Rock Mechanics and Tunneling
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Lecture 28
Analysis of Stresses (Continued)

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Hello everyone! I welcome all of you to the 2nd lecture of Module 6. So, in Module 6, we are discussing about the rock and rock mass failure criteria. Actually, before starting the discussion on the rock mass failure criteria, we should discuss about the analysis of stresses. So, what we have started in our previous class, today also we will continue with that topic only.

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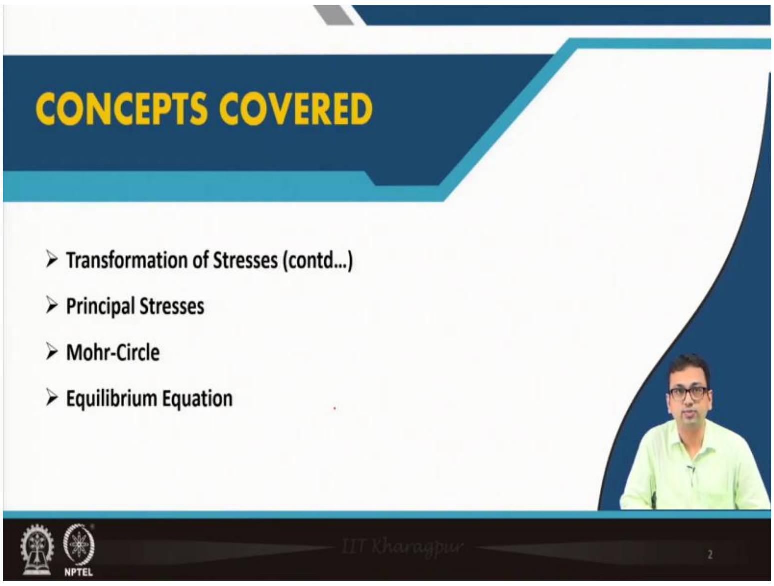

If you remember in our last class, we have discussed towards the end about the transformation of stresses. We will continue with that today also and then we will discuss the principle stresses. After that, Mohr circle and we will also briefly discuss about the equilibrium equation.

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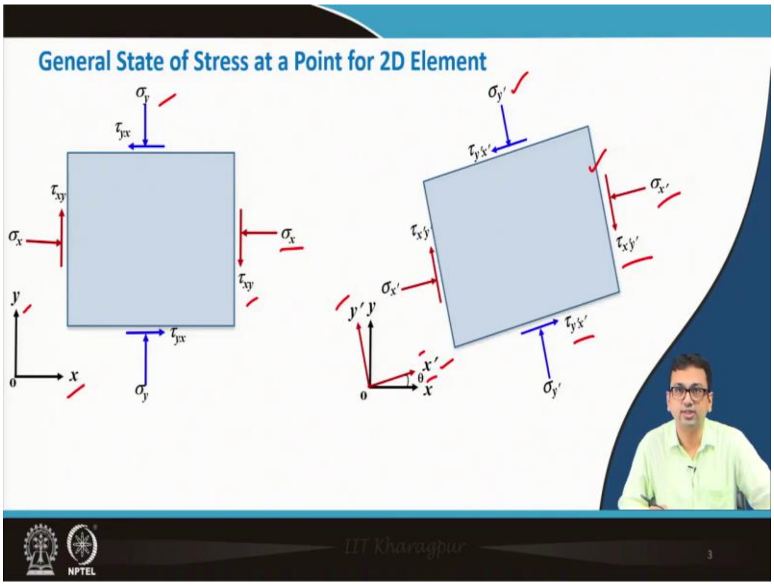

As we know that for the generalized state of stress at a point for 2D element, if this is *xy*-coordinate system is like this as we can see. So, this is an infinitesimal element and if the coordinate system is rotated about an angle θ, and x' and y' come into the picture then the stresses on this infinitesimal element will look like this, i.e., instead of σ_x, σ_y, and τ_{xy}, they become $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ or $\tau_{y'x'}$. We have discussed all these things in our previous class.

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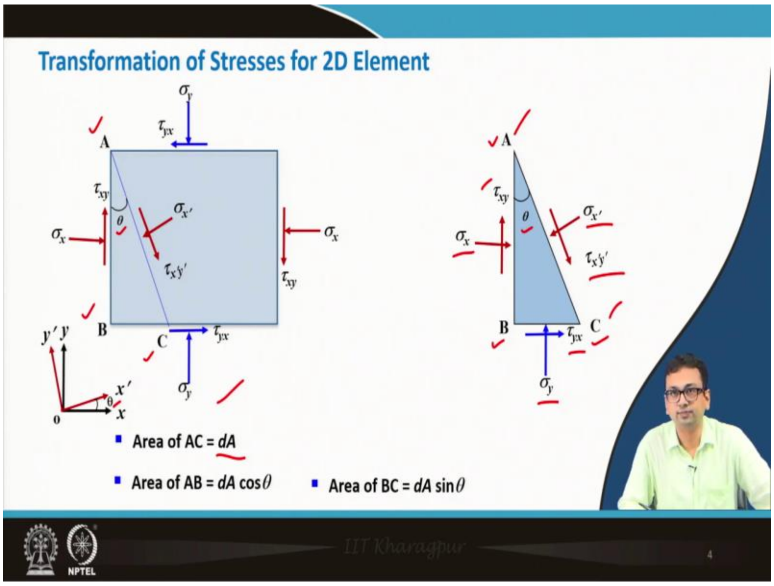

In the previous class, before solving one problem, I have directly given you the transformation equations. However, today, I will discuss in detail about how those equations are coming. So, from the diagram we can see over here that the ABC is an infinitesimal wedge which is cut from the 2D element.

Now, the AC plane is perpendicular to x'. So, the stresses are like the normal stress, $\sigma_{x'}$ and the shear stress, $\tau_{x'y'}$. And, the planes BC and AB are perpendicular to *y*-axis and *x*-axis, respectively. This, is why here the stresses are like σ_y, τ_{yx} and here, they are σ_x, τ_{xy}.

Now, let us consider the area of AC is dA. Since this angle of rotation is θ and the area of the AC is dA. Then, the area of AB will be $dA\cos\theta$ and the area of BC will be $dA\sin\theta$.

