

**Structural Reliability**  
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**Lecture –38**  
**Common Probability Distributions (Part - 09)**

(Refer Slide Time: 00:28)

### Common discrete distributions

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**Example: darts**

You and your friend go into a sports bar where a dart throwing competition is going on. You buy  $m$  darts at 1 rupee each and throw them at the board. Of these,  $N$  darts hit within the inner circle. Your friend picks up these  $N$  darts, and throws them at the board.  $X$  of them hit the inner circle. You and your friend earn 10 rupees for each of the  $X$  hits.

Assume that your throws are independent and each has a probability  $p_1$  of hitting the inner circle. Your friend's throws are also independent, and each has a probability  $p_2$  of hitting the inner circle.

a) What is the distribution of  $N$ ?  
 b) What is the distribution of  $X$ ?  
 c) How much do you expect to earn from this game?  
 d) Say,  $p_1 > p_2$ . Does it matter who goes first?

Clearly,  $N$  is a binomial random variable with parameters  $m$  and  $p_1$ . Given  $N = n$ ,  $X$  too is Binomial:

$$p_{X|N=n}(x;n) = P\{X = x | N = n\} = \binom{n}{x} p_2^x (1 - p_2)^{n-x}$$


The unconditional PMF of  $X$  can be found by theorem of total probability:

$$p_X(x) = \sum_{n=x}^m p_{N=n}(n) p_1^n = \sum_{n=x}^m \binom{m}{n} p_1^n (1 - p_1)^{m-n} \binom{n}{x} p_2^x (1 - p_2)^{n-x} I(x \leq n)$$

where the indicator function ensures that your friend can never have more successes than the darts you win.

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Structural Reliability  
 Lecture 4  
 Common  
 probability  
 distributions



Here we look at a two-stage problem involving Bernoulli trials. So, let us take a minute reading the problem and then we will proceed. So, this is how the problem proceeds you start with a known number of dots trials  $m$ . So, lowercase  $m$  and then from there you filter out  $N$  capital  $N$  successes and you do a filtering once more from those capital  $N$  you come up with capital  $X$  final successes and then there is an expectation issued there.

But we will come to that later. So, the first  $m$  is a deterministic non-random number but the subsequent  $N$  and  $X$  they are random variables. So, the question is what is the distribution of  $N$ ? What is the distribution of  $X$ ? And how much do you expect to earn and what if you know  $p_1$  is greater than  $p_2$ . So, the two success probabilities are different and. So, in that case does it matter who goes first. So, the first would be to understand that the random variable  $n$  the first set of successes from small  $n$  trials is actually a binomial random variable.

And it has its parameter capital its lowercase  $m$  and  $p_1$ . Now given that this capital  $N$  has

attained a certain value a particular value a little n the random variable x is actually a binomial random variable as well. So, that is an important point to note that conditioned on capital N capital X is binomial. So, let us write that out. So, that's what you see on the screen the conditional PMF of X given little m.

Now which means; that we should be able to reduce the unconditional PMF of x by applying the theorem of total probability uh. So, that is what you see in the next block that the PMF of x is the sum as comes from the total probability theorem and you just plug in the known PMF of n into that except we just have to be cautious that n can never be less than x. So, I have put an indicator function there but you just have to keep in mind that anything less than any x less than n the terms in the sum those will be zero. So, let us move on I will go to the next slide and continue with the solution.

**(Refer Slide Time: 04:29)**

## Common discrete distributions

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**Example: darts (contd.)**

You and your friend go into a sports bar where a dart throwing competition is going on....

Substituting  $v = n - x$  so that  $n = x \Rightarrow v = 0$ , and  $n = m \Rightarrow v = m - x$  allows us to rewrite the above summation as:

$$p_x(x) = \frac{m!}{x!} p_1^x \sum_{v=0}^{m-x} \frac{(1-p_1)^v p_2^{m-x-v} (1-p_2)^{m-x-v}}{v!(m-x-v)!}$$

Rearranging and substituting  $m' = m - x$ , we obtain

$$p_x(x) = \frac{m!}{x!(m-x)!} (p_1 p_2)^x \sum_{v=0}^{m'} \binom{m'}{v} [p_1 - p_1 p_2]^v [1 - p_1]^{m'-v}$$

$$= \binom{m}{x} (p_1 p_2)^x (1 - p_1 p_2)^{m-x} \sum_{v=0}^{m'} \binom{m'}{v} \left[ \frac{p_1 - p_1 p_2}{1 - p_1 p_2} \right]^v \left[ \frac{1 - p_1}{1 - p_1 p_2} \right]^{m'-v}$$

Using the binomial identity and seeing that the terms in [...] add up to 1:

$$p_x(x) = \binom{m}{x} (p_1 p_2)^x (1 - p_1 p_2)^{m-x}$$

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Thus, X is Binomial with parameters  $(m, p_1 p_2)$

$$E[\text{earning}] = -m \times 1 + 10 \times E[X]$$

$$= -m + 10 m p_1 p_2 = (10 p_1 p_2 - 1) m$$

Since the solution is symmetric in  $p_1$  and  $p_2$ , it does not matter who goes first.

Structural Reliability  
Lecture 4  
Common  
probability  
distributions



After doing some substitution which you can work through it is going to be a few lines we can arrive at the PMF of x in terms of a sum that we can further simplify and arrive at an expression that is going to be very familiar. And if you want to just work through this all the notations and etc and then let me go to the next line and that looks very familiar. So, what it looks like is nothing but a binomial PMF a binomial PMF whose one parameter is m and the other success probabilities p 1, p 2.

So, now this random variable  $x$  its unconditional PMF is also binomial just like  $n$  and its parameter is also  $m$  like the first case but its success probability is now multiplied by the two. So, it's not  $p_1$  only is  $p_1$  times  $p_2$ . So, now we have done this sort of problems already a few times. So, the earning is the fixed number the non-random number. So, it is minus  $m$  times ones 1 rupee per dot and the random earning.

So, whatever the value of  $x$  is that is what gives me the income? So, 10 times expectation of  $x$  and the earning is  $10 p_1 p_2 - 1$  whole times  $m$  and then you can work out what those values will be once you plug in  $p_1 p_2$  and  $m$ . The final question is that does it matter who goes first it does not because the solution is symmetric in  $p_1$  and  $p_2$ .