Rock Mechanics and Tunneling
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Lecture 29
Analysis of Stresses (Continued)

Hello everyone! I welcome all of you to the 3rd lecture of Module 6. So, in Module 6, we are discussing about the rock and rock mass failure criteria. So, under that, we have basically discussed the analysis of stresses.

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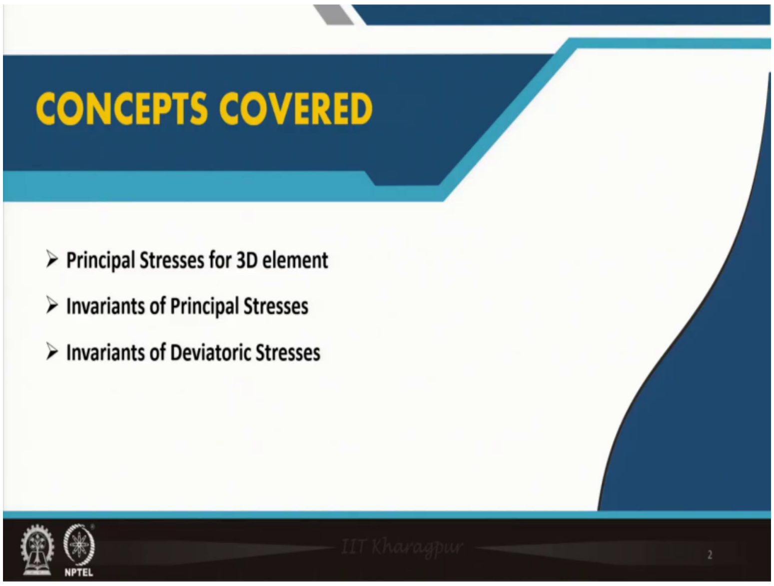

So, these are the things which we will discuss today like Principal Stresses for 3D element, Invariants of Principal Stresses, and Invariants of Deviatoric Stresses.

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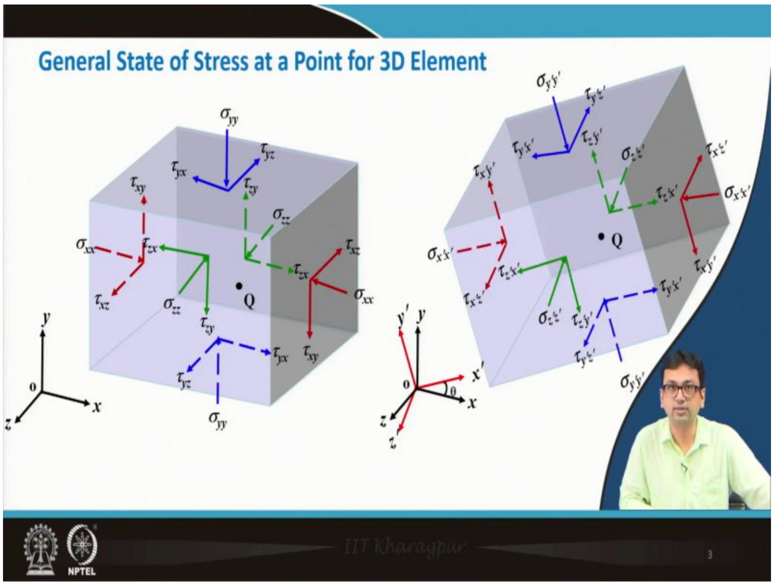

So, for the generalized state of stress at a point for 3D element, the stresses will be as per the first diagram which we have seen in the previous lecture and for an arbitrary orientation of the element, the stresses will be as per the second diagram.

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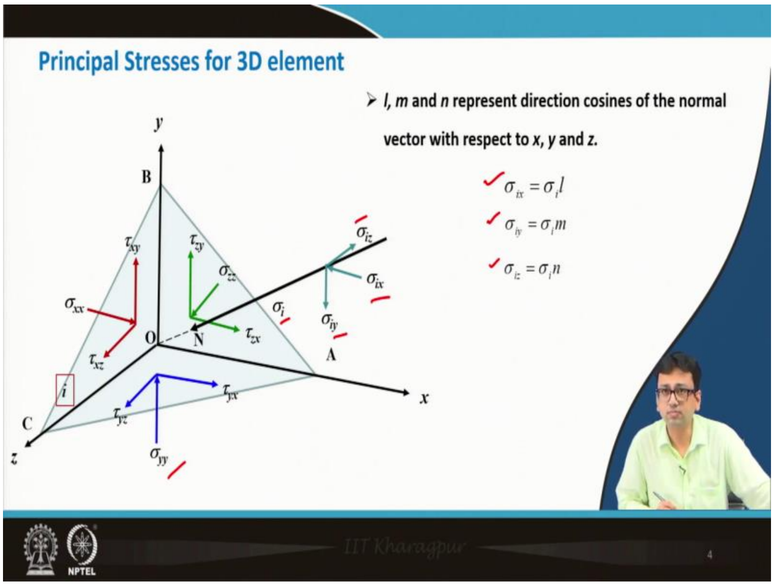

Now, let us discuss about the principal stresses for 3D element. Let us consider it as the *xyz* coordinate system and there is an oblique plane (*i*) which is named as ABC. Now, let us consider that the principal stress (σ_i) is acting on this oblique plane ABC, and ON is the normal vector to the plane.

