Urban Transportation Systems Planning
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Lecture - 46
User Equilibrium Assignment and System Optimum Assignment

Welcome to Module F, Lecture 6. In this lecture we shall continue our discussion about the User Equilibrium Assignment and System Optimum Assignment.

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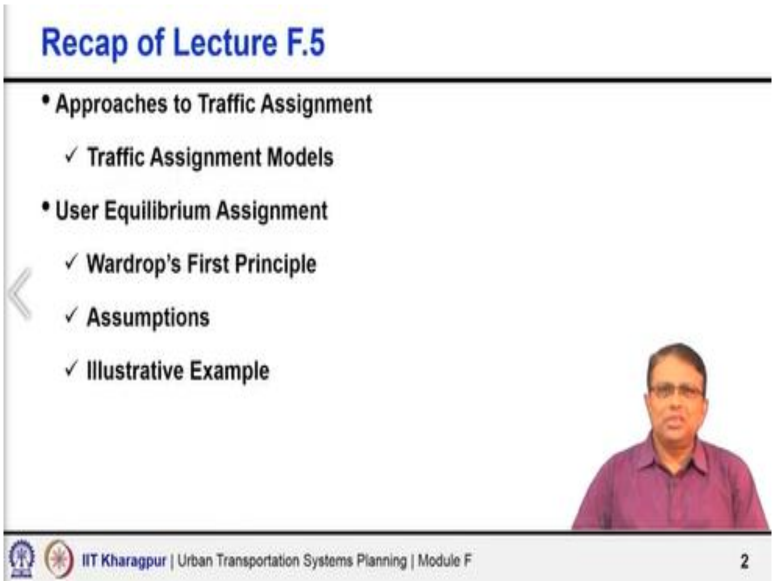

In lecture 5 we discussed about the broad approaches to traffic assignment, first static assignment and the other end is of course the dynamic assignment. And then under static assignment, we discussed about user equilibrium assignment. And mentioned about Wardrop's first principle that generally says people are selfish, everybody is trying to minimize his or her own travel time. And people will keep on changing the route as long as there is any opportunity for individuals to get a better route or get a route with a lower travel time.

So, when no further change will happen when everybody will find that nobody by changing route can reduce the travel time further in the whole network for whole O-D pair to explain that further we took only one O-D pair for example, because it is a kind of classroom example. And then took three alternative paths told you what would be the ideal or optimal solution. I hope you were convinced that would be the best solution;

Because you can try any other alternative distribution you will find that there will be scope for some people to switch over to some other route to reduce the travel time further.

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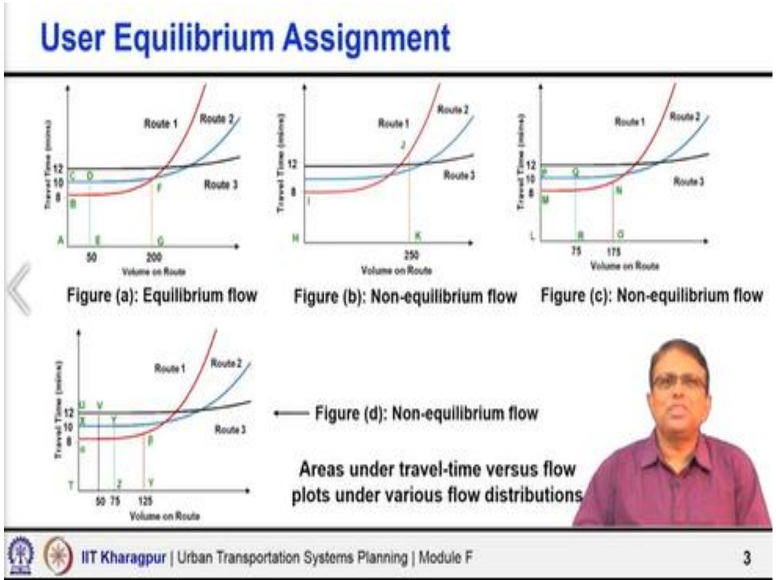

Then one more interesting thing we said that if you consider different possible distribution of demand in all the three routes. You will realize that the area under these curves the way that if there is zero flow then under that curve it will be zero. But otherwise travel time into the demand what is really coming. One thing also was said and it is true of course that, the area under these curves will be minimum when there is distribution of flow as per user equilibrium distribution or equilibrium user equilibrium condition. So, taking a Q from this, we proceed further.

