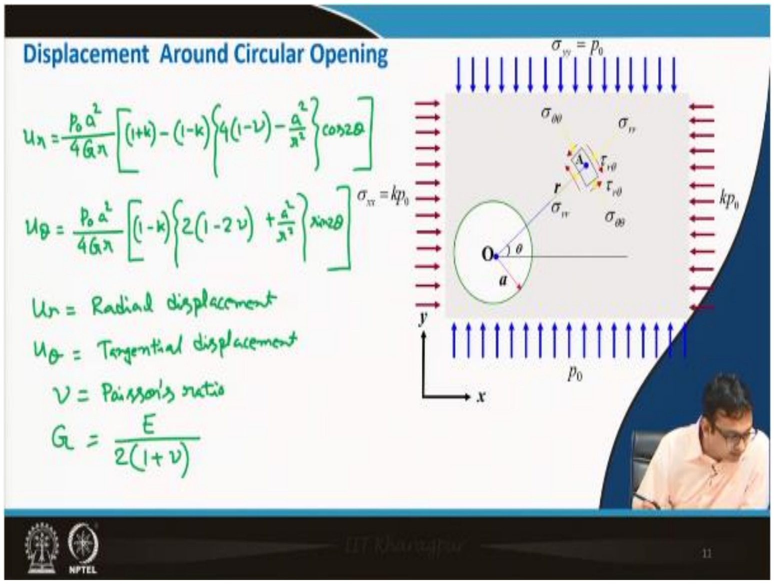

Now, we have learnt of the stresses, now regarding displacement also we have some expressions available as follows.

$$u_r = \frac{p_0 a^2}{4Gr}\left[(1+k)-(1-k)\left(4(1-\upsilon)-\frac{a^2}{r^2}\right)\cos 2\theta\right]$$

$$u_\theta = \frac{p_0 a^2}{4Gr}\left[(1-k)\left(2(1-2\upsilon)+\frac{a^2}{r^2}\right)sin 2\theta\right]$$

where u_r is displacement in r direction and u_θ is the displacement in $\boldsymbol{\theta}$ direction.

So, this way one can derived or determine the radial displacement as well as the tangential displacement, so again these things you can see also in the book of Deb and Verma also.

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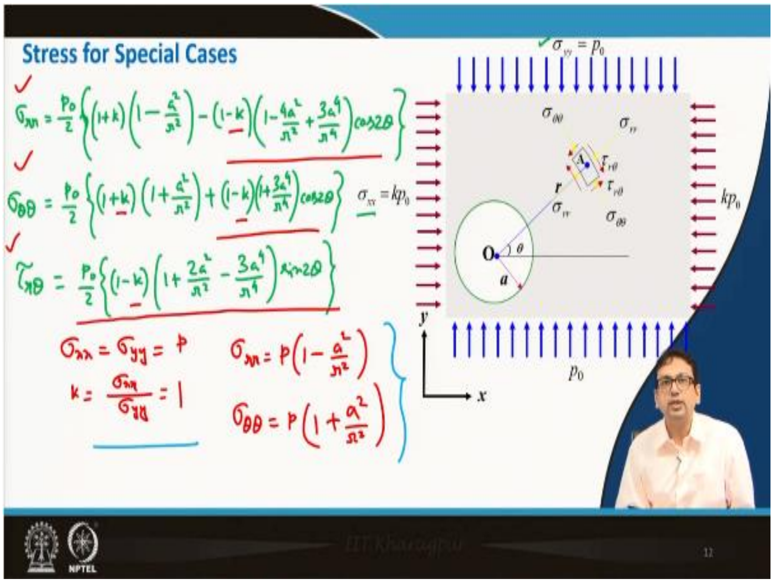

So, now another thing is a special case for like stress, for the special case, special case means what I mean to say here is suppose your σ_{xx} and σ_{yy} is equal, so then that means k will become equal to 1. So, now just let me just quickly write down the basic expression what we have written earlier also,

$$\sigma_{rr} = \frac{p_0}{2}\left\{(1+k)\left(1-\frac{a^2}{r^2}\right)-(1-k)\left(1-\frac{4a^2}{r^2}+\frac{3a^4}{r^4}\right)\cos 2\theta\right\}$$

$$\sigma_{\theta\theta} = \frac{p_0}{2}\left\{(1+k)\left(1+\frac{a^2}{r^2}\right)+(1-k)\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta\right\}$$

$$\tau_{r\theta} = \frac{p_0}{2}\left\{(1-k)\left(1+\frac{2a^2}{r^2}-\frac{3a^4}{r^4}\right)\sin 2\theta\right\}$$

So, we know these equations, now for special case what I am telling is like σ_{xx} is equal to suppose σ_{yy} is equal to p.

Therefore, for these equations k is equal to 1. So, entire shear stress will vanish, so that means under this situation, σ_{rr} is equal to

$$\sigma_{rr} = p\left(1-\frac{a^2}{r^2}\right)$$

Here $\sigma_{\theta\theta}$ is becoming as follows

$$\sigma_{\theta\theta} = p\left(1+\frac{a^2}{r^2}\right)$$

So, these are the special cases where k is equal to 1.

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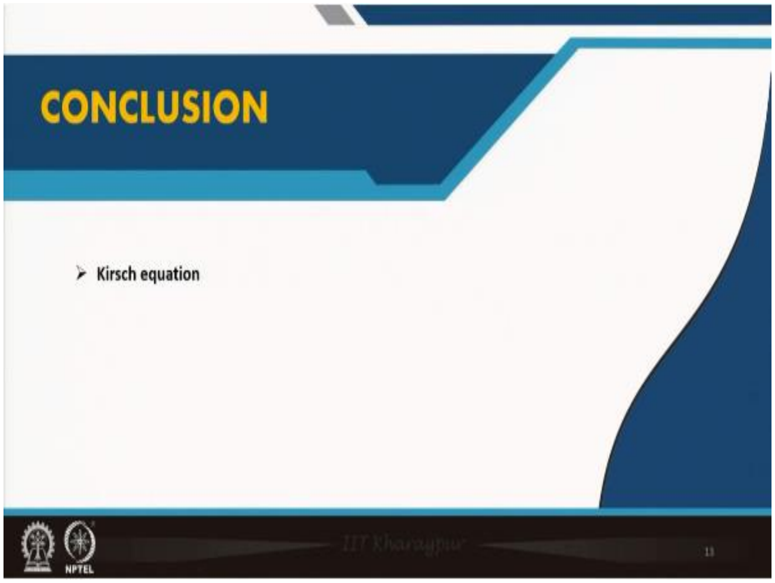

So, with this let us conclude our today's lecture basically we have mainly focused discussed the Kirsch equation only today and in our next lecture we will discuss, first we will take a problem related to this Kirsch equation and we learn few more things like the zone of influence also we learn in our next lecture, thank you.