

Structural Reliability
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Lecture –64
Joint Probability Distributions (Part - 15)

The joint normal distribution the joint normal is the most commonly used joint probability description in almost every branch of applied prop stats and one of the main reasons of its popularity is the fact that this joint density is completely specified by the only the first two moments the mean vector and the covariance matrix. And it is also not a coincidence that if we have real observed data on a bunch of dependent random variables.

It is the first two moments the means the variances and the correlation coefficients are the statistics that we can most reliably get from the data. So, this is the only named joint distribution that we will study in this lecture. Other joint distributions in many cases can be derived from or be approximated by the joint normal. So, let us start with the bivariate normal.

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Jointly distributed random variables

Bivariate normal

Recall that X is said to have a normal distribution with mean μ and variance $\sigma^2 > 0$ if its density function is of the form:

$$N(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}Q(x; \mu, \sigma^2)\right]$$

where, $Q(x; \mu, \sigma^2) = \frac{1}{\sigma^2}(x - \mu)^2 = (x - \mu)\sigma^{-2}(x - \mu)$. It is related to the standard normal form ϕ through:

$$N(x; \mu, \sigma^2) = \frac{1}{\sigma} \phi\left(\frac{x - \mu}{\sigma}\right).$$

In a parallel manner, a two dimensional RV $\underline{X} = (X_1, X_2)^T$ is said to have a non-singular bivariate normal distribution if its density function is of the form:

$$N_2(x; \mu, V) = \frac{1}{2\pi|V|^{1/2}} \exp\left[-\frac{1}{2}Q_2(x; \mu, V)\right]$$


where,

$$Q_2(x; \mu, V) = (x - \mu)^T V^{-1} (x - \mu)$$

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad V = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

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In expanded form:

$$N_2(x_1, x_2; \mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)}{\sigma_1} - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

Its standard form (0 means and unit variances) is:

$$\phi_2(x_1, x_2; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}(x_1^2 - 2\rho x_1 x_2 + x_2^2)\right\}$$

The bivariate normal distribution is singular if:
 $|V| = 0$, i.e., if $\rho^2 = 1$

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Let us recall that the random variable X is said to have a normal distribution with two parameters the mean μ and the variance σ^2 if its density function is given as you see on the screen and this reason that we have Q_1 in the exponent because we are going to generalize that

the form of Q1 we all remember it is the inverse of the variance pre-multiplied and post-multiplied by the deviation of the value of the of the normal random variable from its mean.

And you can also relate the arbitrary normal density with the standard normal density function which is phi. Now in a parallel manner we can define the bivariate normal where we introduce Q2. So, x_1 and x_2 have the single they have the non-singular bivariate normal distribution if their joint density looks like what you see on the screen and Q 2 is written parallelly as we did for Q1 is the inverse of the covariance matrix pre-multiplied and post-multiplied by x minus the mean vector.

And note also that there is the square root of the determinant of the covariance matrix in the denominator. So, we can expand this in terms of the 5 parameters that govern the density function the two means the two standard deviations and the correlation coefficient. In standard form the means are 0 the variances are 1 and it is governed by the by the single parameter rho.

The bivariate normal distribution is said to be singular if if the determinant of the covariance matrix is 0 or in other words if the correlation coefficient is either plus or minus 1.

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Jointly distributed random variables

Bivariate normal

Conditional and Marginal distributions

If $f_{X_1, X_2}(x_1, x_2) = N(x; \mu, \Sigma)$, then the marginal distribution of X_1 , regardless of the correlation coefficient, is univariate normal:

$$f_{X_1}(x_1) = N(x_1; \mu_1, \sigma_1^2)$$

Likewise, for X_2 :

Say, X_1, X_2 are standard bivariate normal.

The marginal density of X_1 :

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1, X_2}(x_1, x_2) dx_2$$

Expressing $x_2^2 - 2\rho x_1 x_2 + x_1^2(1-\rho^2) + (x_1 - \rho x_2)^2$ in the standard bivariate density function, we get:

$$f_{X_1}(x_1) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x_2 - \rho x_1)^2}{2(1-\rho^2)}\right\} \exp\left\{-\frac{x_1^2}{2}\right\} dx_2$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x_1^2}{2}\right\} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x_2 - \rho x_1)^2}{2(1-\rho^2)}\right\} dx_2$$

Substituting $x_2 = \frac{x_2 - \rho x_1}{\sqrt{1-\rho^2}}$, we get:

$$f_{X_1}(x_1) = \frac{1}{2\pi} \exp\left\{-\frac{x_1^2}{2}\right\} \sqrt{2\pi} = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x_1^2}{2}\right\} = N(0, 1)$$

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If X is bivariate normal, then the conditional density of X_1 given a fixed value of $X_2 = b$ is, once again, normal.

