

Structural Reliability
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Lecture –18
Review of Random Variables (Part - 01)

(Refer Slide Time: 00:33)

Review of random variables

Structural Reliability
Lecture 3
Review of
random variables

- Random variables: definition
- Probability laws
- Expectations and moments
- Common discrete distributions
- Common continuous distributions
- Derived distribution
- Jointly distributed random variables
- Functions of random variables
- Convergence

Further reading:

Probability Concepts in Engineering by A. H-S Ang and W. H. Tang, Wiley, any edition.
A First Course in Probability by Sheldon Ross, Pearson, any edition
Probability Random Variables and Stochastic Processes, Papoulis and Pillai, McGraw Hill, any edition.

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90

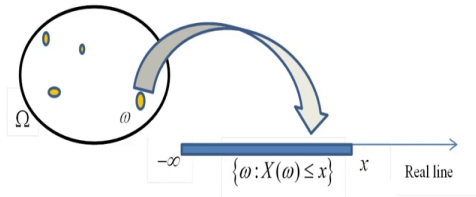


We are going to review random variables in this and the next 3 lectures. I would like to cover the definition, the probability laws that govern random variables, expectations and moments the common discrete and continuous distributions that we will encounter in this course and how they arise and their interrelations, change of variables, joint distributions, functions of random variables and the convergence of a sequence of random variables.

In addition to the recommended text by Ang and Tang I would also like to recommend the book by Sheldon Ross and the book by Papulis and Pillai for further reading.

(Refer Slide Time: 01:22)

Review of random variables



X is a random variable if:

- (i) $X = X(\omega)$ is a function defined on the sample space Ω ,
- (ii) and for every real x , the set $\{\omega: X(\omega) \leq x\}$ is an event in Ω .

That is, we confine ourselves to σ -algebra of events of the type $X \leq x$ (and suppress the argument ω)

Informally, when the outcome of a random experiment (trial) can be described numerically, and the outcomes can be assigned probabilities (subject to the three axioms), we have a random variable at hand.



Now a random variable is a measurable function defined on the sample space. So, it maps events from the sample space on to intervals that you see on the screen and we discussed measurable functions in the previous lecture. Informally when the possible outcomes of an experiment or trial can be given in numerical terms and they can be assigned probabilities obeying the actions of probability then we have a random variable in hand.

So, when an experiment is performed the outcome of the random variable is called a realization a random variable can be discrete or it can be continuous and as I said it is governed by its probability laws. If a quantity varies randomly with time we model it as a stochastic process. A stochastic process can be viewed as a family of jointly distributed random variables if a quantity varies randomly in space we model it as a random field which can be thought of as a generalization of a stochastic process in two or more dimensions.

(Refer Slide Time: 02:43)

Review of random variables

Probability laws

- A random variable (RV) is governed by its probability laws
- Four ways to describe the probability law of an RV:
 - CDF (cumulative distribution function)
 - PDF/PMF (probability density function for continuous RVs, probability mass function for discrete RVs)
 - CF (characteristic function)
 - MGF (moment generating function) – if it exists



Now the probability laws of a random variable there are 4 ways to describe it the cumulative distribution function or the CDF, the probability density of the probability mass function depending on whether the random variable is continuous or discrete in nature. The characteristic function and the moment generating function if it exists.

(Refer Slide Time: 03:09)

Review of random variables

Cumulative distribution function:

The cumulative distribution function of the random variable X is defined as:

$$F_X(x) = P[X \leq x]$$

It starts from 0, ends at 1, and is a non-decreasing function of x . It is piecewise continuous for discrete RVs, and continuous for continuous RVs.

Properties of CDF:

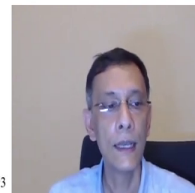
$$F_X(-\infty) = 0$$

$$F_X(\infty) = 1$$

$$F_X(x) \text{ is a non-decreasing function of } x$$

Thus, the probability of finding the random variable X in the semi-open interval $(a, b]$ is:

$$P[a < X \leq b] = F_X(b) - F_X(a) \quad (0.3)$$



Now the CDF is the probability that our random variable assumes values less than or equal to little x . Our convention is to use uppercase letters to denote random variables like X, Y, Z and lowercase letters like a, b, c and little x, y, z to denote its realizations. So, the CDF it

starts from zero ends at one and it is a non-decreasing function. If you want to find the probability that the random variable X takes on values between a and b strictly greater than a and less than or equal to b it is the difference of the CDF's evaluated at b and a .

(Refer Slide Time: 03:58)

Structural Reliability
Lecture 3
Review of
random variables

Review of random variables

Cumulative distribution function:

The cumulative distribution function of the random variable X is defined as:

$$F_X(x) = P[X \leq x]$$

↑ 1st Axiom

It starts from 0, ends at 1, and is a non-decreasing function of x .

It is piecewise continuous for discrete RVs, and continuous for continuous RVs.