The stresses acting on the OBC, OAB, and AOC planes are shown in the diagram.

Now, we can resolve σ_i in x, y, and z directions. So, σ_{ix} is the component of σ_i in x direction, σ_{iy} is the component of σ_i in y direction, and σ_{iz} is the component of σ_i in z direction. Now, l, m, and n represent the direction cosines of the normal vector with respect to x, y, and z direction, respectively.

Thus, we can write,

$$\sigma_{ix} = \sigma_i l$$

$$\sigma_{iy} = \sigma_i m$$

$$\sigma_{iz} = \sigma_i n$$

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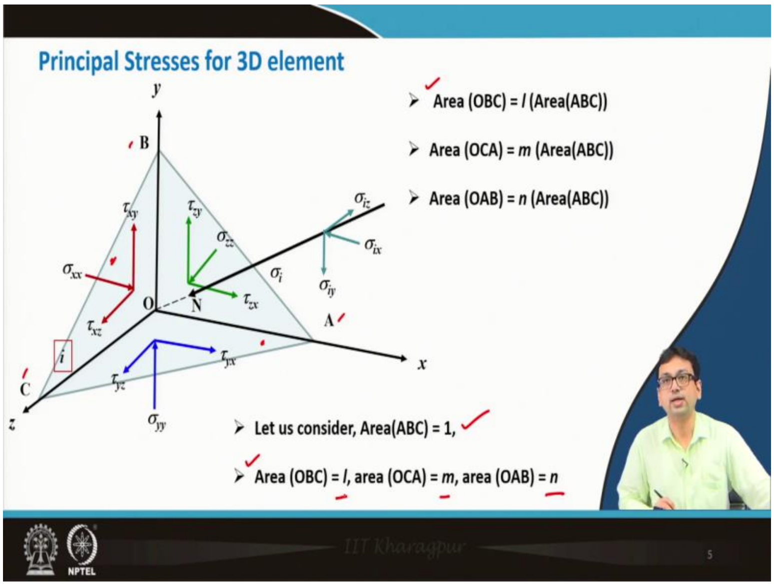

Now, the area of OBC = $l \times$ Area of ABC.

Similarly, area of OCA = $m \times$ Area of ABC.

And, the area of OAB = $n \times$ Area of ABC.

Now, if we consider area of ABC as 1, so accordingly the area of OBC will be equal to l, the area of OCA will be m, and the area of OAB will be equal to n.

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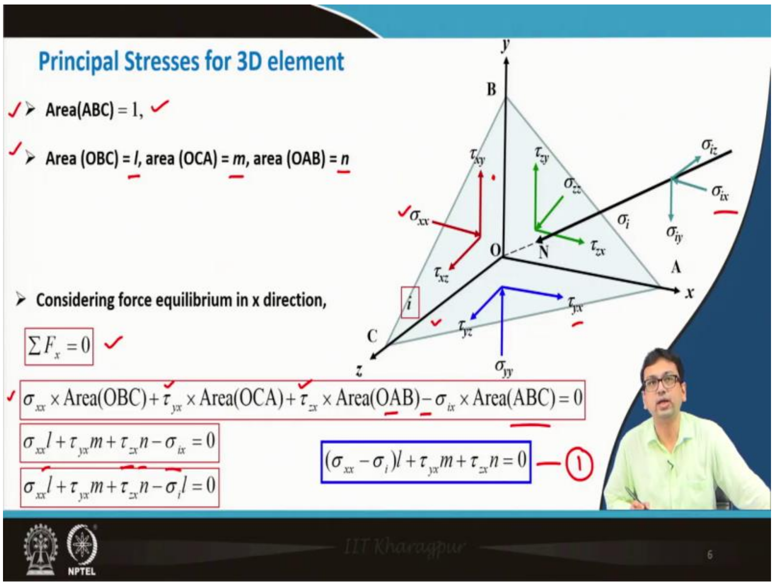

Now, we will consider the force equilibrium in the *x* direction, i.e., $\sum F_x = 0$

$$\Rightarrow \sigma_{xx} \times \text{Area(OBC)} + \tau_{yx} \times \text{Area(OCA)} + \tau_{zx} \times \text{Area(OAB)} - \sigma_{ix} \times \text{Area(ABC)} = 0$$

$$\Rightarrow \sigma_{xx} l + \tau_{yx} m + \tau_{zx} n - \sigma_{ix} = 0$$

$$\Rightarrow \sigma_{xx} l + \tau_{yx} m + \tau_{zx} n - \sigma_i l = 0$$

$$\Rightarrow (\sigma_{xx} - \sigma_i) l + \tau_{yx} m + \tau_{zx} n = 0 \ \ldots (1)$$