Now we shall go for mathematical formulation of user equilibrium problem. And what will be our then objective? We know that for user optimal equilibrium or UE, the total area under these curves will be minimum. So, naturally now we can formulate it that we want to minimize this area under these curves with the flow distribution along different paths. And I know that, if I can get that solution which will minimize this area under these curves.

Then whatever will be my flows on different roads or demand in different roads, distribution of demand in different roads that is my optimal solution, that is my equilibrium flows under user equilibrium. So, let us go that.

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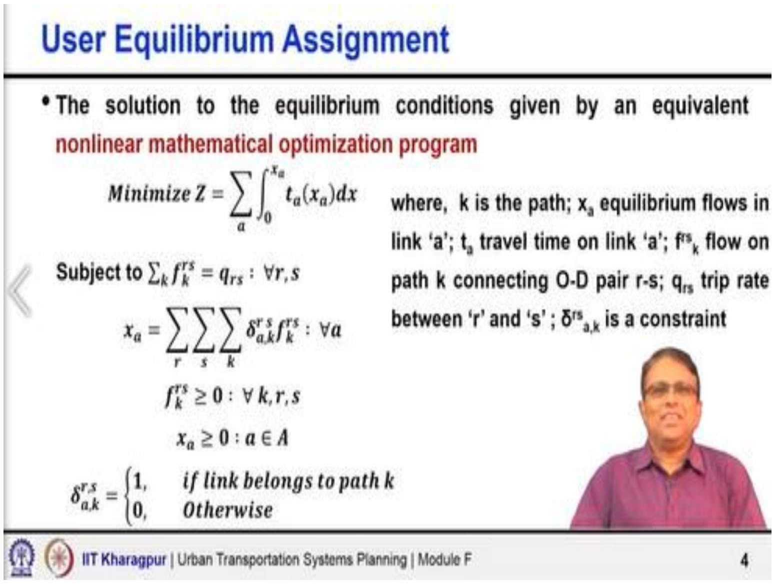

So, the solution to the equilibrium condition may be given by an equivalent nonlinear mathematical optimization program. We can say travel time on route a is t a and t is a function of x a. So, t a is a function of x a into d x take the integral do the integration with a limit 0 to x a, that is for path a. And I want to say the total area, so sum over all a all the paths or all the routes. So, if I try to minimize this then whatever will be the values of x a, if there are three routes or three paths then x 1, x 2, x 3 if there are 5 paths x 1, x 2, x 3, x 4, x 5.

So, whatever will be this x value that is my user optimal equilibrium and the corresponding distribution of flow in the three routes or demand in the three routes. That is the basic formulation so minimize z, x a t a, as a function of x a, d x integral 0 to x a, sum over all routes a, subject to certain constants. What are those constants? The one particular link might be a part of multiple paths for multiple O-D pairs and like this;

And some flow is non-negative, cost is non-negative. So, all or the time is non-negative like all these constraints will be there they are very logical and simple you can understand it very easily.
And if a link belongs to a path then I will take it take that path flow, when I am calculating the total demand or total flow on that link. If that link does not belong to a path then that path flow will not add so it is one or zero all such kind of simple.

But basic formulation is constraints are understood easily. Basic formulation is minimized what we are trying to use the area under these curves, that is what is the key.

