Rock Mechanics and Tunneling
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Lecture 27
Analysis of Stresses

Hello everyone! I welcome all of you to the 1st lecture of Module 6.

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In module 6, we will discuss about the rock and rock mass failure criteria. So, let us see what we will discuss today.

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Today we will discuss about the stress analysis. Then Normal Stress and Shear Stress, then Stress Tensor, and Transformation of Stresses.

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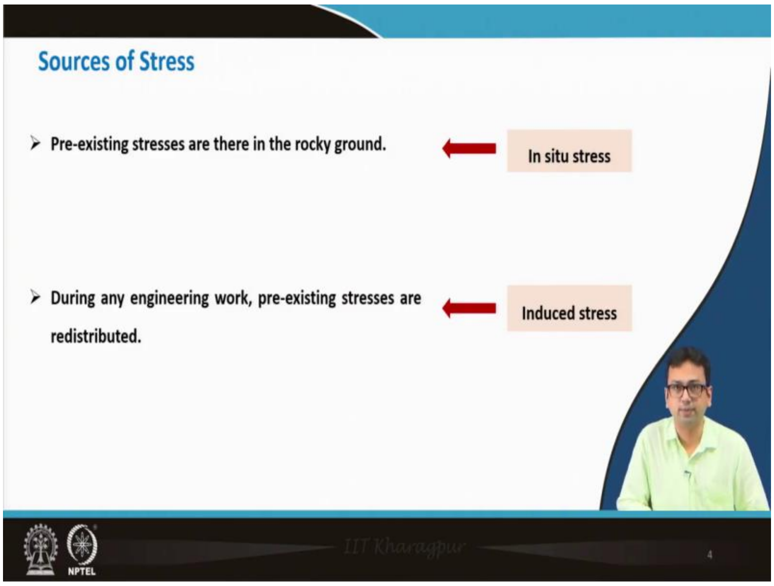

The pre-existing stresses are there in the rocky ground, which is mainly the in-situ stress. Other than that, during any engineering work, pre-existing stresses are redistributed. As a result, the induced stresses come into the picture.

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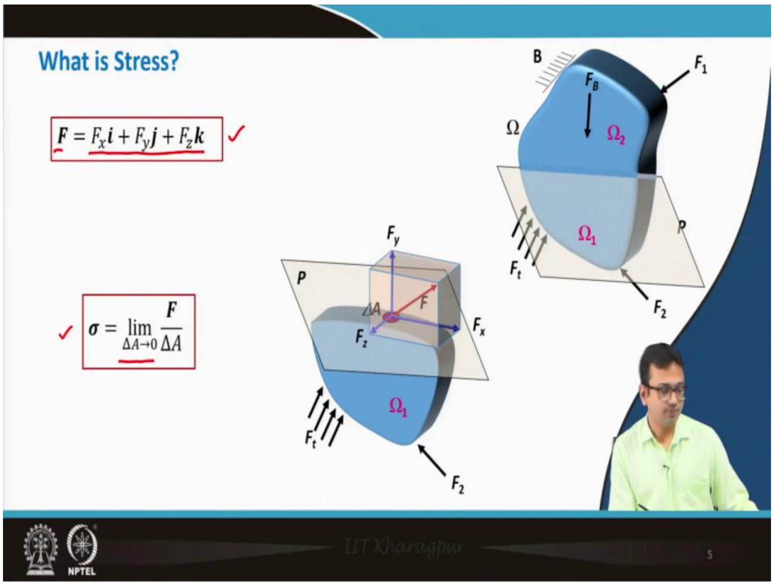

Now what is stress? Let us consider a three-dimensional body and it is constrained at B.

Now these are the different forces, like traction (F_t) is acting like this, then the body forces (F_b) are like these, then the point loads are acting like these, F_1 and F_2. If these forces act over this body, the stresses will develop. Now let us take an arbitrary plane P, which is actually bisecting this body into two parts like Ω1 and Ω2. Now, if we only consider the Ω1 part then and if we consider an infinitesimally small area, then let us consider the resultant force (F).