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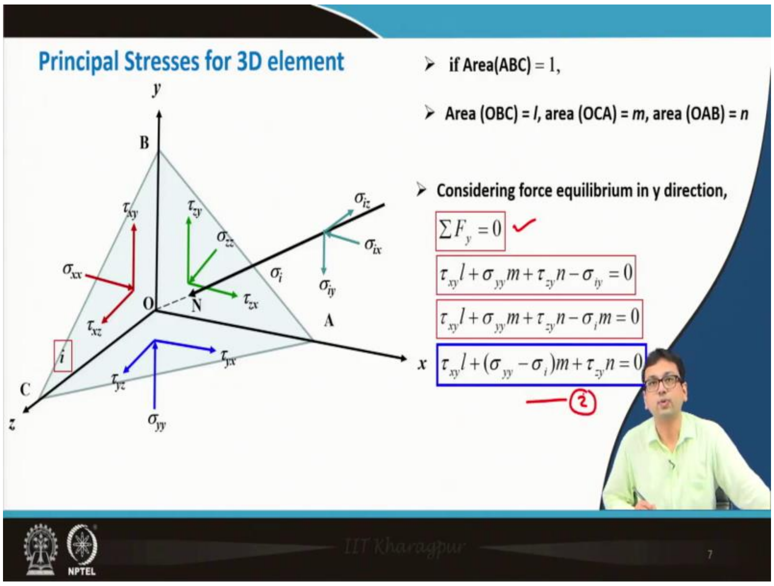

Similarly, if we go for $\sum F_y = 0$

Then, $\tau_{xy} l + \sigma_{yy} m + \tau_{zy} n - \sigma_{iy} = 0$

$\Rightarrow \tau_{xy} l + \sigma_{yy} m + \tau_{zy} n - \sigma_i m = 0$

$\Rightarrow \tau_{xy} l + (\sigma_{yy} - \sigma_i) m + \tau_{zy} n = 0 \;\ldots (2)$

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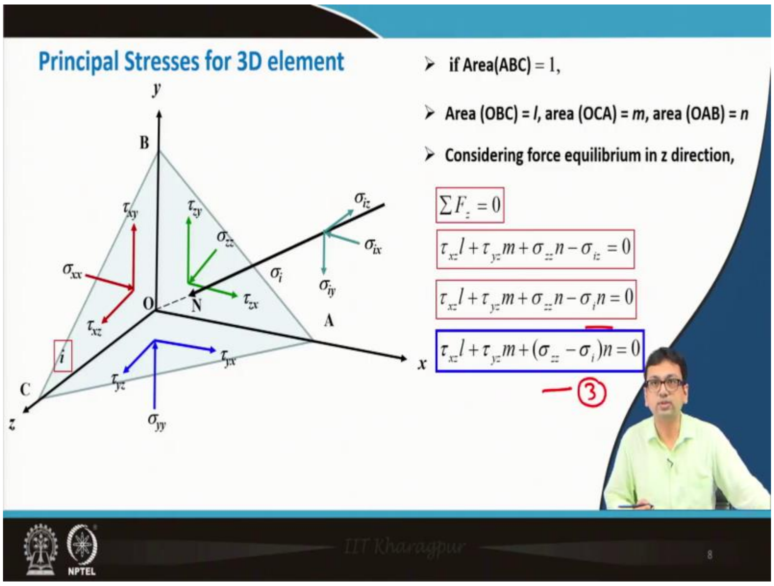