Properties of CDF:


$$F_X(-\infty) = 0$$

$$F_X(\infty) = 1$$

$F_X(x)$ is a non-decreasing function of x

Thus, the probability of finding the random variable X in the semi-open interval $(a, b]$ is:

$$P[a < X \leq b] = F_X(b) - F_X(a) \quad (0.3)$$

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93


And if you would like to find out which of these corresponds to which actions of probabilities. So, this statement corresponds to the first action. This statement corresponds to the second action and the last is the equivalent of the third action.

(Refer Slide Time: 04:22)

Structural Reliability
Lecture 3
Review of
random variables

Review of random variables

Probability density function:

The probability density function (of continuous random variables) is defined as the derivative of the CDF:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

PDF may also be interpreted as the small probability of observing the random variable X around the point x :

$$F_X(x) = \int_{-\infty}^x f_X(t) dt$$


$$f_X(x) dx \approx P[x + dx \leq X < x]$$

$$\int_a^b f_X(x) dx = F_X(b) - F_X(a)$$

It is a non-negative function that integrates to unity:

$$f_X(x) \geq 0$$

$$\int_{-\infty}^{\infty} f_X(x) dx = 1 \quad (0.1)$$

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94


The density function is the derivative of the CDF or equivalently the CDF is the integration of the density function the area under the density curve is the probability that the random variable takes on values between the limits as I said. You can also think of the density function as it gives a measure of the random variable taking on values around that particular x . So, it is $F_x dx$ is a small probability its properties are it is a non-negative function that integrates to unity.

(Refer Slide Time: 05:12)

Review of random variables

Probability mass function:

The probability mass function, PMF, (of discrete random variables) is defined as the probability that the random variable assumes a particular value:

$$p_x(x_i) = P[X = x_i]$$

It can be derived from discrete jumps in the CDF, $F_x(x_i) = P[X \leq x_i]$, as:

$$p_x(x_i) = F_x(x_i) - F_x(x_i - \varepsilon) \text{ where } \varepsilon < \min(x_i - x_j), i \neq j$$


Like PDF, it is non-negative and sums to unity:

$$p_x(x_i) \geq 0 \forall i$$

$$\sum_{\text{all } x_i} p_x(x_i) = 1$$

Structural Reliability
Lecture 3
Review of
random variables

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95 

For discrete random variables we have the mass function. So, it is the probability that the random variable takes on that particular value x_i lowercase x_i you can also identify the PMF from discrete jumps in the CDF as you see here and if you sum all the PMF's these are all non-negative numbers. If you sum all the PMF's you get the value of 1.

(Refer Slide Time: 05:51)

Review of random variables

Probability mass function expressed as density function:

The delta function is defined as

$$\delta(x) = dU(x) / dx$$

where U is the unit step function:

$$U = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

The delta function is symmetric: $\delta(x) = \delta(-x)$. Integrating the delta function from a to b , if $a < 0 \leq b$, gives unity. More generally:

$$g(0) = \int_a^b \delta(x)g(x)dx \tag{0.3}$$

If X is a discrete RV, and $p_i = p_x(x_i)$, we can write an equivalent pdf as

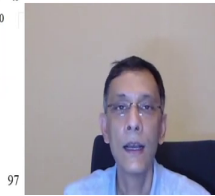
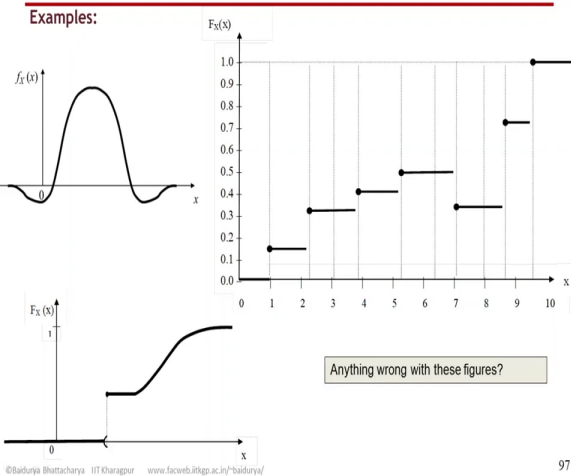
$$f_x(x) = \sum_i p_i \delta(x - x_i)$$



It is possible to describe mass functions as density functions by the help of the delta function. So, as you know the definitions of the delta function as I have put on the screen. So, you can describe the density function equivalently as the sum of the individual PMF's multiplied by the delta functions as you see on your screen.

(Refer Slide Time: 06:21)

Review of random variables



So, I have I am going to put 3 figures here and let us see if you can identify if there is anything wrong with these. So, the first one is a density function the second one is a CDF of discrete random variables and the third one is also another kind of CDF. So, the first one is actually not

right the density function can never be negative. The second one also has a problem because the CDF can never go down as you can see at 7 it goes down.

And the last one it is fine it is non-decreasing. So, it is okay and this is a particular case of a mixed distribution. So, this is where sometimes the random variable takes discrete values and in other cases it takes continuous values.