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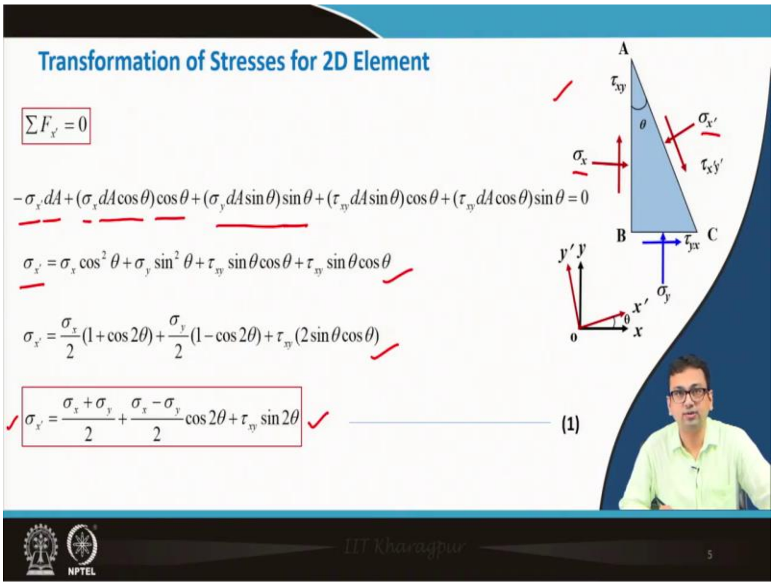

Now, from the force equilibrium, i.e., $\sum F_{x'} = 0$, we will get,

$$-\sigma_{x'} dA + (\sigma_x dA\cos\theta)\cos\theta + (\sigma_y dA\sin\theta)\sin\theta + (\tau_{xy} dA\sin\theta)\cos\theta + (\tau_{xy} dA\cos\theta)\sin\theta = 0$$

$$\Rightarrow \sigma_{x'} = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + \tau_{xy}\sin\theta\cos\theta + \tau_{xy}\sin\theta\cos\theta$$

$$\Rightarrow \sigma_{x'} = \frac{\sigma_x}{2}(1+\cos 2\theta) + \frac{\sigma_y}{2}(1-\cos 2\theta) + \tau_{xy}\times(2\sin\theta\cos\theta)$$

$$\Rightarrow \sigma_{x'} = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta \ldots (1)$$

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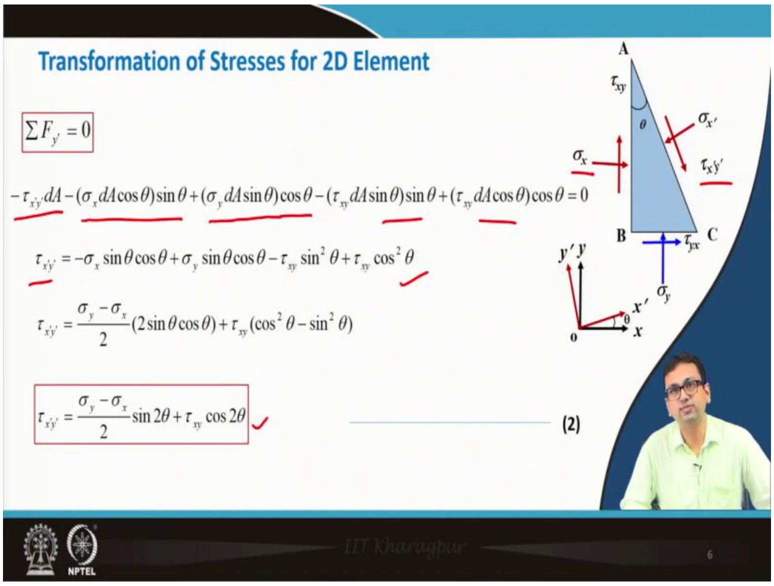

Now similarly, if I go for the $\sum F_{y'} = 0$, then

$$-\tau_{x'y'}dA - (\sigma_x dA\cos\theta)\sin\theta + (\sigma_y dA\sin\theta)\cos\theta - (\tau_{xy} dA\sin\theta)\sin\theta + (\tau_{xy} dA\cos\theta)\cos\theta = 0$$

$$\Rightarrow \tau_{x'y'} = -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta - \tau_{xy}\sin^2\theta + \tau_{xy}\cos^2\theta$$

$$\Rightarrow \tau_{x'y'} = \left(\frac{\sigma_y - \sigma_x}{2}\right)(2\sin\theta\cos\theta) + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\Rightarrow \tau_{x'y'} = \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \ldots (2)$$

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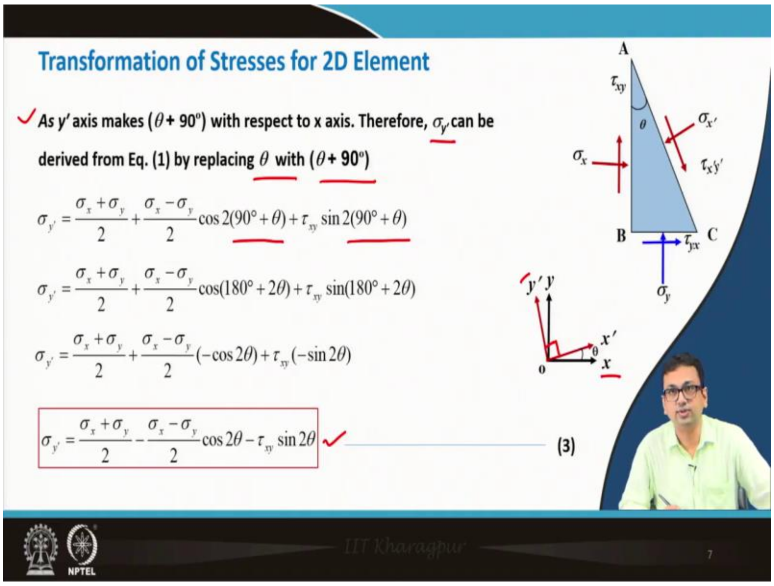