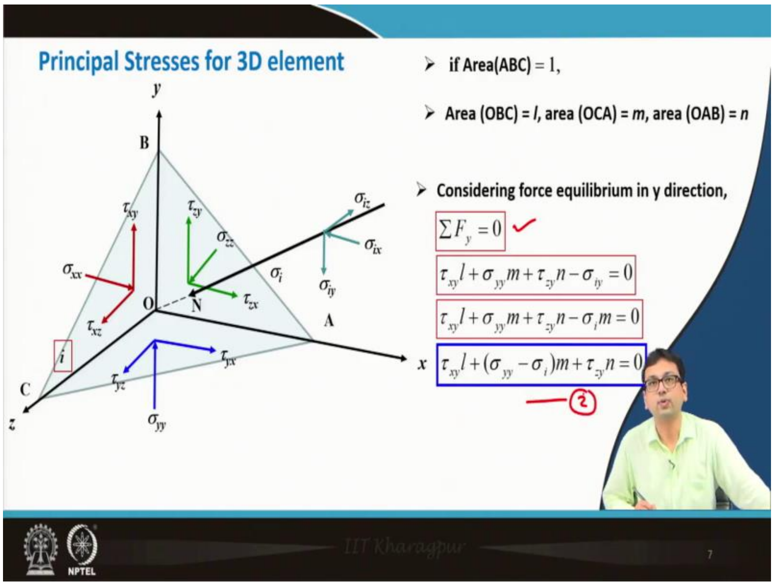

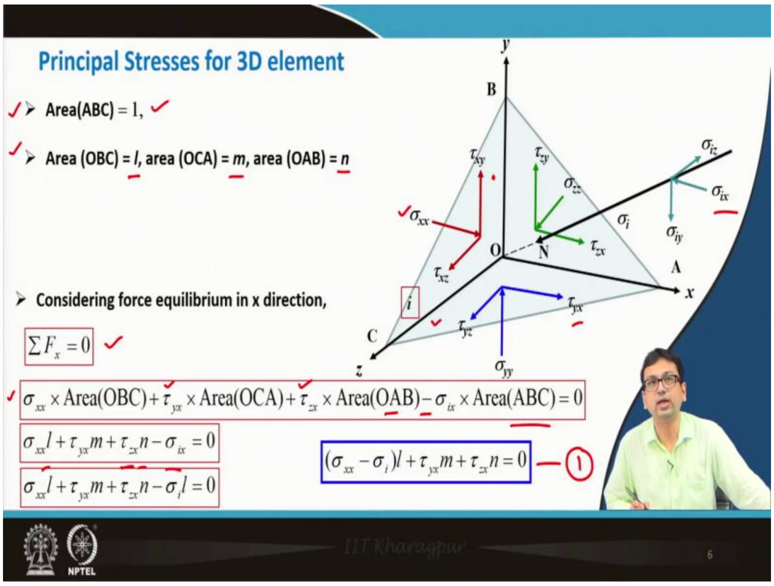

Similarly, if we consider $\sum F_z = 0$,

$$\tau_{xz} l + \tau_{yz} m + \sigma_{zz} n - \sigma_{iz} = 0$$

$$\Rightarrow \tau_{xz} l + \tau_{yz} m + \sigma_{zz} n - \sigma_i n = 0$$

$$\Rightarrow \tau_{xz} l + \tau_{yz} m + (\sigma_{zz} - \sigma_i) n = 0 \ \ldots (3)$$

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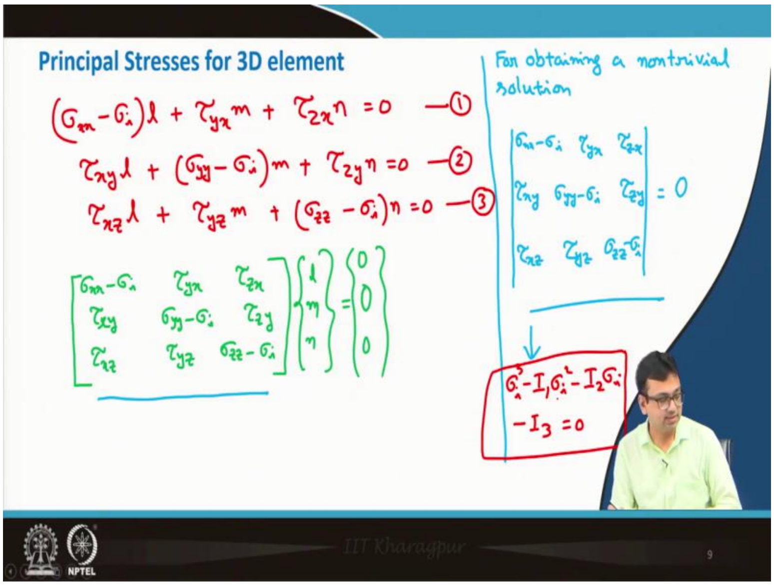

Thus, we can write the three equations as:

$$(\sigma_{xx} - \sigma_i) l + \tau_{yx} m + \tau_{zx} n = 0 \ldots (1)$$

$$\tau_{xy} l + (\sigma_{yy} - \sigma_i) m + \tau_{zy} n = 0 \ldots (2)$$

$\tau_{xz} l + \tau_{yz} m + (\sigma_{zz} - \sigma_i) n = 0 \ldots (3)$

These equation can be written in the matrix form given as follows:

$$\begin{bmatrix} \sigma_{xx} - \sigma_i & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} - \sigma_i & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} - \sigma_i \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Now, to achieve the bon-trivial solution, the determinant of the coefficient matrix must be equal to zero.

i.e., $$\begin{vmatrix} \sigma_{xx} - \sigma_i & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} - \sigma_i & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} - \sigma_i \end{vmatrix} = 0$$

After simplifying, the above equation can be written as: $\sigma_i^3 - I_1 \sigma_i^2 - I_2 \sigma_i - I_3 = 0$

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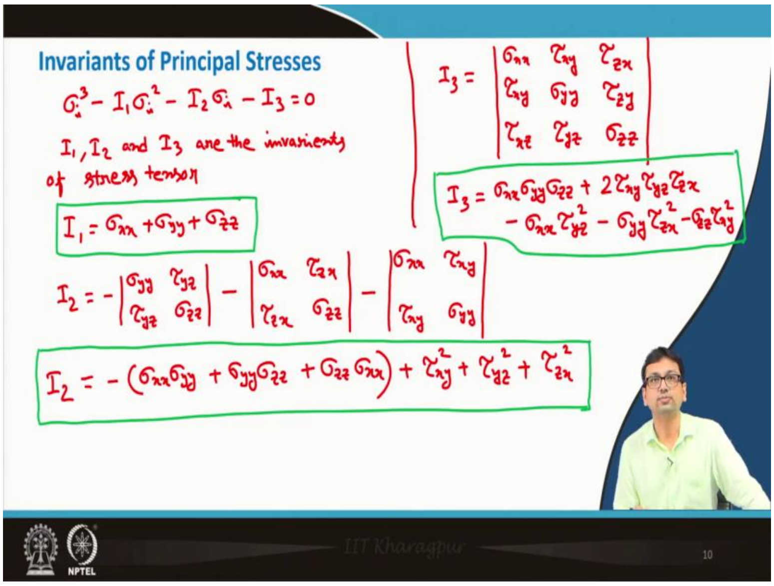