Now, if we have this XYZ coordinate system then we can resolve this force, F into three components like F_x, F_y, and F_z. So now and ΔA is the small area as I have stated. So, we can represent the force vector as $F = F_x i + F_y k + F_z k$. Here, i, j, and k are the unit vectors, i.e., i is for x-direction, j is for y-direction, and k is for z-direction.

So, this is the representation of force. Now, what is stress? The stress is generally represented by the symbol σ, which can be expressed as $\sigma = \lim_{\Delta A \to 0} \frac{F}{\Delta A}$

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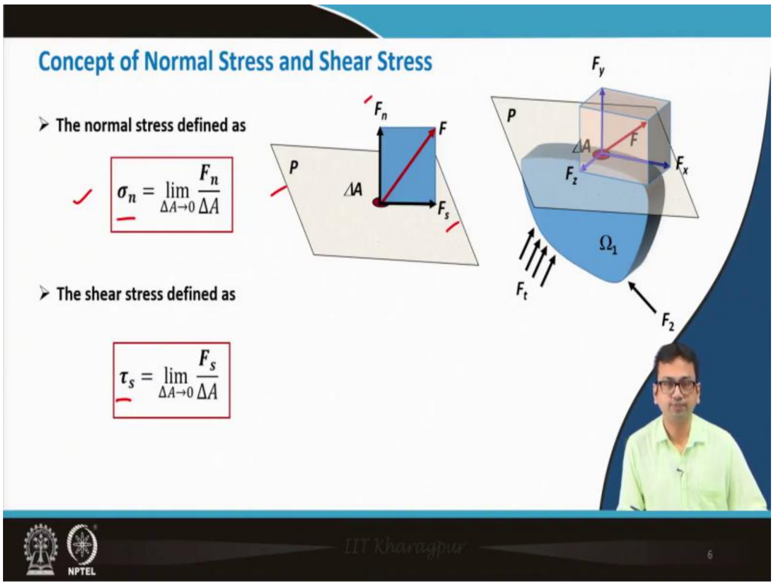

So, form the figure, we can see that this is the P plane and ΔA is the small area, and F is the resultant force. Now, it is resolved into two components, one is the normal component, F_n (in the normal direction), i.e., normal to this ΔA or normal to this plane P and the other one is the shear component, F_s.

If F_n is the normal component of this force vector, F then $\sigma_n = \lim_{\Delta A \to 0} \frac{F_n}{\Delta A}$ and similarly the shear stress can be defined as $\tau_n = \lim_{\Delta A \to 0} \frac{F_s}{\Delta A}$.

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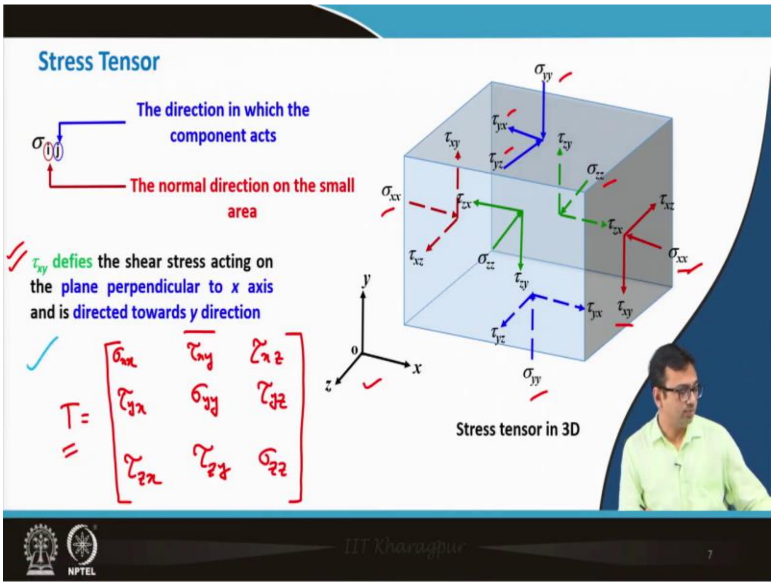

Now, let us learn about the stress tensor. So, let us see the small 3D element. What are the stresses will act in the element? If this is the coordinate system, we will have one normal stress and two shear stresses, σ_{xx}, τ_{xz}, and τ_{xy}. Now, here you can see the suffix xx or xy. What do they represent? let us try to understand.