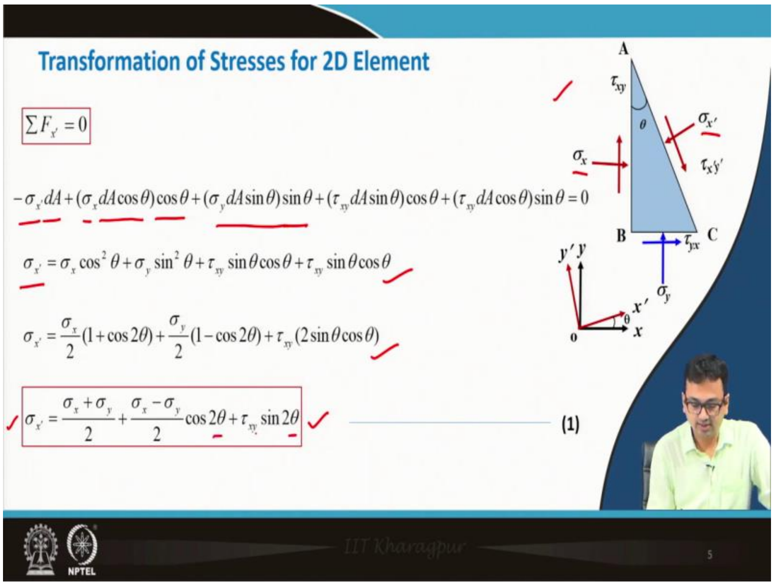

Now, we can see that y' axis makes an angle (θ + 90º) with the x axis. Therefore, $\sigma_{y'}$ can be derived from Eq. (1) by replacing the θ with (θ + 90º). `

So, $\sigma_{y'} = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2(90^\circ+\theta) + \tau_{xy}\sin 2(90^\circ+\theta)$

$$\Rightarrow \sigma_{y'} = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos(180^\circ+2\theta) + \tau_{xy}\sin(180^\circ+2\theta)$$

$$\Rightarrow \sigma_{y'} = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)(-\cos 2\theta) + \tau_{xy}(-\sin 2\theta)$$

$$\Rightarrow \sigma_{y'} = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta - \tau_{xy} \sin 2\theta \;\ldots (3)$$

So, in this way, we can get these three transformation equations from our fundamental knowledge.

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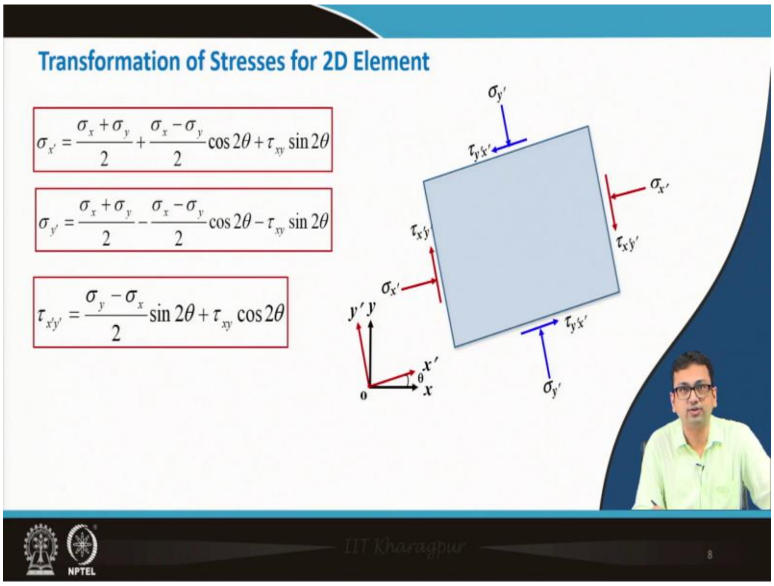

Now, in this slide, the three equations have been written in the compact form.

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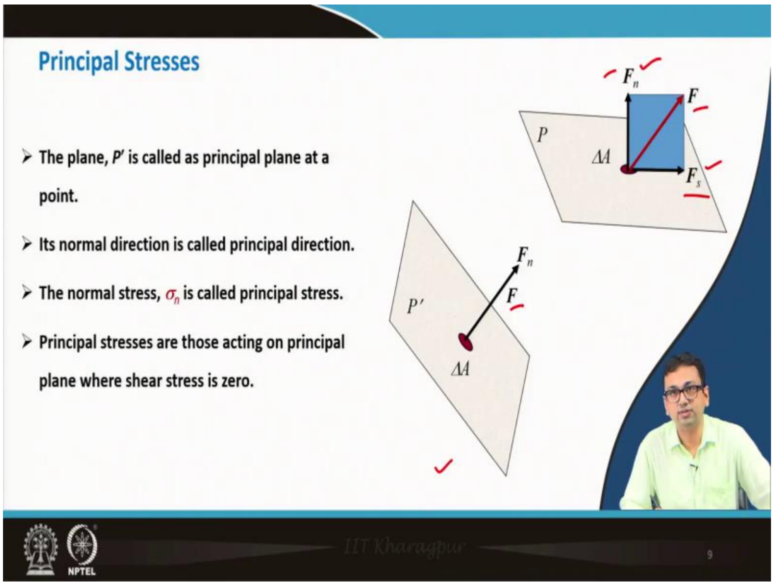

Now, let us discuss about the principal stresses. As we can see over here that the area (ΔA) on plane P and the force (F) is acting on that area. So, we can resolve the force (F) into two components F_s and F_n, where F_n is normal to that plane and F_s is along to the plane P. So, we F_n is the normal force and F_s is the shear force.

Now, if we look into the diagram, we can see that the force (F) is having only the normal component which is actually the F_n and there is no shear component (i.e., F_s). So, we can say that the plane P' is called the principal plane at a point. The normal direction is called the principal direction and the normal stress (σ_n) is called the principal stress.

So, the principal stresses are those acting on principal plane where shear stress is zero. That is why, the shear force (F_s) is zero and the shear stress is also zero. So, this plane is the principal plane and the stress (σ_n) acting on that plane is the principal stress.