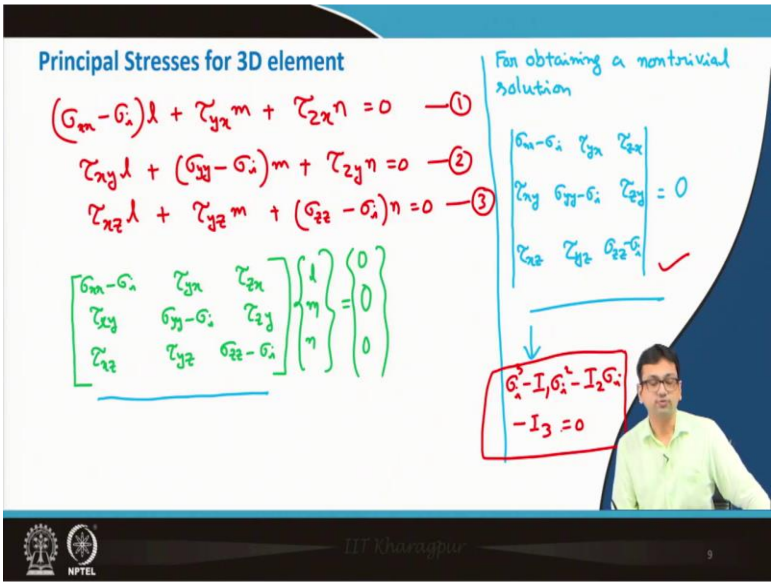

Here, I_1, I_2, and I_3 are the invariants of the stress tensor. What these invariants are that we all obviously discuss in the subsequent slides.

After simplifying, we will get the first invariant of stress tensor as $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$

I_2 can be represented as follows:

$$I_2 = -\begin{vmatrix} \sigma_{yy} & \tau_{yz} \\ \tau_{yz} & \sigma_{zz} \end{vmatrix} - \begin{vmatrix} \sigma_{xx} & \tau_{zx} \\ \tau_{zx} & \sigma_{zz} \end{vmatrix} - \begin{vmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{vmatrix}$$

$$\Rightarrow I_2 = -(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2$$

So, it the second invariant of the stress tensor.

The third invariant of the stress tensor can be represented as follows

$$I_3 = \begin{vmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{vmatrix}$$

$$\Rightarrow I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2$$

So, this is the final expression for I_3.

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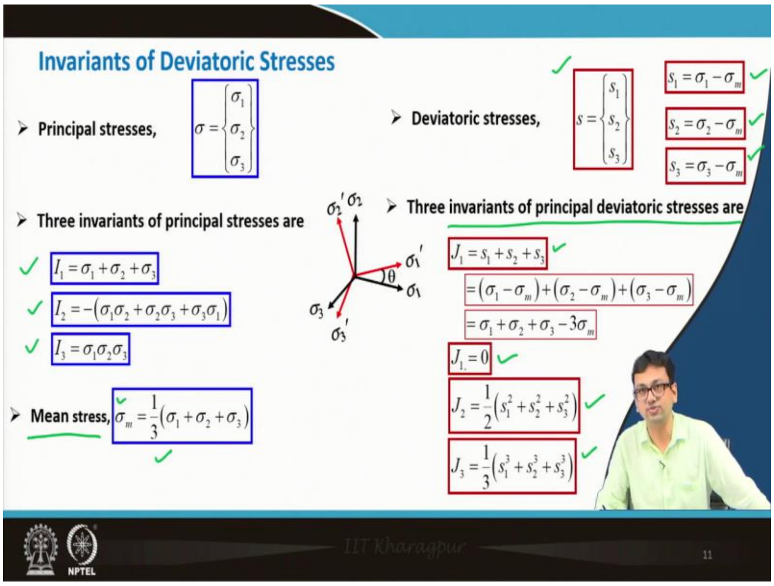

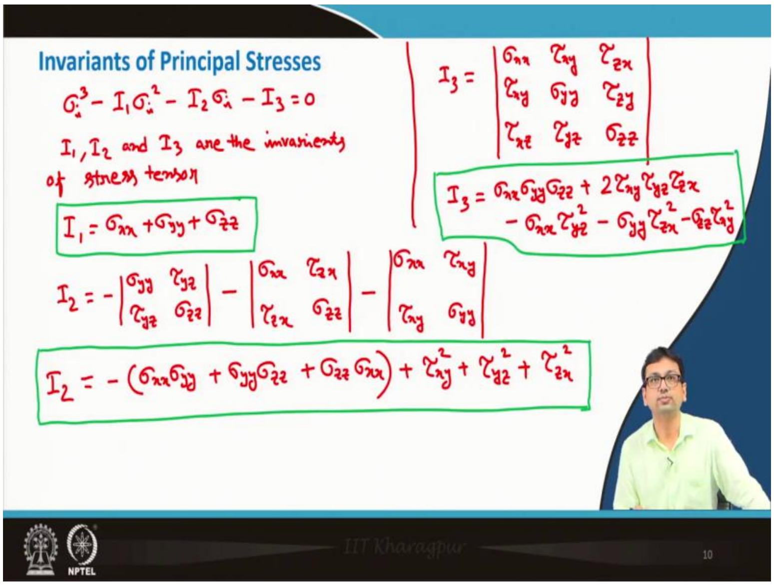

Now, if the principal stresses (i.e., σ_1, σ_2, and σ_3) are present, i.e., there is no shear stresses, the expressions for the invariants become simpler as σ_{xx} will become σ_1, σ_{yy} will become σ_2, likewise, σ_{zz} will become σ_3, and the shear stresses terms will be zero.