Starting with the definition of the conditional density function, and using the above result that the marginal density of X_1 is normal:

$$f_{X_1|X_2=b}(x_1) = \frac{f_{X_1, X_2}(x_1, b)}{f_{X_2}(b)}$$


$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \frac{\exp\left\{-\frac{x_1^2 - 2\rho x_1 b + b^2}{2(1-\rho^2)}\right\}}{\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{b^2}{2}\right\}}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left\{-\frac{(x_1 - \rho b)^2}{2(1-\rho^2)}\right\} = N(\rho b, 1-\rho^2)$$

Generalizing, for any arbitrary bivariate normal, given $X_2 = x_2$, the conditional density of X_1 is normal:

with conditional mean: $\mu_{1|2} = \mu_1 + \rho \frac{\sigma_1}{\sigma_2}(x_2 - \mu_2)$

and conditional variance: $\sigma_{1|2}^2 = \sigma_1^2(1-\rho^2)$



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Let us just look at the conditional and marginal distributions from the bivariate normal. So, if we

have the bivariate normal N_2 given by it is the mean vector and the covariance matrix. Then the marginal distribution of either one X_1 or X_2 and regardless of the correlation coefficient between them is univariate normal. So, f_{X_1} is normal $\mu_1 \sigma_1^2$ and likewise f_{X_2} is normal $\mu_2 \sigma_2^2$.

Unfortunately and here I use the bouncing animation the converse is not necessarily true. So, if you have two marginal normal's if X_1 and X_2 are normal and variables and they have a certain degree of dependence between them it is not necessary that their joint density function is going to be bivariate normal. And there are several quite famous examples of that and they can be looked up from standard text.

And it is quite straightforward to show that from the bivariate normal we get the marginal density as the normal if we just integrate out one of the two random variables. So, here we look at the marginal of X_1 and we integrate the joint density in terms of X_2 and if we do the right substitutions and if we go through the steps then we are going to end up with the normal density function for X_1 .

And here we started with the standard bivariate normal and we ended with the standard univariate normal the same logic can be taken for the conditional density function. So, if x is bivariate normal we already know that the marginal density of either one is univariate normal. So, if x is bivariate normal and we fix the value of say x_2 then the conditional density of x_1 is again normal.

And that can simply be achieved by dividing the joint normal expression by the marginal density of x_2 evaluated at the particular value b . If we go through the steps it is interesting to see that the conditional density of x_1 is normal with a mean that depends on b the particular value that X_2 has taken and a variance that depends on the correlation coefficient it does not depend on b and it is also to be noted that $1 - \rho^2$ the conditional variance is less than the original variance of one.

And if we generalize this for any bivariate normal where say X_2 takes on the value x_2 then the conditional density of X_1 is normal with the conditional mean which is given in terms of the original mean plus a correction factor which depends both on the deviation of X_2 from its own mean and the correlation coefficient between the two random variables. The conditional variance depends on the original variance and the correlation coefficient.

And it is easy to see that the conditional variance is less than the original variance and it does not depend on what particular value the other one X_2 takes.

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Jointly distributed random variables

Multivariate normal

Definitions

An n -dimensional random variable X with mean μ and cov. matrix V is said to have a nonsingular multivariate normal distribution if V is positive definite and the joint PDF of X is :

$$f_X(x; \mu, V) = \frac{1}{(2\pi)^{n/2} |V|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T V^{-1}(x-\mu)\right), |V| > 0$$

which is symbolically written as $X \sim N_n(\mu, V)$.