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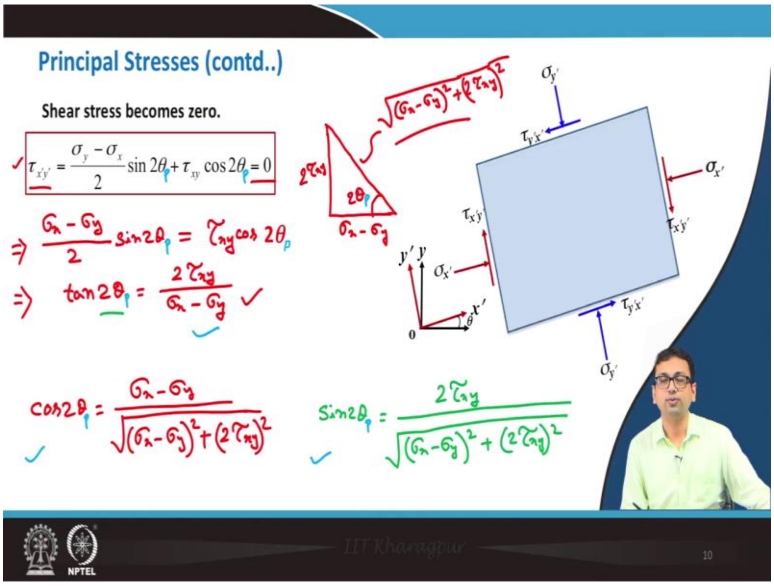

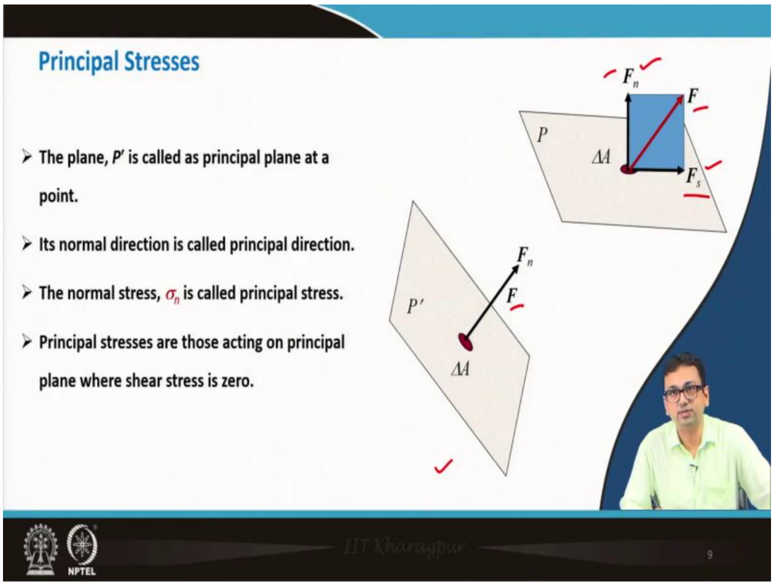

So, this is the principal stress (σ_n) on the principle plane where the shear stress is equal to zero. So, following the Eq. (2), $\tau_{x'y'} = 0$

Hence, we can write, $\Rightarrow \left(\dfrac{\sigma_y - \sigma_x}{2} \right) \sin 2\theta_p + \tau_{xy} \cos 2\theta_p = 0$

$$\Rightarrow \left(\frac{\sigma_x - \sigma_y}{2} \right) \sin 2\theta_p = \tau_{xy} \cos 2\theta_p$$

$$\Rightarrow \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Now, if I draw a simple triangle so that from here, we should able to get some idea about the cos $2\theta_p$ and sin $2\theta_p$. If this angle is 2θ, the vertical side is $2\tau_{xy}$ and the horizontal side is ($\sigma_x - \sigma_y$). So, the hypotenuse side is $\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}$.

Hence, $\cos 2\theta_p = \dfrac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}}$ and $\sin 2\theta_p = \dfrac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}}$

Hence, θ_p gives the orientation of the principal plane.

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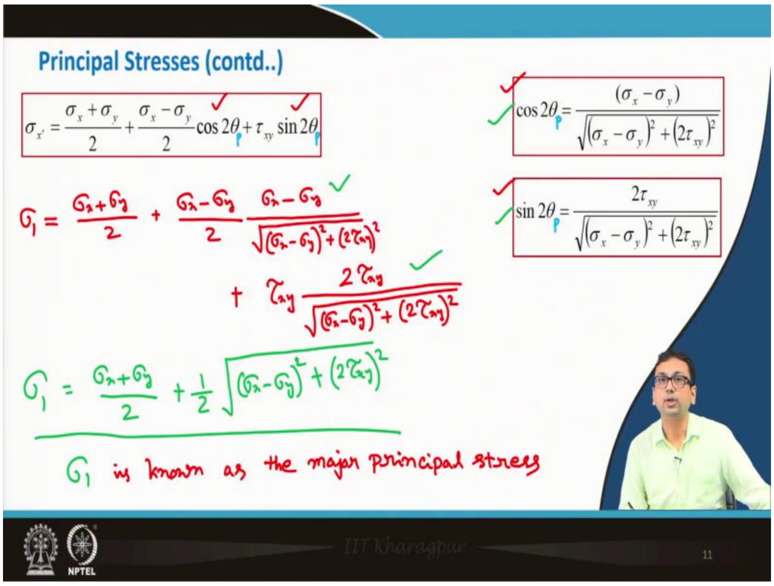

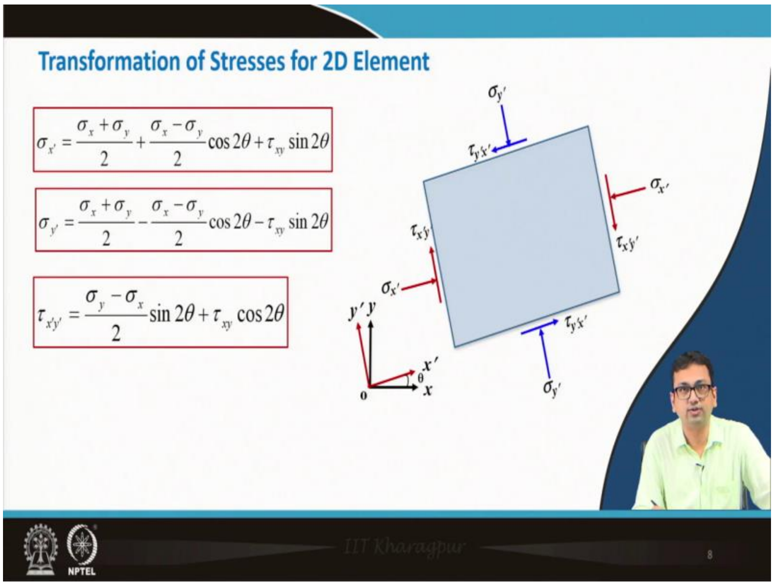

Now, we will try to find out the principal stresses with respect to the principal plane.