So here σ_{ij}, i is the first suffix and j is the second suffix; i is basically the normal direction on the small area (ΔA), over which the force is acting. So, i indicates the normal direction on the small area and j denoted the direction in which the stress component is acting.

Now, i is the normal direction on the small area and j is the direction in which component acts. Now accordingly let us complete all the other stress components acting on the faces of the element. In the face normal to the y-axis, the stress components are σ_{yy}, τ_{yx}, and τ_{yz}. Likewise, in the face normal to the z-axis, the stress components are σ_{zz}, τ_{zy}, and τ_{zx}. Here, they are the σ_{zz}, τ_{zy}, and τ_{zx}, equal and opposite stresses, in the hidden faces.

So, as we can clearly notice that the shear stresses are also acting in the opposite directions.

The τ_{xy} defines the shear stress acting on the plane perpendicular to x-axis and is directed towards y direction. So, the first x is the plane perpendicular to x-axis and y is the direction towards which the stress is acting.

Now, what is the stress tensor? The tensor is a quantity which has the magnitude, direction, and a plane under consideration. That is what I have stated that a tensor is a quantity with magnitude, direction, and a plane under consideration.

Now we can write all these stresses in the matrix form given as follows:

$$T = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

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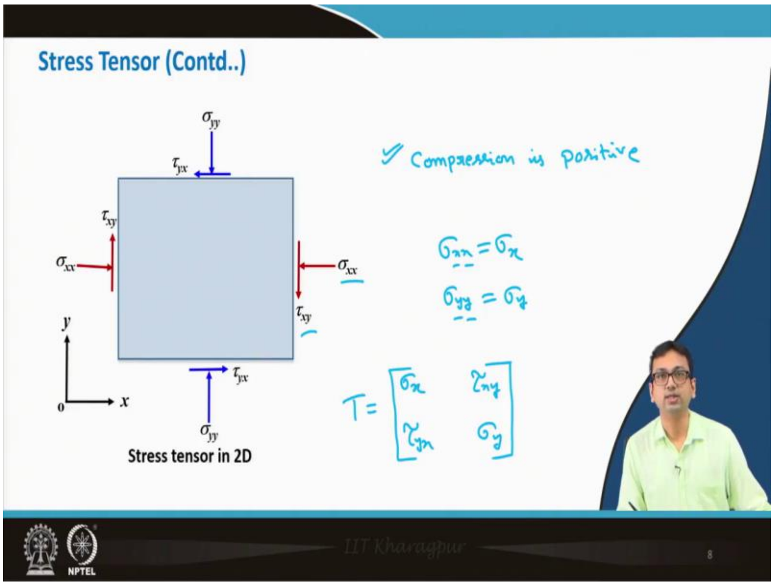

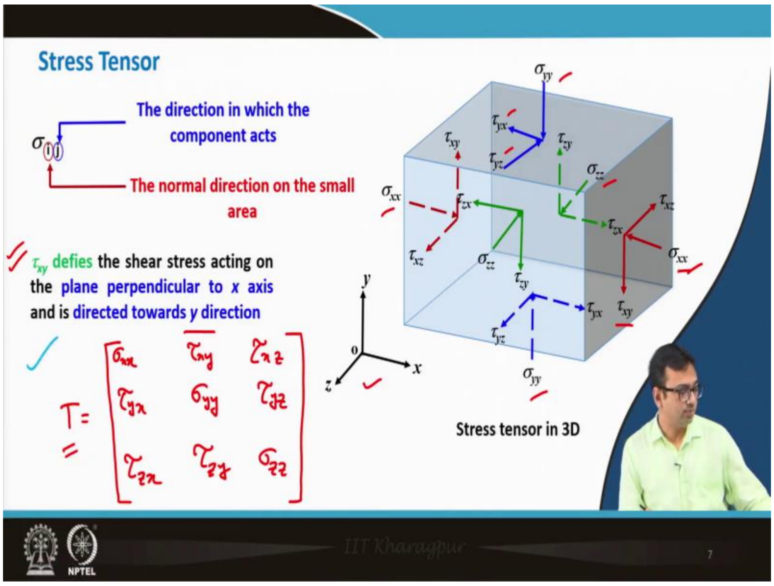

For our convenience, we generally analyze the problem as a plane strain problem, i.e., in 2D plane.

