Structural Reliability
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Lecture –56
Joint Probability Distributions (Part - 07)

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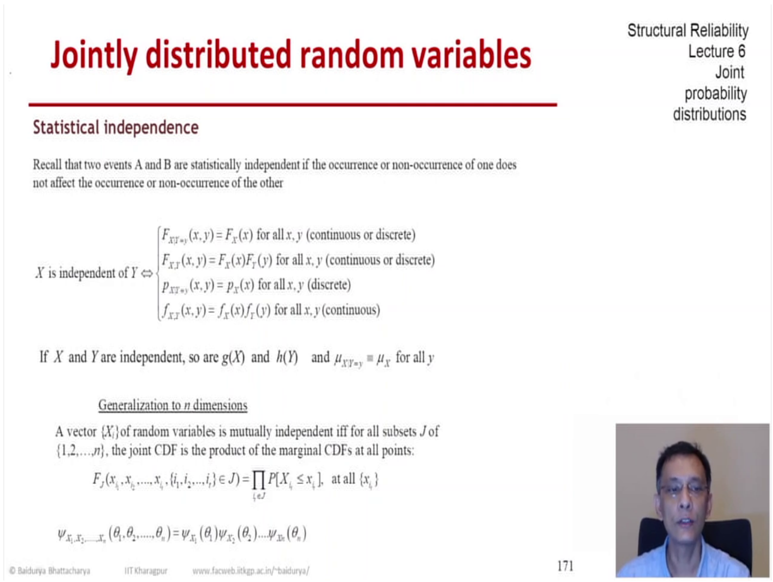

The independence of random variables recall when we are discussing probabilities of events 2 events were said to be statistically independent if the occurrence or non-occurrence of one did not affect the occurrence or non-occurrence of the other. So, we take the same concept to defining the independence of two random variables and ensure that happens at all possible pairs of points.

So, here you have 4 equivalent statements of independence between x and y and each of these is both necessary and sufficient for independence we can extend this idea to functions of random variables. So, if x and y are independent. So, our g of x and h of y g and h being a new function and like the conditional mean of x given any value of y would be the unconditional mean of x and so on.

$$X \text{ is independent of } Y \Leftrightarrow \begin{cases} F_{X|Y=y}(x,y) = F_X(x) \text{ for all } x, y \text{ (continuous or discrete)} \\ F_{X,Y}(x,y) = F_X(x)F_Y(y) \text{ for all } x, y \text{ (continuous or discrete)} \\ p_{X|Y=y}(x,y) = p_X(x) \text{ for all } x, y \text{ (discrete)} \\ f_{X,Y}(x,y) = f_X(x)f_Y(y) \text{ for all } x, y \text{ (continuous)} \end{cases}$$

Now just as we did for independence among 3 or more events we would do that for the mutual independence of 3 or more random variables. So, not only would the probability of a, b, c be equal to p a, p b, p c, we would also need to establish that for all subsets. So, p a b would also need to be equal to p a p b and so on. So, looking at that for the n dimensional random variables we have one expression for the joint CDF.

1.1.1.1 Generalization to *n* dimensions

A vector $\{X_i\}$ of random variables is mutually independent iff for all subsets J of $\{1,2,\ldots,n\}$, the joint CDF is the product of the marginal CDFs at all points:

$$F_J(x_{i_1}, x_{i_2}, \ldots, x_{i_t}, \{i_1, i_2, .., i_t\} \in J) = \prod_{i_t \in J} P[X_{i_t} \le x_{i_t}], \text{ at all } \{x_{i_t}\}$$

So, it must be the product of the individual CDF for all possible subsets of the index set and we could write this in terms of the joint density function in terms of the joint characteristic function and so, on.

$$\psi_{X_1, X_2, \ldots\ldots, X_n}(\theta_1, \theta_2, \ldots., \theta_n) = \psi_{X_1}(\theta_1)\psi_{X_2}(\theta_2)\ldots\psi_{Xn}(\theta_n)$$