So, we know that $\cos 2\theta_p = \dfrac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}}$

$$\sin 2\theta_p = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}}$$

We also know that $\sigma_{x'} = \left(\dfrac{\sigma_x + \sigma_y}{2}\right) + \left(\dfrac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$. However, θ is for the generalized cases, whereas for the principal plane, we have to replace θ with θ_p and $\sigma_{x'}$ will be replaced by σ_1.

Thus, $\sigma_{x'} = \left(\dfrac{\sigma_x + \sigma_y}{2}\right) + \left(\dfrac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta_p + \tau_{xy}\sin 2\theta_p$

$$\Rightarrow \sigma_1 = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\left(\frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}}\right) + \tau_{xy}\left(\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}}\right)$$

$$\Rightarrow \sigma_1 = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}$$

Here, the σ_1 is known as the major principal stress.

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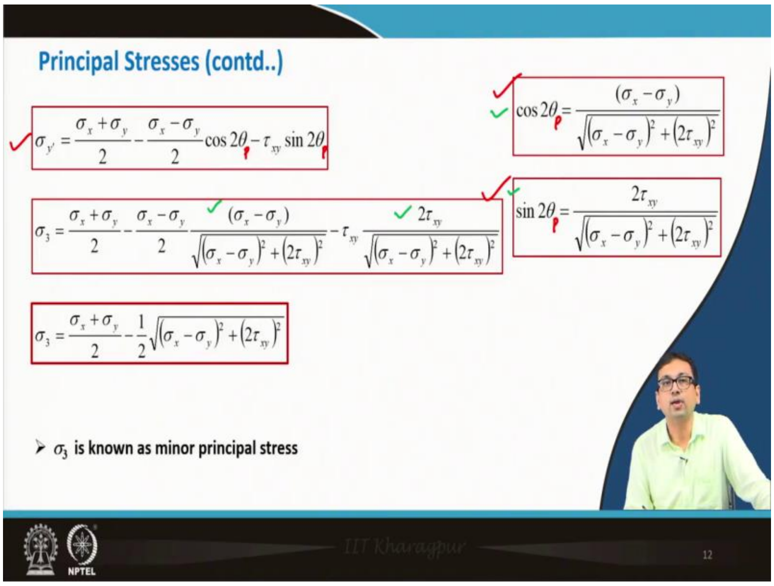

Similarly, $\sigma_{y'} = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$

$$\Rightarrow \sigma_3 = \left(\frac{\sigma_x+\sigma_y}{2}\right) - \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta_p - \tau_{xy}\sin 2\theta_p$$

$$\Rightarrow \sigma_3 = \left(\frac{\sigma_x+\sigma_y}{2}\right) - \left(\frac{\sigma_x-\sigma_y}{2}\right)\left(\frac{\sigma_x-\sigma_y}{\sqrt{(\sigma_x-\sigma_y)^2+(2\tau_{xy})^2}}\right) - \tau_{xy}\left(\frac{2\tau_{xy}}{\sqrt{(\sigma_x-\sigma_y)^2+(2\tau_{xy})^2}}\right)$$

$$\Rightarrow \sigma_3 = \left(\frac{\sigma_x+\sigma_y}{2}\right) - \frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+(2\tau_{xy})^2}$$

Here, the σ_3 is known as the minor principal stress.

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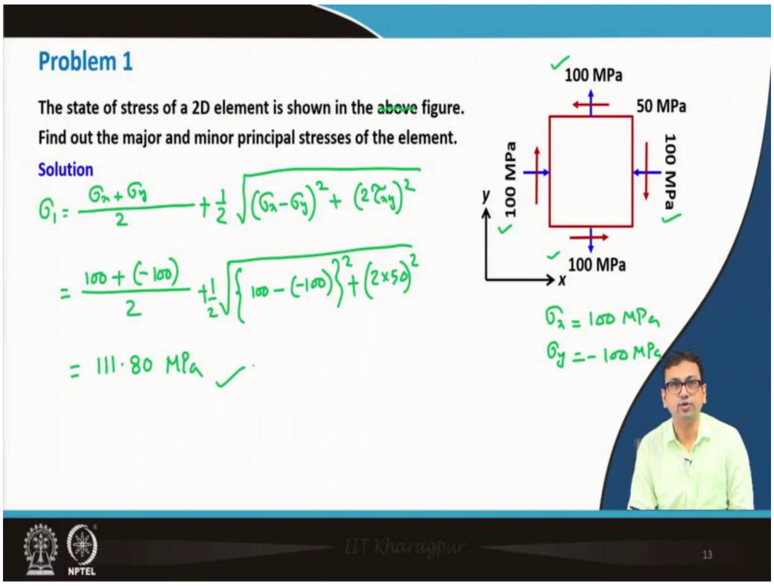

Now, let us take this problem. The state of stress of a 2D element shown in the figure. Find out the major and minor principle stresses of the element. In this problem, the σ_x normal stress is compressive in nature which is considered to be positive, whereas σ_y is tensile in nature. So, it is negative.

The shear is also considered as positive and τ_{xy} is equal to 50 MPa.

So, σ_x = 100 MPa, σ_y = -100 MPa, and τ_{xy} = 50 MPa

Now, the major principal stress, σ_1 can be calculated as follows:

$$\sigma_1 = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}$$

$$\Rightarrow \sigma_1 = \frac{100 + (-100)}{2} + \frac{1}{2}\sqrt{[100 - (-100)]^2 + (2 \times 50)^2}$$

$$\Rightarrow \sigma_1 = 111.80 \text{ MPa}$$

So, the major principal stress (σ_1) is 111.8 MPa.