Thus, the three invariants of the principal stresses will be

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$

$$I_3 = \sigma_1\sigma_2\sigma_3$$

Now, the mean stress (σ_m) is the average of σ_1, σ_2, and σ_3. So, $\sigma_m = \dfrac{\sigma_1 + \sigma_2 + \sigma_3}{3}$

Now if we consider the principal stresses, the corresponding deviatoric stresses will be

$$S = \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \end{Bmatrix}$$

where, $S_1 = \sigma_1 - \sigma_m$, $S_2 = \sigma_2 - \sigma_m$, and $S_3 = \sigma_3 - \sigma_m$

Now, we have the invariants of principal stresses which are I_1, I_2, and I_3. Similarly, we can have the three invariants of principal deviatoric stresses which are J_1, J_2, and J_3.

So, $J_1 = S_1 + S_2 + S_3$

$$\Rightarrow J_1 = (\sigma_1 - \sigma_m) + (\sigma_2 - \sigma_m) + (\sigma_3 - \sigma_m)$$

$$\Rightarrow J_1 = (\sigma_1 + \sigma_2 + \sigma_3) - 3\sigma_m$$

Since $\sigma_m = \dfrac{\sigma_1 + \sigma_2 + \sigma_3}{3}$, $J_1 = (\sigma_1 + \sigma_2 + \sigma_3) - 3 \times \left(\dfrac{\sigma_1 + \sigma_2 + \sigma_3}{3} \right)$

So, $J_1 = 0$

The expression for J_2 can be expressed as follows:

$$J_2 = \frac{1}{2}(S_1^2 + S_2^2 + S_3^2)$$

$$J_3 = \frac{1}{2}(S_1^3 + S_2^3 + S_3^3)$$

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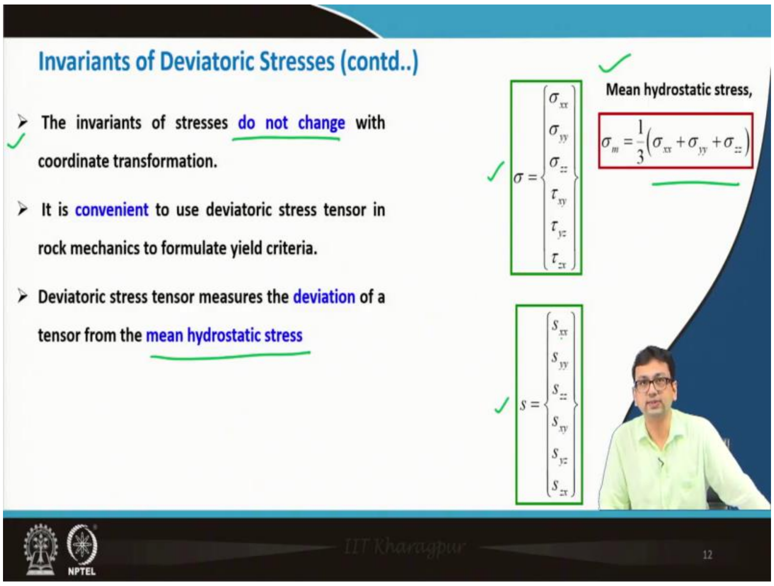

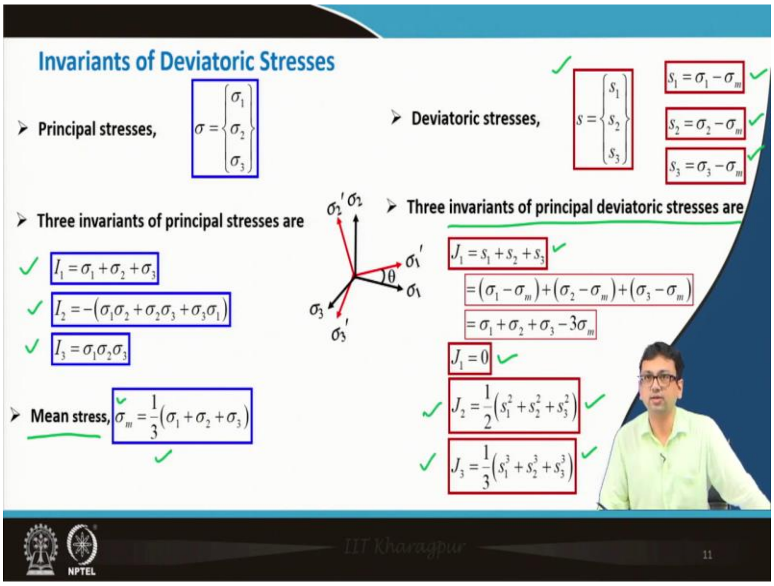

So now, let us discuss about this invariants, so what are the invariants? The invariants of stresses do not change with the coordinate transformation, which is the biggest advantage of using these invariants. We will see in the latter module that when you present the failure criterion, then we will try to use these invariants, so that it should not be dependent on the coordinate transformation. It is very important that the invariants of stresses do not change with coordinate transformation.