What are the stress components that will come for the 2D element? They will be σ_{xx} and τ_{xy}. So, here why is it *xx*? Because, the face or the plane on which the stress is acting, is perpendicular to the *x* direction and the next *x* is for the direction in which the stress is acting.

So, we can see that the normal stresses are compressive in nature. So, in this discussion, we are considering the compression as the positive stress convention.

So, this is the σ_{xx} which is the normal stress, both the plane as well as the direction are *x*. Generally, what we can do instead of writing it like σ_{xx}, we can write only σ_x. Likewise, instead of σ_{xx}, we can generally write only σ_y, because the y is repeating twice because first one is for plane and the second one is for direction, both are y.

But in the case of shear stresses, xy it is important for tau τ_{xy} as the first suffix x is related to the plane and the second suffix y is related to its direction. So, when we solve a problem in 2D, then the matrix reduces drastically. In the case of 3D element, the stress matrix was a 3×3 matrix. So, there were nine stress components.

Now, if we only deal with the 2D problem, $T = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix}$. It will be a 2×2 matrix having four stress components.

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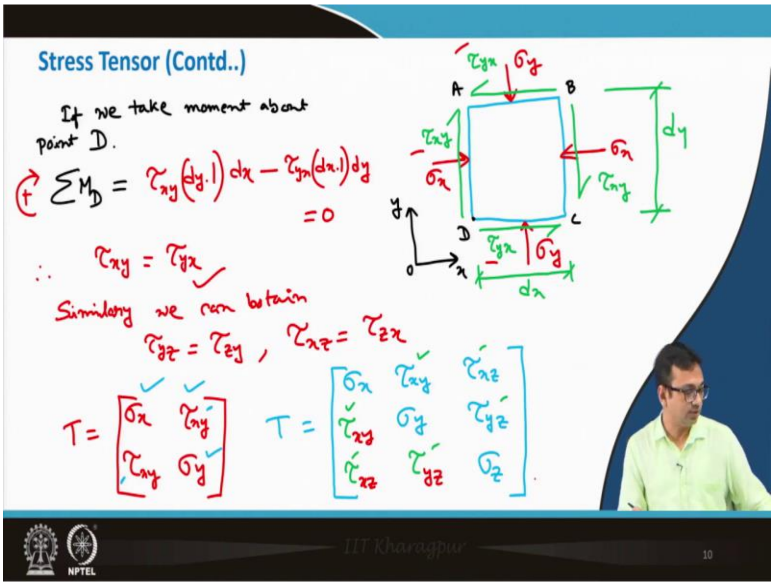

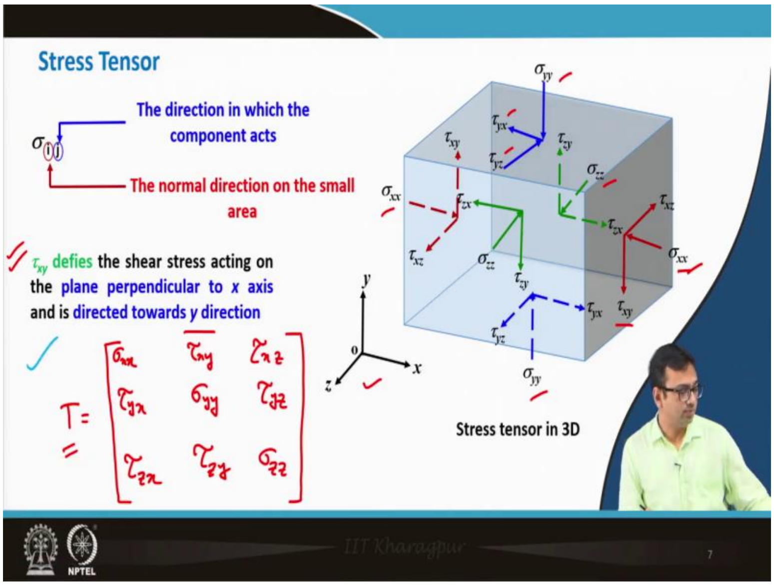

Now, let us draw a 2D element. So, here they are σ_x, σ_x, σ_y, σ_y. So, this one is τ_{xy}, whereas this one is τ_{yx}. Similarly, this one is τ_{yx} and this one is τ_{xy}. We know that the first suffix represents the plane and the second suffix represents the direction.