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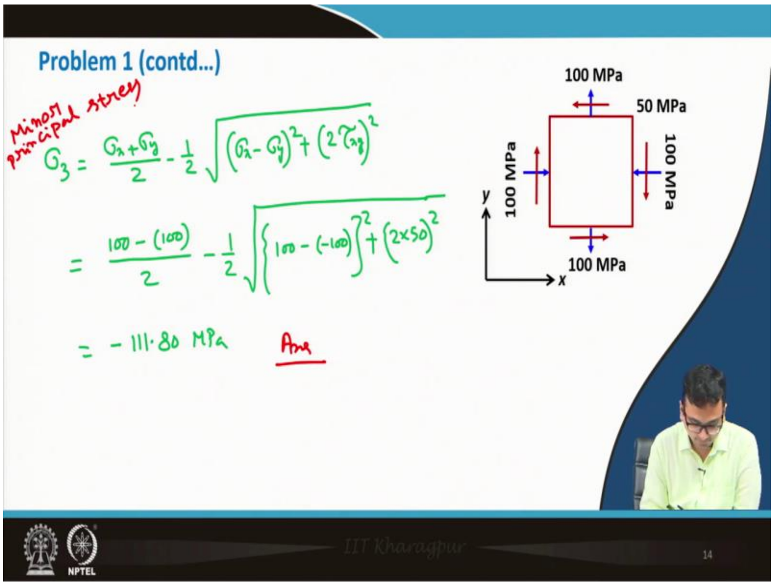

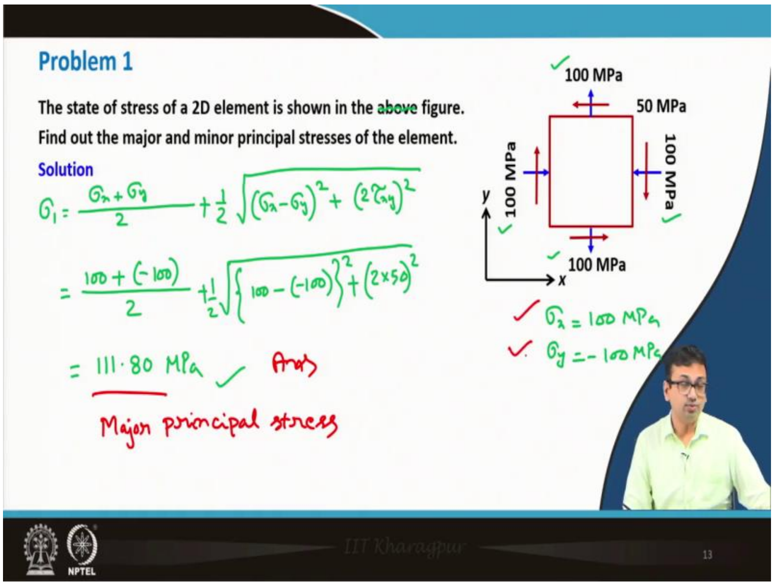

And similarly, we can get the minor principal stress (σ_3). The major principal stress, σ_3 can be calculated as follows: $\sigma_3 = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}$

$$\Rightarrow \sigma_3 = \frac{100 + (-100)}{2} - \frac{1}{2}\sqrt{[100 - (-100)]^2 + (2 \times 50)^2}$$

$$\Rightarrow \sigma_3 = -111.80 \text{ MPa}$$

So, the minor principal stress (σ_3) is -111.8 MPa.

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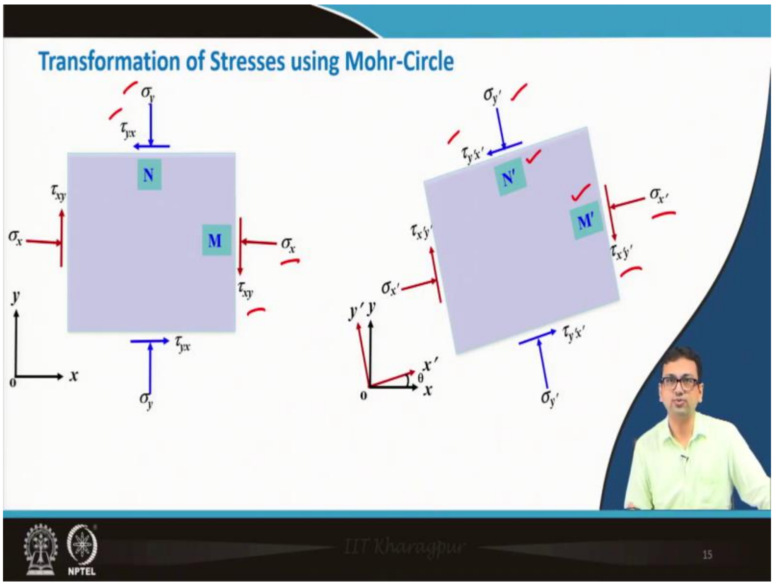

Now, we will just briefly discuss about the Mohr-Circle. Using the Mohr-Circle, we can very easily find out the transformation of stresses. So, what we have learned with the help of equations, we can also obtain the same thing with the help of Mohr-circle also. So, let us consider the planes which are M plane and N plane.

So, in the M plane, σ_x, τ_{xy} are the normal stress and the shear stress, respectively, and in the N plane, σ_y, τ_{yx} are the normal stress and the shear stress, respectively. Now, let us take this orientation where $x'y'$ is the new coordinate system, and the angle between x and x' is θ. So, on the plane perpendicular to x', the normal stress and the shear stress are $\sigma_{x'}$ and $\tau_{x'y'}$, respectively. Now, let us consider this plane as M' plane and here, it is the N' plane where the normal and the shear stresses are $\sigma_{y'}$ and $\tau_{y'x'}$, respectively.