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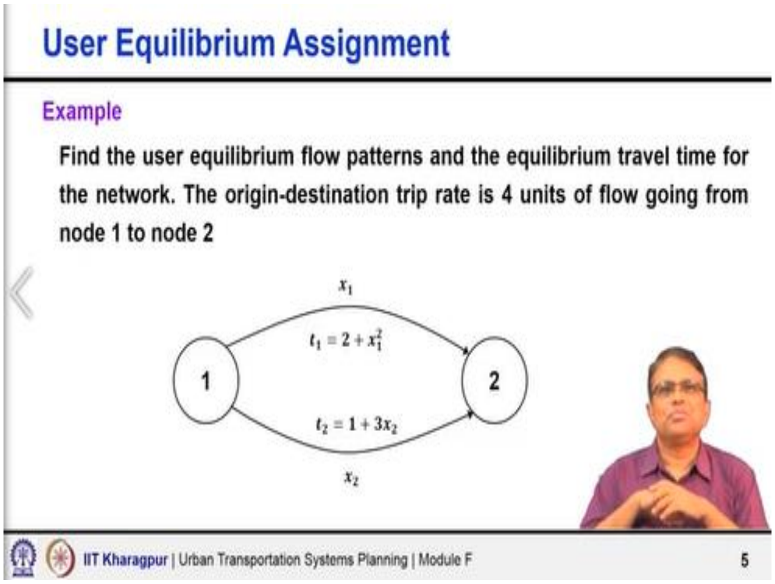

So, let us take an example. This example you do not need to apply that mathematical formula or the formulation to solve you can solve it very easily. But that is not the objective here. There is objective is to get the solution using this formulation whatever I said. And of course, it is a classroom example, so there is no point in enhancing or increasing the complexity of calculation that is again not the objective.

So, there are two parts t 1 2 + x 1 square, t 2 1 + 3x 2. That is the difference sensitivity, different zero flow thing because the free flow speed may be different, the route length may be different, the capacities are different. So, the sensitivity as the volume increases the increase in travel time is also not same. So, any example any it could have been anything else as well but I have taken this two travel time functions.

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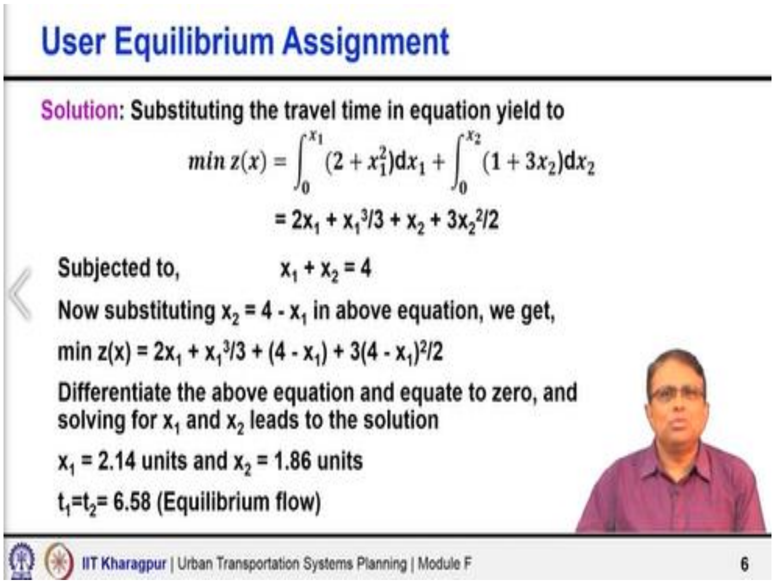

Now, 4 units of flow we want to distribute. So, what I would do 2+ x 1 square into dx 1, 0 to x 1 + 1 + 3x 2 dx 2, 0 to x 2. So, I find out what the values you know 2x 1 + x 1 to the power 3 divided by 3 + x 2 3x 2 square by 2. Now I know x 1 + x 2 has to be 4. So, x 1 + x 2 is 4. So, then I can now replace x 2 by 4 - x 1 simply I do that and then I have only one variable that is x and, in this case, it is easy.

So, you find out that minimum thing and you know, so I know what is the z value I can differentiate and I can calculate make it zero. And that tells me x 1 equal to 2.14 and x 2 equal to 1.86. So, and with those the t 1 equal to t 2 equal to 6.58. So, each path or both paths will have then equal travel time of 6.58. So, nobody can switch to any other route further, to reduce travel time.