Now, let us consider the dimensions which are dx in x-direction and dy in y-direction. This is the xy-coordinate system and let us give some name of the elements like $ABCD$. Now, this σ_x, σ_y, τ_{xy}, and τ_{yx} are acting like this, i.e., σ_x is acting on BC face which is equal to the σ_x, same amount of stress acting on the AD face.

Here, the force equilibrium is getting satisfied as the acting forces are equal and opposite. Now, if we take moment about point D, $\sum M_d = 0$. so what will happen? If we consider τ_{xy}, this side is dy and if we consider in out of plane direction, the length or width is 1.

The total force is $\tau_{xy}.(dy \times 1)$. Now, to take moment about D, we need to be multiply the force with dx. Here, we are considering the clockwise moment which is considered to be positive and now what we can see that τ_{yx} component acting on AD face and τ_{xy} component acting on CD face are passing through D. Hence, they will have no contribution.

Now, the force $\tau_{yx}.(dx \times 1)$ is creating the anticlockwise moment.

We can write, $\tau_{xy}.(dy \times 1) \times dx - \tau_{yx}.(dx \times 1) \times dy = 0$. So, finally we can get, $\tau_{xy} = \tau_{yx}$.

Thus, we can get, $\tau_{yz} = \tau_{zy}$ and $\tau_{zx} = \tau_{xz}$ from yz and zx planes, respectively.

Since $\tau_{xy} = \tau_{yx}$ for the 2D element, we need to deal with only these three stress components.

Thus, $T = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ & \sigma_{yy} \end{bmatrix}$

Similarly, $\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$, and $\tau_{zx} = \tau_{xz}$ for the 3D element. So, we need to deal with only these six stress components for the 3D element.

Thus, $T = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ & \sigma_{yy} & \tau_{yz} \\ & & \sigma_{zz} \end{bmatrix}$

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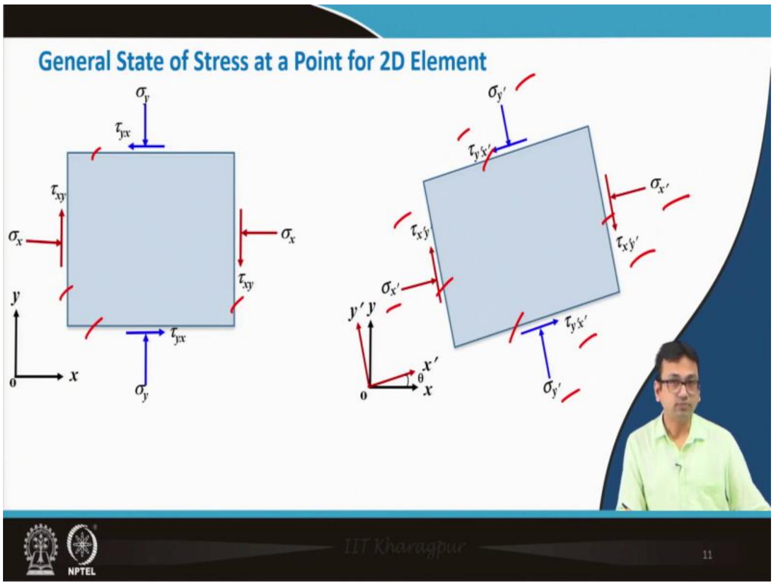

Now, let us see the general state of stress at a point for a 2D element. This side and this side are parallel to *x*-axis and this and this sides are parallel to *y*-axis. Now, the coordinate system is rotated by an angle θ. Thus, this side is making an angle θ with the *x*-axis, similarly, this side is also making an angle θ with the *x*-axis and these sides are making also θ with the *y*-axis.

