# Structural Reliability
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## Lecture –64
## Joint Probability Distributions (Part - 15)

The joint normal distribution the joint normal is the most commonly used joint probability description in almost every branch of applied prop stats and one of the main reasons of its popularity is the fact that this joint density is completely specified by the only the first two moments the mean vector and the covariance matrix. And it is also not a coincidence that if we have real observed data on a bunch of dependent random variables.

It is the first two moments the means the variances and the correlation coefficients are the statistics that we can most reliably get from the data. So, this is the only named joint distribution that we will study in this lecture. Other joint distributions in many cases can be derived from or be approximated by the joint normal. So, let us start with the bivariate normal.

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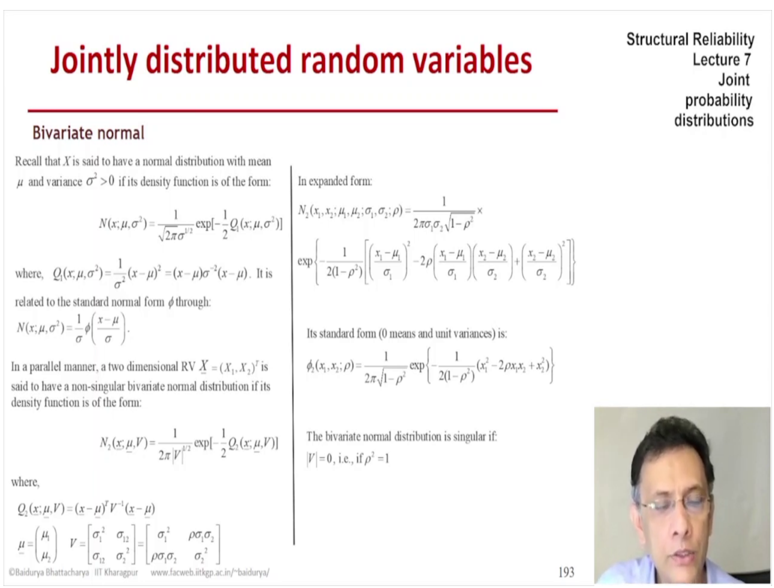


Let us recall that the random variable X is said to have a normal distribution with two parameters the mean mu and the variance sigma squared if its density function is given as you see on the screen and this reason that we have Q1 in the exponent because we are going to generalize that the form of Q1 we all remember it is the inverse of the variance pre-multiplied and post-multiplied by the deviation of the value of the of the normal random variable from its mean.

And you can also relate the arbitrary normal density with the standard normal density function which is phi. Now in a parallel manner we can define the bivariate terminal where we introduce Q2. So, x1 and x2 have the single they have the non-singular bivariate normal distribution if their joint density looks like what you see on the screen and Q 2 is written parallelly as we did for Q1 is the inverse of the covariance matrix pre-multiplied and post-multiplied by x minus the mean vector.

And note also that there is the square root of the determinant of the covariance matrix in the denominator. So, we can expand this in terms of the 5 parameters that govern the density function the two means the two standard deviations and the correlation coefficient. In standard form the means are 0 the variances are 1 and it is governed by the by the single parameter rho.

The bivariate normal distribution is said to be singular if if the determinant of the covariance matrix is 0 or in other words if the correlation coefficient is either plus or minus 1.

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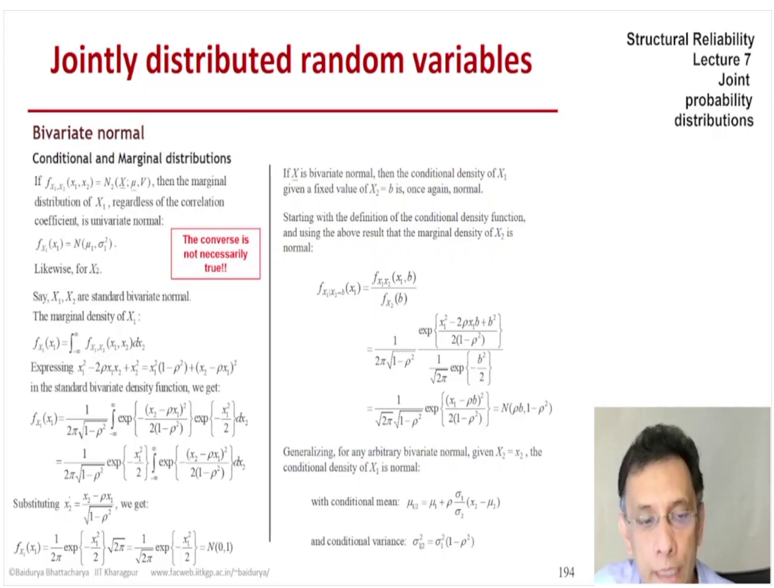


Let us just look at the conditional and marginal distributions from the bivariate normal. So, if we have the bivariate normal N 2 given by it is the mean vector and the covariance matrix. Then the marginal distribution of either one X 1 or X 2 and regardless of the correlation coefficient between them is univariate not. So, f x 1 is normal mu 1 sigma 1 squared and likewise f x 2 is normal mu 2 sigma 2 squared.

Unfortunately and here I use the bouncing animation the converse is not necessarily true. So, if you have two marginal normal's if X 1 and X 2 are normal and variables and they have a certain degree of dependence between them it is not necessary that their joint density function is going to be bivariate normal. And there are several quite famous examples of that and they can be looked up from standard text.