So, nobody will switch to other route and it will be equilibrium. And you check that it is the equilibrium time because you can see that two times are becoming equal with the corresponding x 1 and x 2. And that again shows that if I try to minimize the area under these curves, then whatever x 1 x 2 I will get those will be the equilibrium flows on the corresponding routes.

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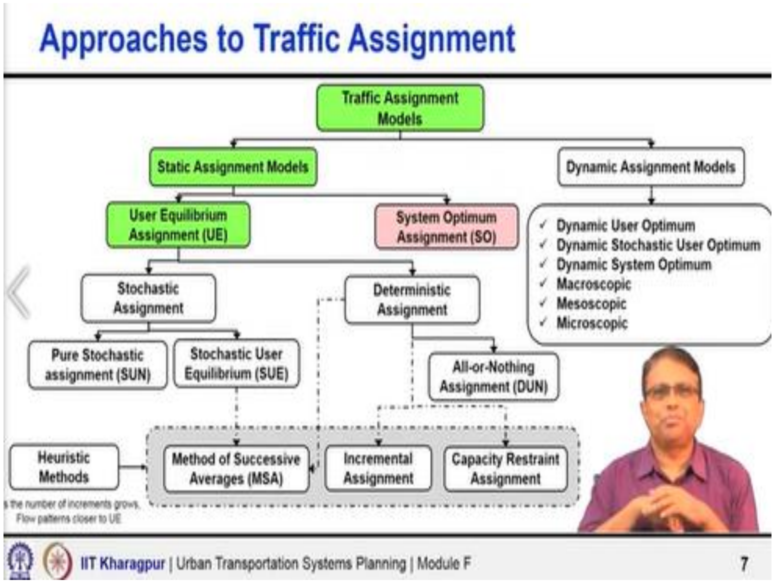

Now, we go to the other part so my user equilibrium the bigger box is also green now. Now we go to the right side system optimal or system optimum equilibrium. What is that? Till now we said that people act selfishly. Every individual when I want to travel, I am thinking about how I can choose a route so I get a deal. That means I get the least travel time when I want you know travel accordingly, I am taking a path.

Every individual is taking a path like this, nobody is trying to you know everybody is very much individualistic. And everybody is thinking about his or her benefit that gives the user optimal. But what we are saying here the whole system is still not getting optimized. Why it is not getting optimized is something you have to understand clearly but let me tell you first what system optimum.

System optimal is that, what distribution of flows will give me total aggregate travel time as minimum. Understand that, people from different O-D pairs are travelling they are taking different paths some links may be common to the multiple path and all such kind of things are there. I am saying that way I want the distribution I use the word I want or rather we want the distribution the transport planets we want the distribution where, the aggregate travel time will be minimum.

That means when individuals are saying that they are taking the best route there is no other route alternative route, where they can shift to get a better time travel time. Still the overall system is not getting optimized through that user equilibrium. So, this is the solution a different solution where the aggregate travel time for all the users using the network is minimum. So, the vehicle minute time is minimum for example.

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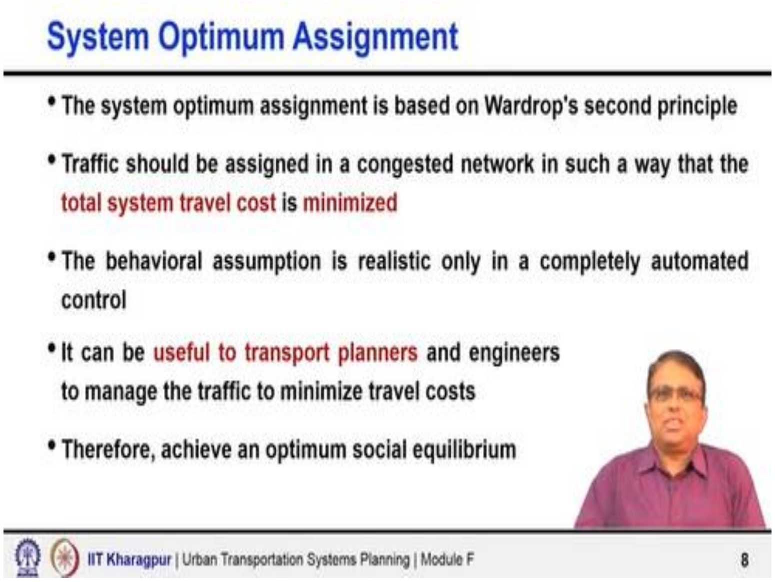