So, now we will consider a new coordinate system, i.e., x'-y' coordinate system. So, accordingly this is $\sigma_{x'}$ and this is also $\sigma_{x'}$, equal and opposite, likewise $\tau_{x'y'}$, $\tau_{x'y'}$. Now, $\sigma_{y'}$, $\sigma_{y'}$, similarly, $\tau_{y'x'}$, $\tau_{y'x'}$.

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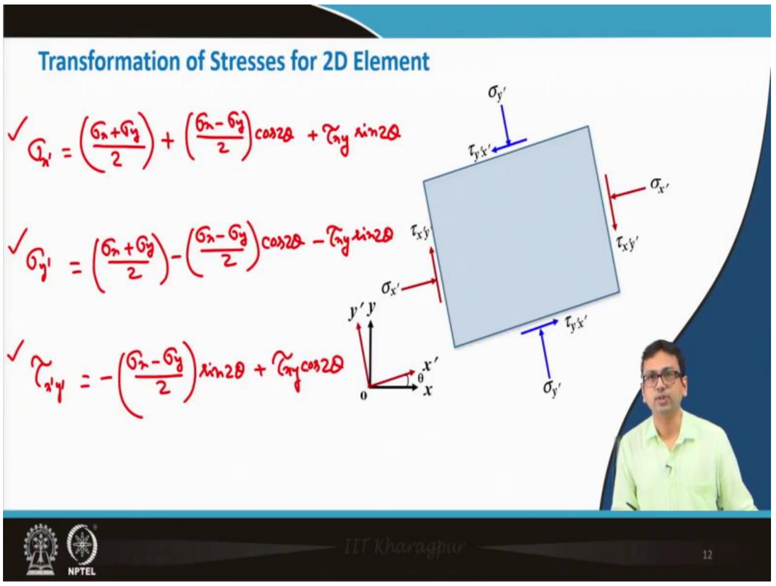

What are the expressions of $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ in terms of the known stress components like σ_x, , σ_y, and τ_{yx}? It can be obtained by doing a simple small derivation.

I am not going into that derivation, as we can derive it from the first year mechanics. However, it is a very derivation. Here, I am writing the expressions only as this transformation of stresses will be quite useful in our future classes.

So, the expressions for $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ are as follows:

$$\sigma_{x'} = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma_{y'} = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy} \cos 2\theta$$

So, these three are the expressions which we can derive very easily but anyway here I have directly written the expressions due to the time constraint.

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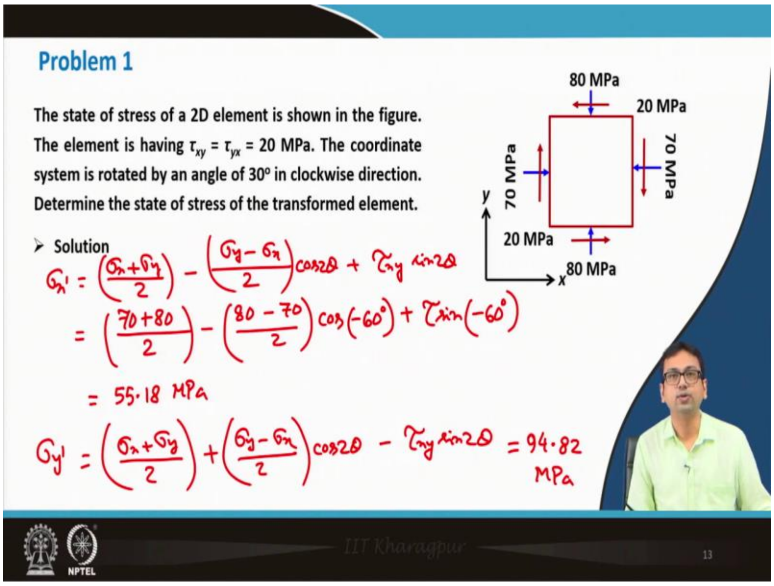