Other than that it is convenient to use deviatoric stress tensor in rock mechanics to formulate the yield criteria.

Now what we have seen that $J_1 = 0$, i.e., out of these three invariants, one invariant is becoming 0. So, we need to think about only J_2 and J_3. So, it is one of the advantages of using the invariants of deviatoric stress tensor.

So, the third point shows that the deviatoric stress tensor measures the deviation of a tensor from the mean hydrostatic stress. So, we know that $S_1 = \sigma_1 - \sigma_m$, $S_2 = \sigma_2 - \sigma_m$, and $S_3 = \sigma_3 - \sigma_m$, where σ_m is the mean stress.

So, the deviatoric stress tensor measures the deviation of a tensor from the mean hydrostatic stress, where mean hydrostatic stress is σ_m which is equal to $\dfrac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}$ for the generalized condition of stresses (not for the principal stresses).

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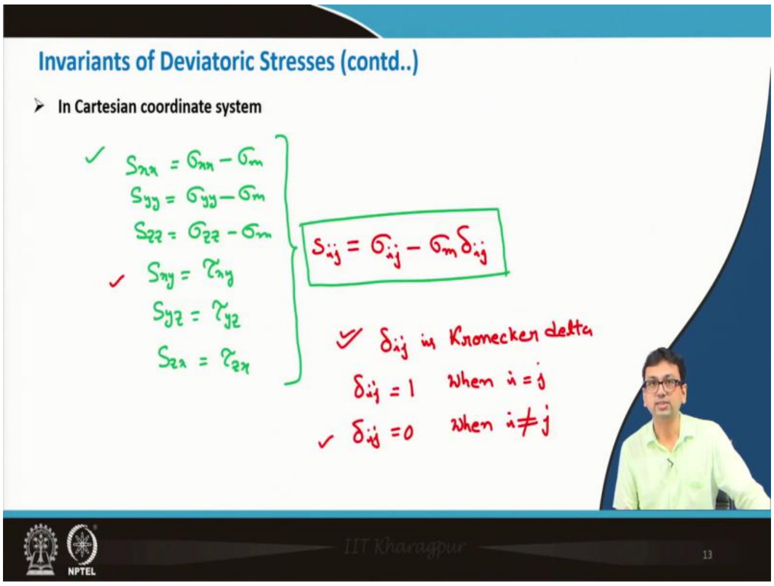

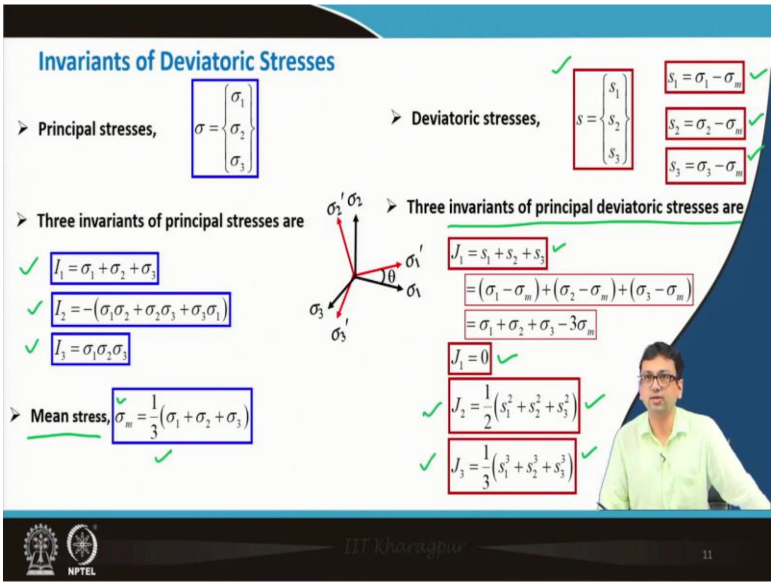

Now, let us try to see the generalized case where $S = \begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{xy} \\ S_{yz} \\ S_{zx} \end{Bmatrix}$.

In the case of the Cartesian coordinate system, $S_{xx} = \sigma_{xx} - \sigma_m$, whereas we have seen for the principal stress, $S_1 = \sigma_1 - \sigma_m$. Similarly, $S_{yy} = \sigma_{yy} - \sigma_m$, and $S_{zz} = \sigma_{zz} - \sigma_m$.

Again, $S_{xy} = \tau_{xy}$, $S_{yz} = \tau_{yz}$, and $S_{zx} = \tau_{zx}$. Now, these six equation can be written in a generalized form which is $S_{ij} = \sigma_{ij} - \sigma_m \delta_{ij}$.

Here, the δ_{ij} is known as the Kronecker delta.