So, that is what I said the system optimum assignment is based on Wardrop's second principle. The first one is first principle as I say user equilibrium. Second one; system equilibrium is as per Wardrop's second principle. Traffic should be assigned in a congested network in such a way that the total system travel cost is minimized, I say aggregate it is the system cost. The behavioral assumption is realistic only in a completely automated control.

What is that why suddenly is a completely automated control? Because in system what exists today nobody can force you to choose a route. And everybody will think selfishly and likely to act more like user equilibrium. But system equilibrium, makes a much deeper meaning because at a network level if I consider myself as a system engineer as a planner, I would be very happy to see that my total system travel cost is minimized or total system travel time is minimized vehicle mini travel. So, normally you are not expected to get system equilibrium.

It can only happen in a fully automatic control where every user so maybe we sit in the inside the car and select our destination and you say, yes take me to this destination. And the system decides which path I should take so that my system travel time is optimized. Then only it can happen. If we individually try to make a choice each of us will act in a selfish manner. And we will go back to somewhat nearer to user equilibrium.

It can be useful to the transport planners and engineers to manage the traffic to minimize travel cost. Why? To achieve an optimal social equilibrium that means now the question is we know that in a setup like the present setup what exist in most cities' Indian cities of course all cities. People will try to go for user equilibrium but we know that user equilibrium is not going to give me the system efficiency. So, and I also know that I cannot force people to take route or everybody to choose a route.

But to some extent something is in our hand as well as a traffic engineer. What is that? Traffic management. I can do my traffic management in a manner of course still I will not be able to achieve completely system optimal. But all what I am saying if my user optimal is here and the system optimal is here I can still make this user optimal or the operation to happen somewhere.
Or I can push the whole thing towards system equilibrium still it is user equilibrium but I am pushing it towards system equilibrium by traffic management technique.

Because you can say that, this route only bus can enter here the signal setting you can do to control the flow and therefore indirectly people make another route attractive to people. So, traffic management can influence the behaviour of course it can influence the behaviour it has that potential. So, to transport planner and traffic management experts, this is interesting because they know the present solution is here as I said and what I would like to see is here I cannot anyhow see this one it is not possible.

But through traffic management and control and everything can I push it towards to some extent whatever I can push is my success because I know that I am moving towards an optimal social equilibrium I am saying social equilibrium, because the system is getting optimized.

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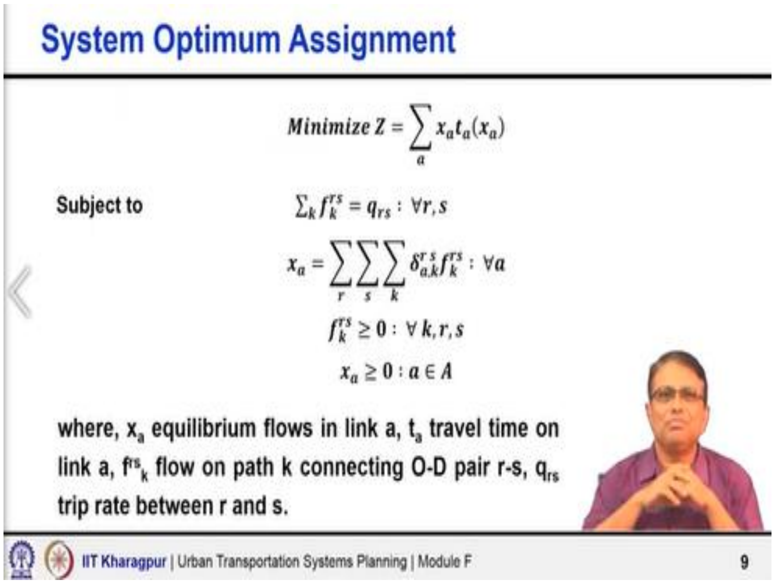