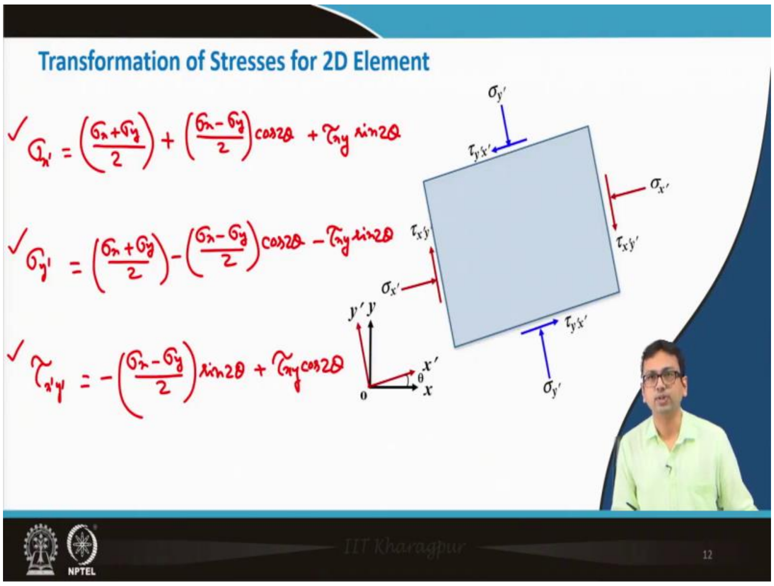

Now, let us solve a small problem with that will conclude our today's class. So, it states that the state of stress of a 2D element is shown in this figure. The element is having τ_{xy} or τ_{yx} is equal to 20 MPa. The coordinate system is rotated by an angle of 30° in clockwise direction. Determine the state of stress of the transformed element.

So, as per the problem, the rotation is 30° in clockwise direction. But, the expressions are given based on the anti-clockwise rotation. So, in this problem, the θ value will be negative. So let us solve it.

So, first $\sigma_{x'} = \left(\dfrac{\sigma_x + \sigma_y}{2}\right) + \left(\dfrac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$

which can be rewritten as $\sigma_{x'} = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \left(\frac{\sigma_y - \sigma_x}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$

As per the problem, σ_x = 70 MPa, σ_y = 80 MPa, τ_{xy} = 20 MPa, and θ = -30°.

So, $\sigma_{x'} = \left(\frac{70+80}{2}\right) - \left(\frac{80-70}{2}\right)\cos[2\times(-30^o)] + 20\times\sin[2\times(-30^o)]$

$$\sigma_{x'} = \left(\frac{70+80}{2}\right) - \left(\frac{80-70}{2}\right)\cos(-60^o) + 20\times\sin(-60^o)$$

$$\sigma_{x'} = 55.18 \text{ MPa}$$

Similarly, $\sigma_{y'} = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_y - \sigma_x}{2}\right)\cos 2\theta - \tau_{xy}\sin 2\theta$

$$\sigma_{y'} = \left(\frac{70+80}{2}\right) + \left(\frac{80-70}{2}\right)\cos[2\times(-30^o)] - 20\times\sin[2\times(-30^o)]$$

$$\sigma_{y'} = \left(\frac{70+80}{2}\right) + \left(\frac{80-70}{2}\right)\cos(-60^o) - 20\times\sin(-60^o)$$

$$\sigma_{y'} = 94.82 \text{ MPa}$$

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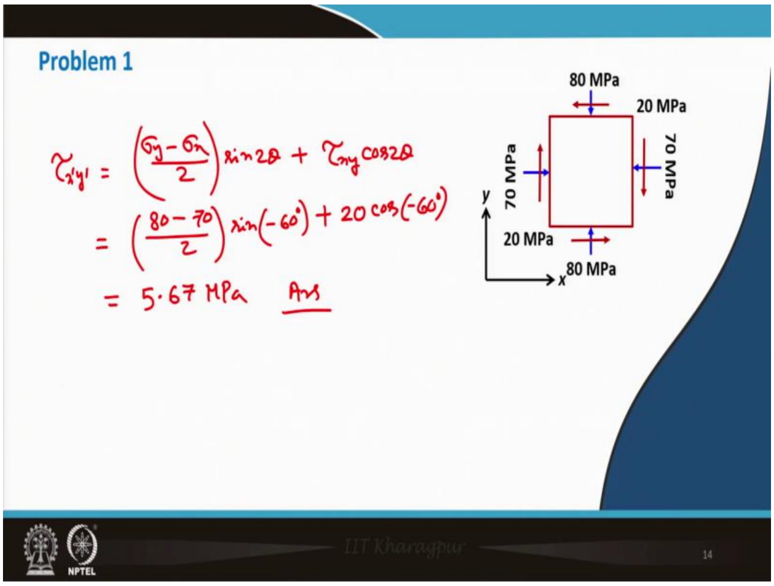