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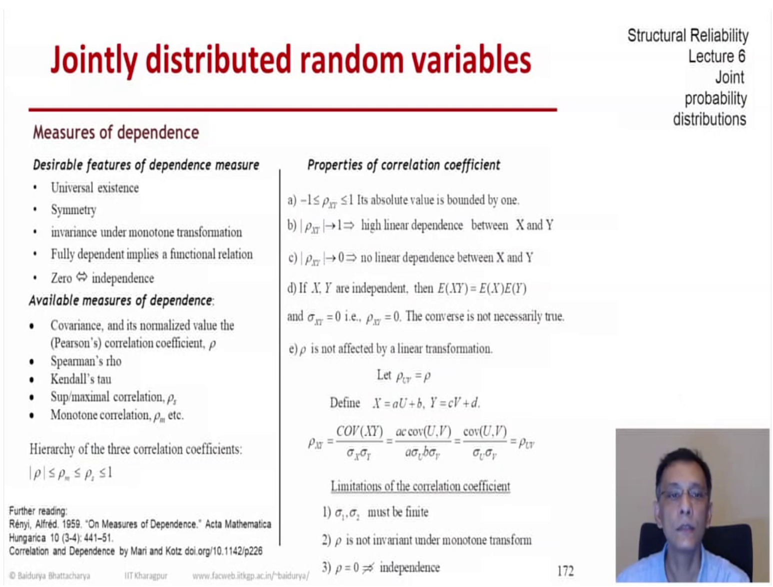

Now after discussing independence let us also talk about dependence and how to measure dependence. An ideal measure of dependence between two random variables should have a set of desirable properties I have listed a few here universal existence symmetry meaning the dependence measure between x and y should be the same as that between y and x if I know the dependence measure between x and y .

If I exponentiate both or one of them that should not change if my measure shows that they are fully dependent that should imply an underlying function relationship between x and y if the measure is zero then it should imply independence and vice versa. So, there are several measures of dependence available the first is covariance which we have already seen or covariance normalized by the product of the standard deviations.

The correlation coefficient or more completely Pearson's correlation coefficient there are others like Spearman's Rho, Kendall's Tau, the super maximal correlation, monotone correlation and so on. These 3 rho's rho, rho s and rho m have a hierarchical relationship which if x and y are jointly normal then these three are equal. Now the one correlation measure that is universally popular and which is what we are going to use in this course also is Pearson's correlation coefficient.

a) $-1 \le \rho_{XY} \le 1$ Its absolute value is bounded by one.

b) $|\rho_{XY}| \to 1 \Rightarrow$ high linear dependence between X and Y

c) $|\rho_{XY}| \to 0 \Rightarrow$ no linear dependence between X and Y

d) If X, Y are independent, then $E(XY) = E(X)E(Y)$

and $\sigma_{XY} = 0$ i.e., $\rho_{XY} = 0$. The converse is not necessarily true.

And we are going to discuss some of its pros and cons next but if you would like to know more I have listed a couple of resources for further reading. Now the sum of the properties of Pearson's correlation coefficient it's bounded between -1 and +1. If it is close to 1 in absolute value then it implies there is a high linear dependence between x and y. If it is close to 0 it means there is not much linear dependence between x and y.

Let $\rho_{UV} = \rho$

Define $X = aU + b,\ Y = cV + d.$

$$\rho_{XY} = \frac{COV(XY)}{\sigma_X \sigma_Y} = \frac{ac\,\mathrm{cov}(U,V)}{a\sigma_U b\sigma_V} = \frac{\mathrm{cov}(U,V)}{\sigma_U \sigma_V} = \rho_{UV}$$

If x and y are independent then the covariance is zero which means rho is zero unfortunately the converse is not necessarily true and there are many well-known examples of that. But on a positive note rho is not affected by a linear transformation and it is easy to show that if I know the row between U and V any linear transformation x from U and y from V would give me the same correlation coefficient.

Here are some of the more limitations that we discussed sigma 1 and sigma 2 need to be defined and need to be finite rho is not invariant under a molecular transformation. We if there is time we can go over one example and as I said already that rho correlation coefficient being 0 unfortunately does not imply independence although if the independent rho is 0 but rho equals 0 does not always imply independence.

1.1.1.1 Limitations of the correlation coefficient

1) σ_1, σ_2 must be finite

2) ρ is not invariant under monotone transform

3) $\rho = 0 \not\Rightarrow$ independence