So, what will be the mathematical formulation? Mathematical formulation is simple, but carefully observe why the solution is different because what you are trying to optimize your objective function is very different. There what we were trying to optimize the area under this curve, here something different what we are saying here, the aggregate travel time is minimum. So, t a x a, t a is the travel time on link a or path a or route a. When the flow is x a or the number of vehicle using it x a.

So, t a x a into x a is the travel time aggregate travel time for all the x a vehicle which are using this path a or route a summit over all such path. That gives me that minimum aggregate travel time for all the vehicles which are getting distributed to different paths I want to achieve that. What value of x a is minimizing my z? That is x a t a, t a is a function of x a, sum over a. That is getting optimized.

Whatever flow distribution will optimize this that is my system equilibrium flow. Subject to certain constraints which are again very similar the constraint part is not really difficult.

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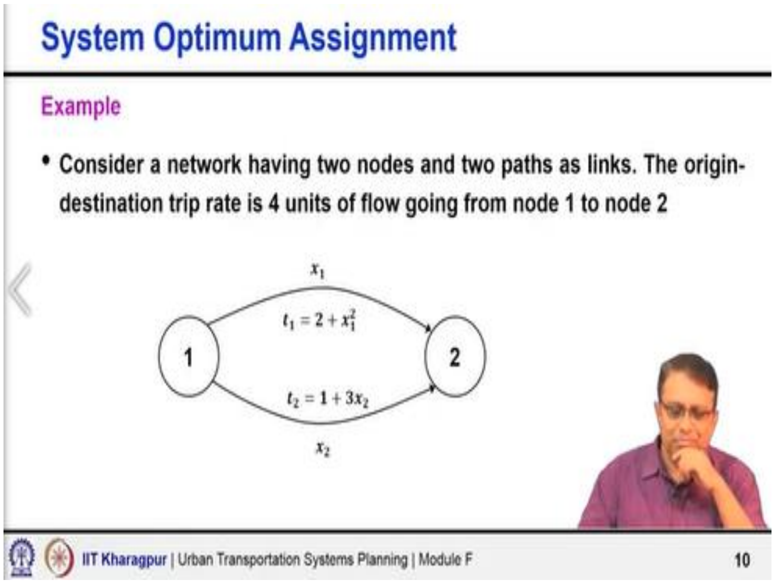

Let us take the same example, what I took earlier in the case of user equilibrium. And now I am saying the four units I want to distribute to route a and route b following the principle of system equilibrium or system optimal distribution.

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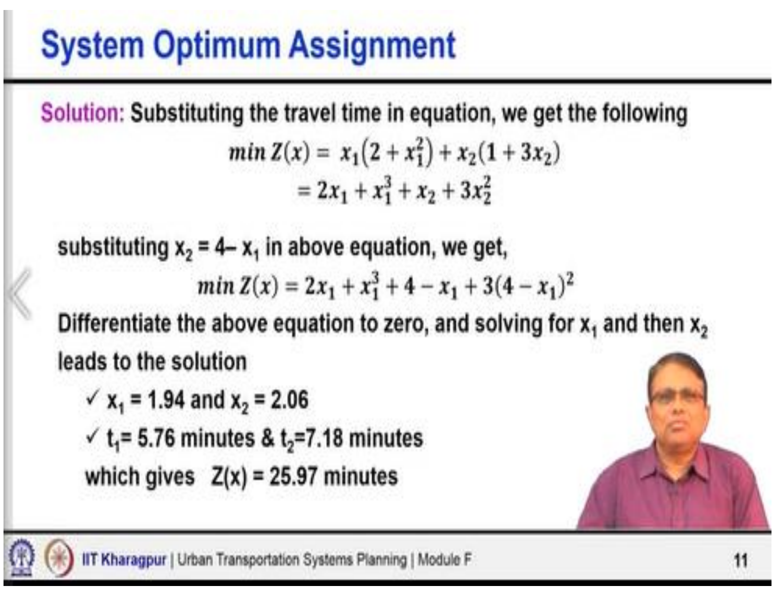