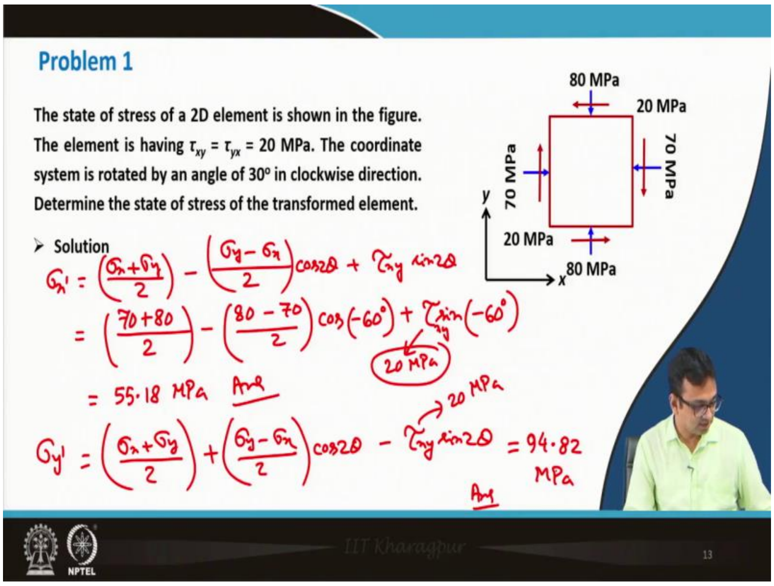

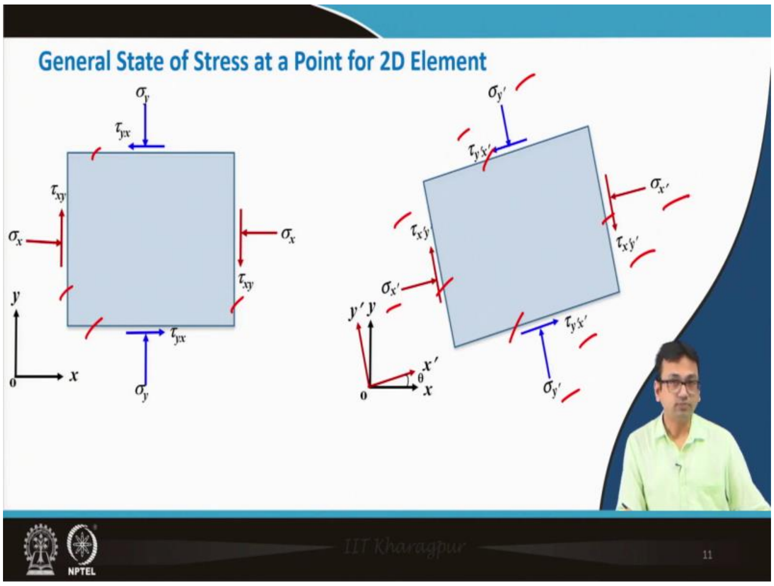

$$\text{Finally, } \tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\text{i.e., } \tau_{x'y'} = \left(\frac{\sigma_y - \sigma_x}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\sigma_{y'} = \left(\frac{80-70}{2}\right)\sin[2\times(-30^o)] + 20\times\cos[2\times(-30^o)]$$

$$\tau_{x'y'} = \left(\frac{80-70}{2}\right)\sin(-60^o) + 20\times\cos(-60^o)$$

$$\tau_{x'y'} = 5.67 \text{ MPa}$$

So, $\sigma_{x'} = 55.18$ MPa, $\sigma_{y'} = 94.82$ MPa, and $\tau_{x'y'} = 5.67$ MPa.

So let us stop here, we will continue our discussion with this stress transformation and we will also discuss about the principal stresses in the next lecture. So thank you.