It defines $\delta_{ij} = 1$, when $i = j$, and $\delta_{ij} = 0$, when $i \neq j$

So, for the case of σ_{xx}, both the subscripts are x, i.e., $i = j$. Hence, we can write $S_{xx} = \sigma_{xx} - \sigma_m \delta_{xx}$. As we know that, $\delta_{ij} = 1$, when $i = j$. So, $\delta_{xx} = 1$.

That is why, $S_{xx} = \sigma_{xx} - \sigma_m$

In the case of S_{xy}, let us assume, $i = x$ and $j = y$. So, in this case, $i \neq j$, i.e., $\delta_{ij} = 0$.

So, $S_{xy} = \sigma_{xy} - \sigma_m \delta_{xy}$

Since $\delta_{xy} = 0$, $S_{xy} = \sigma_{xy}$, i.e., $S_{xy} = \tau_{xy}$

So, the generalized form is $S_{ij} = \sigma_{ij} - \sigma_m \delta_{ij}$ (where, $\delta_{ij} = 1$, when $i = j$, and $\delta_{ij} = 0$, when $i \neq j$), which is very important. Earlier, we have defined J_1, J_2, and J_3 based on the principal deviatoric stresses where we have use only S_1, S_2, and S_3. But, here we will define the J_1, J_2, and J_3 for the generalized case.

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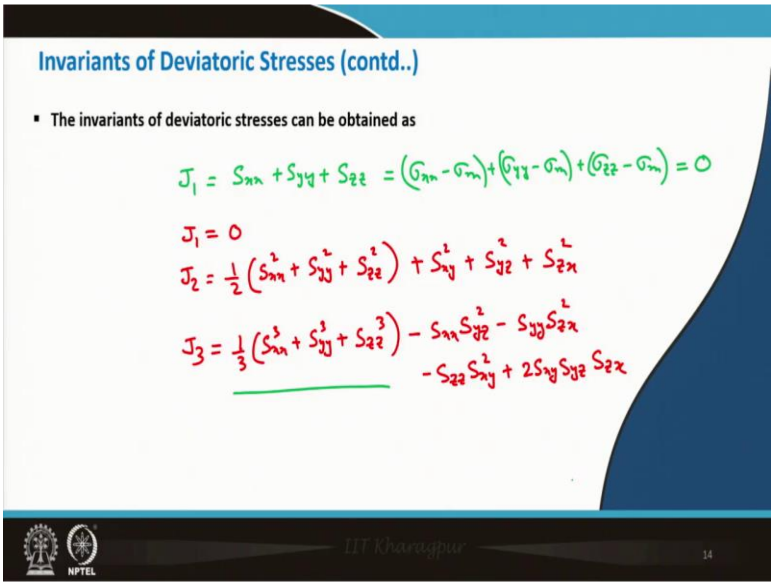

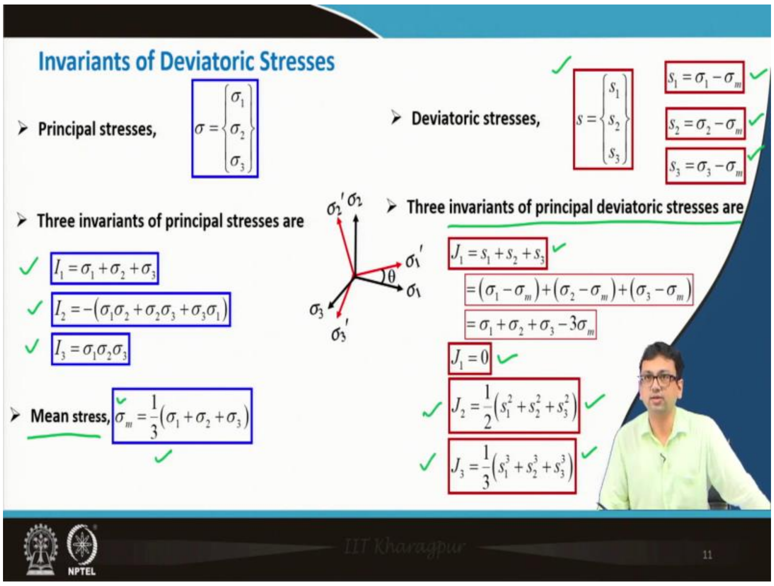

So, the invariants of the deviatoric stresses can be obtained as follows:

$$J_1 = S_{xx} + S_{yy} + S_{zz} = (\sigma_{xx} - \sigma_m) + (\sigma_{yy} - \sigma_m) + (\sigma_{zz} - \sigma_m) = 0$$

$$J_1 = 0$$

Now, $J_2 = \frac{1}{2}(S_{xx}^2 + S_{yy}^2 + S_{zz}^2) + S_{xy}^2 + S_{yz}^2 + S_{zx}^2$

And $J_3 = \frac{1}{3}(S_{xx}^3 + S_{yy}^3 + S_{zz}^3) - S_{xx}S_{yz}^2 - S_{yy}S_{zx}^2 - S_{zz}S_{xy}^2 + 2S_{xy}S_{yz}S_{zx}$

So, in the chase of principal stresses, the S_{xy}, S_{yz}, S_{zx} terms will not be there, and $S_1 = S_{xx}$, $S_2 = S_{yy}$, and $S_3 = S_{zz}$.

So, let us conclude here only our today's class. So thank you.