So, I say 2 + x 1 square is the travel time for path one, so 2 + x 1 square multiplied by x 1, x 1 is the flow that is going to get distributed or the demand that is going to come here. The route two 1+ 3x 2 is the time and the flow is x 2 so multiplied by x 2. So, you get this one to, x 1 + x 1 q or x 1 to the power 3 + x 2 + 3x 2 square. Now again I know, x 1 + x 2 is 4. So, I can do that substitution x 2 is 4 - x 1.

So, I can write this objective function only as a function of x 1. The remaining is simply take the derivative make it zero, get a value of x 1 do the substitution get a value of x 2. But here you can see the x 1 is coming out to be 1.94, x 2 is 2.06 different not the same and not the same means not even similar to what you got there how much you got there let us go back. We got x 1 as 2.14, x 2 as 1.86, 2.14, 1.86 what we got now?

We got now, 1.94 and 2.06 also see this solution is not the user equilibrium solution, why I say? Look at this travel time corresponding travel time, t 1 is 5.76 and t 2 is 7.18. Travel time on both routes are not equal. So, in user equilibrium what will happen? Some more people from path two will shift to t 1 keep on shifting to t 1 till the travel time on both the routes or till t 1 and t 2 becomes same.

But that does not give me system optimal because whatever you get here the value of z the aggregate travel time 25.97 vehicle minutes or so it is written only as minute but if I take x 1 and x 2 number of vehicles. Then the aggregate travel time again of course it is totally in minutes only finally aggregate minute. That aggregate minute here you got 25.97 please do the calculation and see what is the value of this travel time. The total travel time for both the path with the user equilibrium distribution of flow you will find the degree of travel time will be higher.

So, it clearly shows that, user equilibrium in this case is not the system equilibrium because by aggregate travel time can be further reduced and this is the minimum value I can reach with this x 1 as 1.94 and x 2 as 2.06 and the corresponding t 1 as 5.76 and t 2 as 7.18. so, these two solutions are different.

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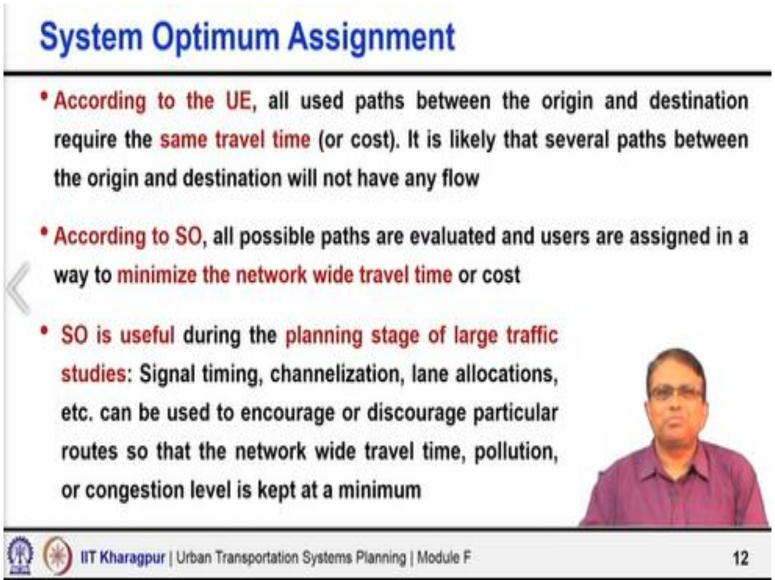

So, what I said here according to the UE, all used path between the origin and destination requires the same travel time or same cost. It is likely that several paths between the origin and destination will have will not have any flow may be if they are higher, nobody is going to use that. But according to system optimal equilibrium all possible paths are evaluated and users are assigned in a way to minimize the network wide travel time.