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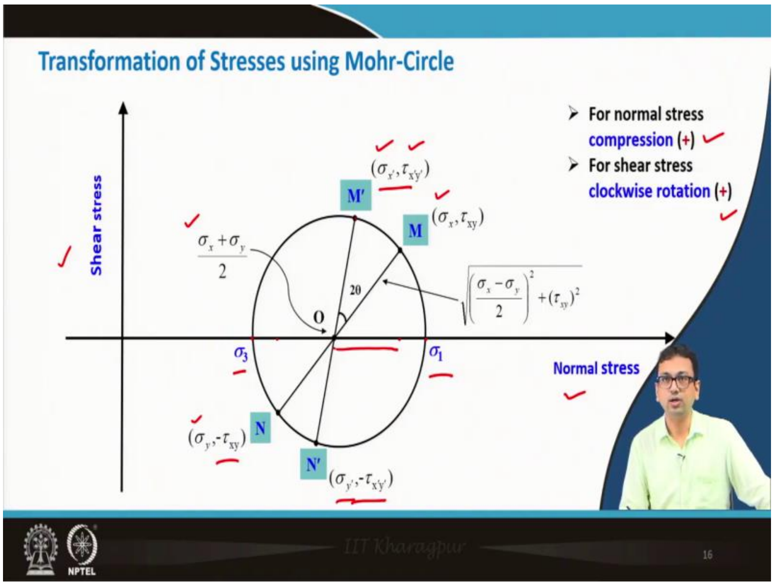

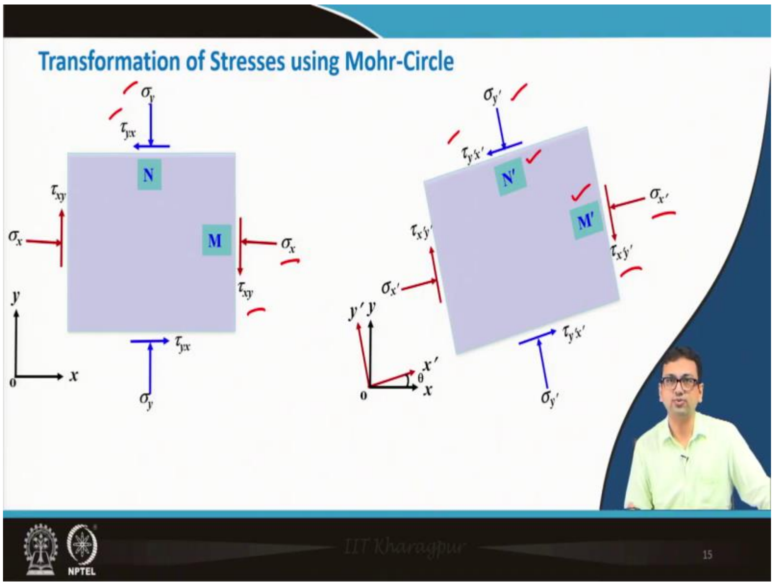

Now using Mohr-circle how can we get the $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$? As we know that we need two axes to draw the Mohr-circle. One is the shear stress axis and another one is the normal stress axis. Now first, we have to identify σ_x, τ_{xy} and σ_y, τ_{yx}. If we look into the M face, the stresses are σ_x, τ_{xy}, and in the N face, the stresses are σ_y, τ_{yx}.

So, we have to locate σ_x, τ_{xy}, which are stresses on M plane, on the σ-τ plane. If the numerical values are given to us, we can easily locate these values.

We have considered the sign convention as for the normal stress, compression is considered to be positive and for the shear stress, the clockwise rotation is considered to be positive. So, if we look into the M face, the shear stress (τ_{xy}) is creating the clockwise rotation which we have considered as the positive.

Similarly, we can identify the state of stress on N face which is ($\sigma_y, -\tau_{yx}$) or ($\sigma_y, -\tau_{xy}$) as we have learnt from the previous class that $\tau_{yx} = \tau_{xy}$. The negative sign is coming as the shear stress on this N plane (τ_{yx}) is creating the anti-clockwise rotation but what we have considered over here that the clockwise rotation is positive for shear stress. That is why, the negative sign is coming.

Now, if we connect (σ_x, τ_{xy}) and ($\sigma_y, -\tau_{xy}$), it will intercept this normal stress axis at a point, suppose O. So, the point O defines $\left(\dfrac{\sigma_x + \sigma_y}{2}\right)$. Now, $\sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$ defines the radius of the Mohr-Circle. We can very easily find out the radius, the horizontal distance from O to M will be $\dfrac{\sigma_x - \sigma_y}{2}$ and the vertical distance between O and M will be τ_{xy}. So, the radius of the circle (OM) will be $\sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$. Thus, considering this distance OM as the radius, we can very easily draw the Mohr-circle.

Now, once we draw this Mohr-circle, then the farthest intercept of the circle in the σ-axis will give the magnitude of major principal stress (σ_1) where the shear stress is zero. Similarly, the nearest intercept of the circle in the σ-axis will give the magnitude of minor principal stress (σ_3) where the shear stress is also zero. So, the magnitude of σ_3 is less than the magnitude of σ_1. Now, if we have to find out the state of stress for this orientation, i.e., for the planes M' and N' where the M' face is oriented at an angle θ with the x-axis. So, the state of stress on M' face is ($\sigma_{x'}, \tau_{x'y'}$) and we can get it graphically from the Mohr-circle.

So, after drawing the Mohr-circle, we will consider an angle 2θ from MN as the angle θ in stress element will be equal to angle 2θ in Mohr-Circle. Thus, if we draw the line OM' which is at angle 2θ from OM line, it will give us the state of stress on the plane M', i.e., ($\sigma_{x'}, \tau_{x'y'}$). Similarly, ON' line will give the state of stress as ($\sigma_{x'}, -\tau_{x'y'}$). So, by using the Mohr-Circle, we can obtain the state of stress graphically at any orientation.