And it is it is quite straightforward to show that the from the bivariate normal we get the marginal density as the normal if we just integrate out one of the two random variables. So, here we look at the marginal of X 1 and we integrate the joint density in terms of X 2 and if we do the right substitutions and if we go through the steps then we are going to end up with the normal density function for X 1.

And here we started with the standard bivariate normal and we ended with the standard univariate number the same logic can be taken for the conditional density function. So, if x is bivariate normal we already know that the marginal density of either one is univariate normal. So, if x is by weight normal and we fix the value of say x 2 then the conditional density of x 1 is again normal.

And that can simply be achieved by dividing the joint normal expression by the by the marginal density of x t evaluated at the particular value b. If we go through the steps it is it is interesting to see that the conditional density of x 1 is normal with a mean that depends on b the particular value that X 2 has taken and a variance that depends on the correlation coefficient it does not depend on b and it is also to be noted that that 1 - rho squared the conditional variance is less than the original variance of one.

And if we generalize this for any bivariate normal where say X 2 takes on the value little x 2 then the conditional density of X 1 is normal with the conditional mean which is given in terms of the original mean plus a correction factor which depends both on the deviation of X 2 from its own mean and the correlation coefficient between the two random variables. The conditional variance depends on the original variance and the correlation coefficient.

And it is easy to see that the conditional variance is less than the original variance and it does not depend on what particular value the other one X 2 takes.

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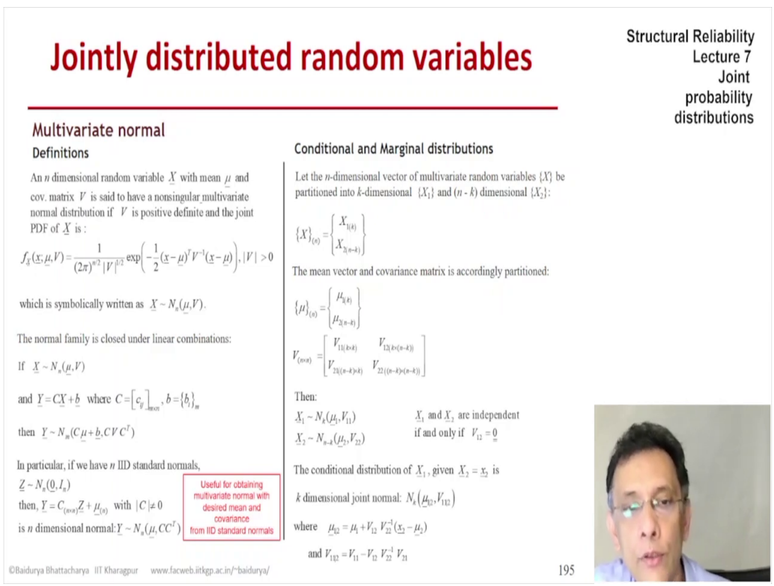


Now let us move on to the multivariate normal distribution generalizing what we saw for the bivariate normal. So, we have the mean vector mu the covariance matrix V and the n-dimensional random variable X has the multivariate normal distribution the non-singular multiplied normal distribution. If the covariance matrix is positive definite and the joint PDF is of the form as we have seen before.

So, here we have Q n V inverse is multiplied pre and post multiplied by x minus mean vector. And note that we have the determinant square root of the determinant in the denominator and 2 pi raised to the power of n over 2. And it is often symbolically written as N subscript little n a very important result is that the normal family is closed under linear combination. So, if I start with this random vector x and combine them in with the help of coefficients C and add another b vector as you see here.

The new random vector that we get y which in general is m dimensional is also not and with the mean vector as you see on the screen and the covariance matrix as also you see on the screen. So the mean of mu is multiplied with c and that b vector is added to obtain the mean of y and the covariance matrix of x is pre-multiplied and post-multiplied by c. So, in particular and this is very useful for the purposes of Monte Carlo simulations which we will see soon.

Is if we start with IID standard normal's. So, independent and identically distributed standard normal's which means that each of the sets z 1 up to z n has a zero mean and unit standard deviation and no correlation between any pair. So, the variance, so, the covariance matrix of the of the z s is the identity matrix. So, we can multiply the z's with the the c matrix the square metric c and add to that any vector mu then what we get is an n dimensional normal y whose mean vector is that vector that you added and whose covariance matrix is C C transpose.

So, this actually points to how we can generate dependent normal's from independent standard normal's when we do Monte Carlo simulations and we will see that. So, let us complete this discussion with the conditional marginal's as we did for the bivariate case. So, let us let us partition the x vector the n-dimensional x-vector which is multivariate normal into X 1 and X 2 of dimensions k and n - k respectively.

And then the mean vector and the covariance matrix is also partitioned in a similar manner. So, you have mu 1 and mu 2 and V 11 V 22 in the diagonal and we want to invert 2 1 in the of diagonal locations. So, if we partition it that way then each of these partitions X 1 and X 2 are individually jointly normal. So, x1 is nk and x2 is n minus k with the mean vector mu 1 and mu 2 respectively and the covariance matrix V 11 and V 2.

Now X 1 and X 2 these two partitions would be independent of each other if and only if V 1 to all the elements are zero. The conditional distribution of X 1 given X 2 would again be k-dimensional joint normal and as we saw in the bivariate case the conditional mean vector is the original mean vector plus a correction term which depends on how far X 2 is from its mean and conditional covariance matrix which is the original covariance matrix v11 and corrected by a term that you see on the screen. In the next few slides we are going to look up we are going to look at some examples.