

Structural Reliability
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Lecture –44
Common Probability Distributions (Part - 15)

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Common Continuous Distributions

Erlang distribution

Erlang: RV describing time to k^{th} arrival in a homogeneous Poisson process with rate λ .

$T_k = \tau_1 + \tau_2 + \dots + \tau_k$, $\tau_i \sim \varepsilon(\lambda)$, and τ_i, τ_j indep for $i \neq j$

The CDF is given by the complementary CDF of the Poisson RV:

$$P(T_k \leq t) = P(N(t) \geq k) = \sum_{m=k}^{\infty} \frac{e^{-\lambda t} (\lambda t)^m}{m!}$$

And the PDF by differentiating the CDF:

$$f_{T_k}(t) = \frac{1}{(k-1)!} \lambda e^{-\lambda t} (\lambda t)^{k-1}, t \geq 0$$

The first two moments of T_k as the sum of k IID RVs:

$$\mu = \frac{k}{\lambda}$$

$$\sigma^2 = \frac{k}{\lambda^2}$$

Gamma distribution

Generalization of Erlang when k is non integer. The density and distribution functions are:

$$f_X(x) = \frac{\lambda (\lambda x)^{k-1}}{\Gamma(k)} e^{-\lambda x}, x \geq 0$$

Integrating:


$$F_X(x) = \frac{1}{\Gamma(k)} \int_0^x \lambda e^{-\lambda t} (\lambda t)^{k-1} dt$$

$$= \frac{1}{\Gamma(k)} \int_0^x e^{-v} v^{k-1} dv, \lambda t = v$$

$\mu = \frac{k}{\lambda}, \sigma^2 = \frac{k}{\lambda^2}$

If $X_1 \sim \text{Gamma}(k_1, \lambda)$, $X_2 \sim \text{Gamma}(k_2, \lambda)$ and $X_1 \perp X_2$ then, $Y = X_1 + X_2$ is $\text{Gamma}(k_1 + k_2, \lambda)$

Structural Reliability
Lecture 5
Common probability distributions



The online distribution describes the time to the k^{th} arrival in a homogeneous Poisson process with rate lambda. So, just as we had in the case of the sequence of Bernoulli's trials. The number of trials to the k^{th} occurrence the negative binomial random variable being the sum of k independent and identical geometric random variables we have the Erlang random variable as a sum of k independent and identical exponentials with the same rate lambda obviously.

So, that would give me the CDF of the Erlang distributions the same way as we derived it for the negative binomial case from the Poisson CDF and differentiating that with respect to time we get the Erlang PDF. Going back to the first equation that you see on the screen the mean and the variance of T_k can be easily found as the mean of the sum of those tau's and the variance of the sum of those tau's as k over lambda and k over lambda squared respectively.

Now what if k is not an integer we can generalize that $k - 1$ factorial in the PDF of T_k and define the gamma distribution. So instead of $k - 1$ factorial we have gamma k in the denominator and

we can integrate that density function to get the CDF which again involves the gamma function and it has the same mean and variance as in the case of the Erlang with the same k and λ .

One interesting property that would come to be useful in some cases is that if you sum two independent gammas with the same λ then their sum is again gamma with the same λ and the k are given in terms of $k_1 + k_2$.