So, is useful during the planning stage of large traffic studies may be signal timings, channelization, lane utilization and so on. So, forth and can be used to encourage or discourage particular routes so the network by travel time and accordingly maybe the pollution or the condition is kept to a minimum. So, there could be benefit, but as I said this is more like an ideal solution not the realistic thing.

We do not expect system equilibrium to happen in practice under this kind of setup where everybody is free to choose the route. It can happen only if we are taking say, fully automated travel you just you do not select the route you just sit inside a vehicle press the switch select your destination in start and the system will decide how you travel. Of course, there will no more be travelling will or driving may not be any may not be fun anymore. Also, it may not be so stressful anymore.

That is a different question there are mixed reaction people sometimes they like sometimes people may not like it is a different it could be a different world altogether. But you can definitely optimize the system.

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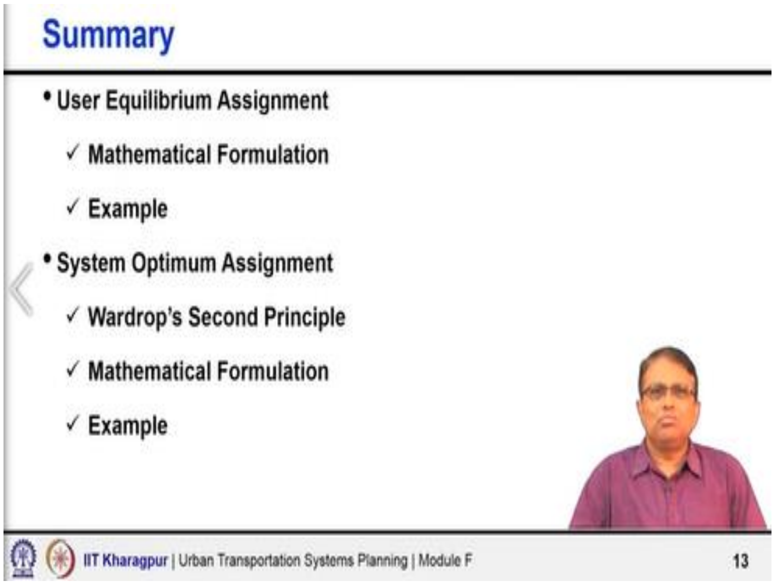

So, generally, before I summarize, I would suggest you to think, why these solutions are different? If individually every individual is able to optimize his or her own travel because he is getting the best deal. So, individual deals are based. But overall system is still not the best one at it best level. I only give you a clue and leave it for you to think further. Cost is non-linear you remember the time cost function look at the non-linearity.

Just imagine a reduction in the travel, reduction in the volume, when the road is operating at a congested condition the curve is very sharp. At that level if you take out some people from that. Then there is a significant reduction because you are at that portion of the curve which is very sharp. So, small reduction in the volume can give lot of benefit. Whereas on the other road you find this is flat so even if we increase the volume the cost or travel time does not increase.

So, what will happen? As long as by shifting some people from a congested situation I will keep shifting to uncongested routes. Then my savings in the travel cost or travel time or aggregate travel time reduction will be lesser will be higher than the increase. Increase may not happen because you are maybe the other you are at a flat path. So, the overall the aggregate travel time will get reduced.

So, that means from congested route generally speaking from congested route if you can shift people to uncongested route even though some people best route will not be there. There is a lesser cost route but still if you shift them then the overall aggregate travel time will reduce. As I said why the reason because when you are in the sharp part of the curve then say, x amount of reduction in volume whatever y reduction in travel time it will give.

Here on an uncongested route x increase in, you know the demand or the vehicle volume will not give you, y increase in travel time it will be much lesser than y and as long as it will be lesser than y the shift will bring the overall aggregate travel time, benefiting all. Because 100 vehicles are travelling with 30 minute travel time and you ship 10 vehicles so the remaining 90 also the benefit is happening because for those 90 also travel time is now reduced.

So, that tradeoff happens. So, with this I close this session, thank you so much. We will continue our discussion in the next lecture. Thank you.