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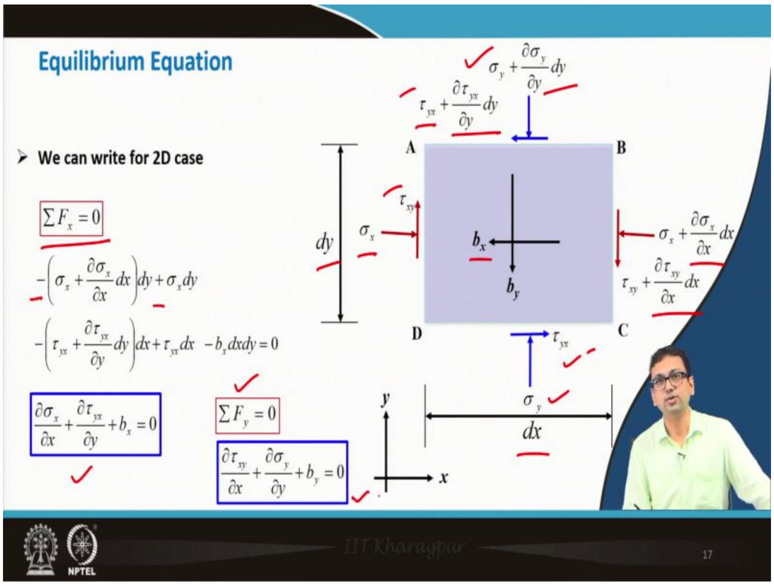

Now, let us discuss about the equilibrium equation. So, for this infinitesimally small element, the sides are dx and dy. The stresses acting on the AD side are (σ_x, τ_{xy}). So, on the BC face, the normal stress will be $\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right)$ and the shear stress will be $\left(\tau_{xy} + \frac{\partial \tau_{yx}}{\partial x} dx\right)$.

Similarly, for DC face, the state of stress will be (σ_y, τ_{yx}) and in AB face, the state of stress will be $\left(\sigma_y + \frac{\partial \sigma_y}{\partial y} dy, \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right)$.

Now, if we consider b_x and b_y are the body forces in x and y directions, respectively, by considering the equilibrium of forces for the 2D element, we can write $\sum F_x = 0$

$$\Rightarrow \sigma_x dy - \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dy + \tau_{yx} dx - \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dx - b_x dx dy = 0$$

$$\Rightarrow \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + b_x = 0$$

Similarly, by considering, $\sum F_y = 0$

We can get, $\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0$

In this way, we can get the equations of equilibrium for 2D element.

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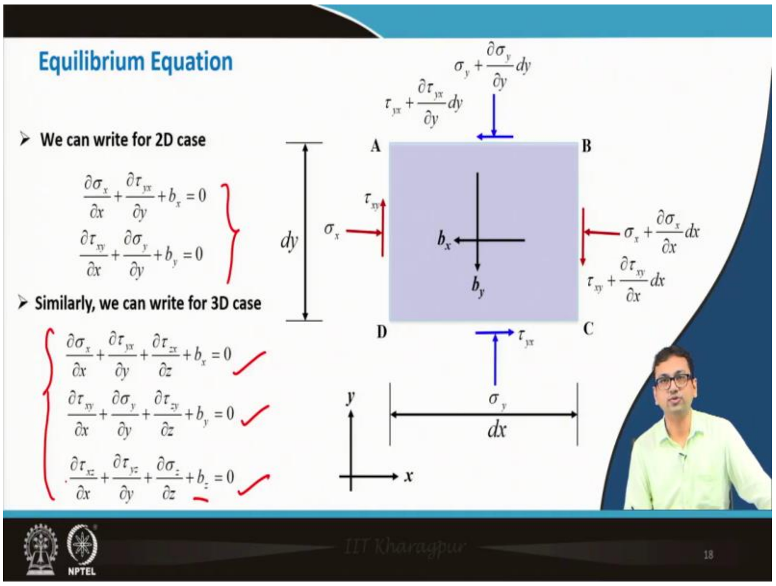

So, we have obtained the two equations for 2D element which are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + b_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0$$

If we extend this logic for the 3D element, the equilibrium equations will be

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + b_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + b_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + b_z = 0$$

Here, the first one is for the x direction. The second one is for the y direction. Similarly, the last one is for the z direction. Hence, σ_z is also coming into picture along with τ_{xz} and τ_{yz}. So, for the 3D case, these three will be the equilibrium equations.

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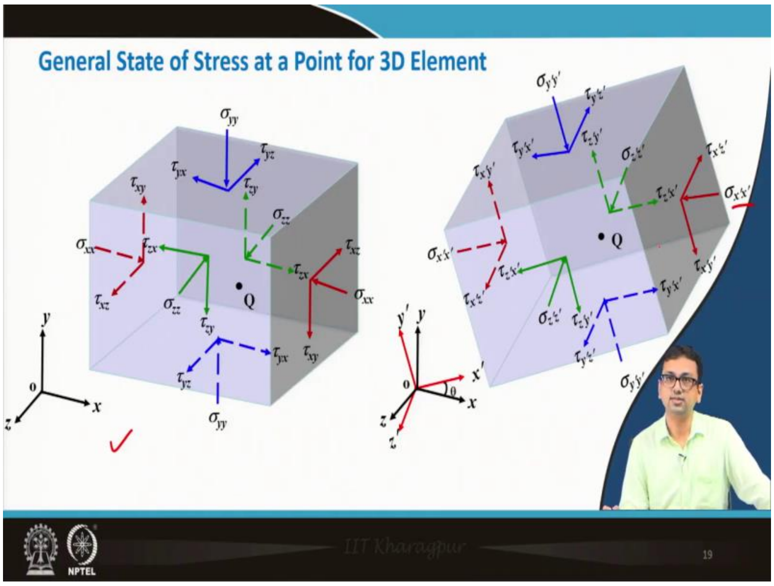

So, the general state of stress at a point for 3D element when the element is at an angle θ with the horizontal: it will be $\sigma_{x'x'}$ instead of σ_{xx}. Here, we can write only x instead of xx as discussed in the previous lecture as we know for the normal stress, if the plane on which the stress is acting, perpendicular to a particular axis and the direction of the stress is also in the same direction, we can write only the one symbol.

So, instead of writing σ_{xx}, you can write σ_x. Similarly, instead of writing $\sigma_{x'x'}$, we can write $\sigma_{x'}$ only.

However, this is the general state of stress at a point for 3D element when this kind of the rotation of axis is happening (refer to the diagram). The stresses will be looking like these.

So, with this let us conclude our today's lecture here only. So we will further discuss about these stresses and subsequently we will enter into our main discussion that is on the rock mass failure criterias. Thank you.