The normal family is closed under linear combinations:

If $X \sim N_n(\mu, V)$

and $Y = CX + b$ where $C = [c_{ij}]_{m \times n}$, $b = \{b_i\}_m$

then $Y \sim N_m(C\mu + b, CV^T C)$

In particular, if we have n IID standard normals,
 $Z \sim N_n(0, I_n)$
then, $Y = C_{(m \times n)} Z + \mu_{(m)}$ with $|C| \neq 0$
is n -dimensional normal: $Y \sim N_m(\mu, CV^T C)$

Useful for obtaining multivariate normal with desired mean and covariance from IID standard normals

Conditional and Marginal distributions

Let the n -dimensional vector of multivariate random variables $\{X\}$ be partitioned into k -dimensional $\{X_1\}$ and $(n-k)$ -dimensional $\{X_2\}$:

$$\{X\}_{(n)} = \begin{Bmatrix} X_{(k)} \\ X_{(n-k)} \end{Bmatrix}$$

The mean vector and covariance matrix is accordingly partitioned:

$$\{\mu\}_{(n)} = \begin{Bmatrix} \mu_{(k)} \\ \mu_{(n-k)} \end{Bmatrix}$$


$$V_{(n \times n)} = \begin{bmatrix} V_{(k \times k)} & V_{(k \times (n-k))} \\ V_{((n-k) \times k)} & V_{((n-k) \times (n-k))} \end{bmatrix}$$

Then:

$X_1 \sim N_k(\mu_1, V_{11})$ X_1 and X_2 are independent
 $X_2 \sim N_{n-k}(\mu_2, V_{22})$ if and only if $V_{12} = 0$

The conditional distribution of X_1 , given $X_2 = x_2$, is
 k -dimensional joint normal: $N_k(\mu_{1|2}, V_{11|2})$
where $\mu_{1|2} = \mu_1 + V_{12} V_{22}^{-1}(x_2 - \mu_2)$
and $V_{11|2} = V_{11} - V_{12} V_{22}^{-1} V_{21}$

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Now let us move on to the multivariate normal distribution generalizing what we saw for the bivariate normal. So, we have the mean vector μ the covariance matrix V and the n -dimensional random variable X has the multivariate normal distribution the non-singular multiplied normal distribution. If the covariance matrix is positive definite and the joint PDF is of the form as we have seen before.

So, here we have $Q = n \times V^{-1}$ is multiplied pre and post multiplied by x minus mean vector. And note that we have the determinant square root of the determinant in the denominator and 2π raised to the power of n over 2. And it is often symbolically written as N subscript n

very important result is that the normal family is closed under linear combination. So, if I start with this random vector x and combine them in with the help of coefficients C and add another b vector as you see here.

The new random vector that we get y which in general is m dimensional is also not and with the mean vector as you see on the screen and the covariance matrix as also you see on the screen. So the mean of μ is multiplied with c and that b vector is added to obtain the mean of y and the covariance matrix of x is pre-multiplied and post-multiplied by c . So, in particular and this is very useful for the purposes of Monte Carlo simulations which we will see soon.

Is if we start with IID standard normal's. So, independent and identically distributed standard normal's which means that each of the sets z_1 up to z_n has a zero mean and unit standard deviation and no correlation between any pair. So, the variance, so, the covariance matrix of the of the z s is the identity matrix. So, we can multiply the z 's with the the c matrix the square metric c and add to that any vector μ then what we get is an n dimensional normal y whose mean vector is that vector that you added and whose covariance matrix is $C C^T$.

So, this actually points to how we can generate dependent normal's from independent standard normal's when we do Monte Carlo simulations and we will see that. So, let us complete this discussion with the conditional marginal's as we did for the bivariate case. So, let us let us partition the x vector the n -dimensional x -vector which is multivariate normal into X_1 and X_2 of dimensions k and $n - k$ respectively.

And then the mean vector and the covariance matrix is also partitioned in a similar manner. So, you have μ_1 and μ_2 and V_{11} V_{22} in the diagonal and we want to invert V_{22} in the of diagonal locations. So, if we partition it that way then each of these partitions X_1 and X_2 are individually jointly normal. So, x_1 is n_k and x_2 is $n - k$ with the mean vector μ_1 and μ_2 respectively and the covariance matrix V_{11} and V_{22} .

Now X_1 and X_2 these two partitions would be independent of each other if and only if V_{12} all the elements are zero. The conditional distribution of X_1 given X_2 would again be k -dimensional joint normal and as we saw in the bivariate case the conditional mean vector is the original mean vector plus a correction term which depends on how far X_2 is from its mean and conditional covariance matrix which is the original covariance matrix v_{11} and corrected by a term that you see on the screen. In the next few slides we are going to look up we are going to look at some examples.