Advanced Foundation Engineering
Prof. Kousik Deb
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture No -35
Beams on Elastic Foundation- IX

So, last class I have discussed that how we can express this beam equation in finite difference form. And I was discussing the case when beam is resting on Winkler spring and then how to use the boundary conditions. So, this was the expression.

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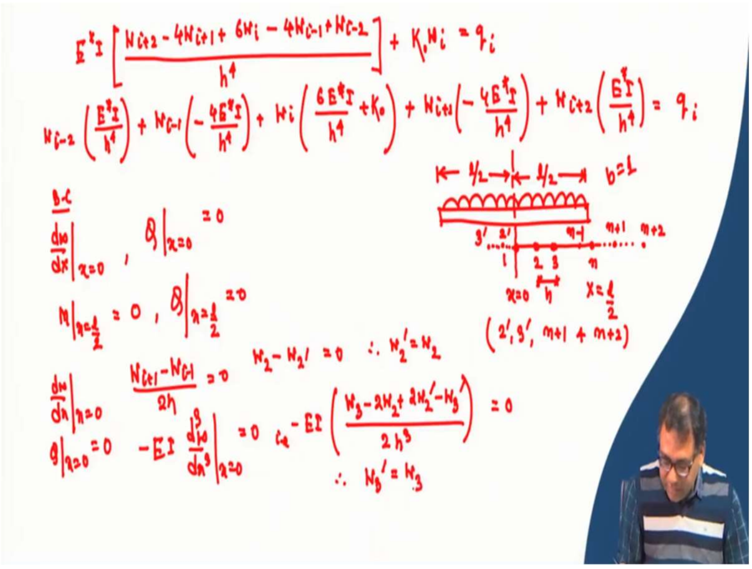

So, these equations which are expressed in central difference scheme we have to apply at all the nodes. So, this beam which has finite length is divided into number of nodes and this 3rd node to $(n-2)^{\text{th}}$ node we can apply this equation, but for other four nodes we cannot apply directly this equation. So, we have to apply the boundary conditions. So, these are the four boundary conditions due to symmetry the slope is zero.

And as we are applying the UDL the shear force at the centre is zero and it is a free end beam. So, bending moment at the edge of the beam is zero and shear force is also zero. So, now if I apply these equations, so that means to apply this equation we need the help of some imaginary nodes. So, these nodes are called imaginary nodes. So, this is your $2'$ node, this is $3'$ node and then this is your $(n+1)^{\text{th}}$ node, this is $(n+2)^{\text{th}}$ node.

So, these $2'$, $3'$ and $(n+1)^{\text{th}}$ and $(n+2)^{\text{th}}$ these four are imaginary nodes because actual nodes are from 1 to n that means your total beam half portion is divided by $n-1$ equal parts if you divide to n equal parts then the number of node will be $n+1$ but here the node is given up to n. So, it is divided into $n-1$ equal parts.

And, that equal part distance is h or you can say Δh or Δx also depending upon your requirement you can express that. So, now these four are the imaginary nodes. Now, if I apply this boundary condition, which is at centre; that means $i = 1$ first node we have to apply first two boundary conditions because it is $= 0$. So, first node if I apply this boundary conditions that $\frac{dw}{dx}$ at $x = 0$.

So, then I can write that $\frac{W_{i+1} - W_{i-1}}{2h} = 0$. So, now here $i+1$ is your second node that is W_2 node and W_{i-1} is this $2'$ imaginary node, $W_{2'} = 0$. So, if I apply this boundary condition then $W_{2'} = W_2$. So, that means the settlement at node 2 will be equal to the settlement at imaginary node $2'$. So, that problem is solved.

Now, if I apply the second boundary condition, that is Q at $x = 0$ that is 0. So, this is $EI\frac{d^3W}{dx^3} = 0$ at $x = 0$. So, now I can express that as $EI\left(\frac{W_3 - 2W_2 + 2W_{2'} - W_{3'}}{2h^3}\right) = 0$ because I am now expressing this $\frac{d^3W}{dx^3}$ in finite difference scheme. So, this is the expression. So, now here I know $W_{2'} = W_2$.

So, this 2 part will cancel. So, I can write that $W_{3'} = W_3$. So, that has also been obtained by using second equation.

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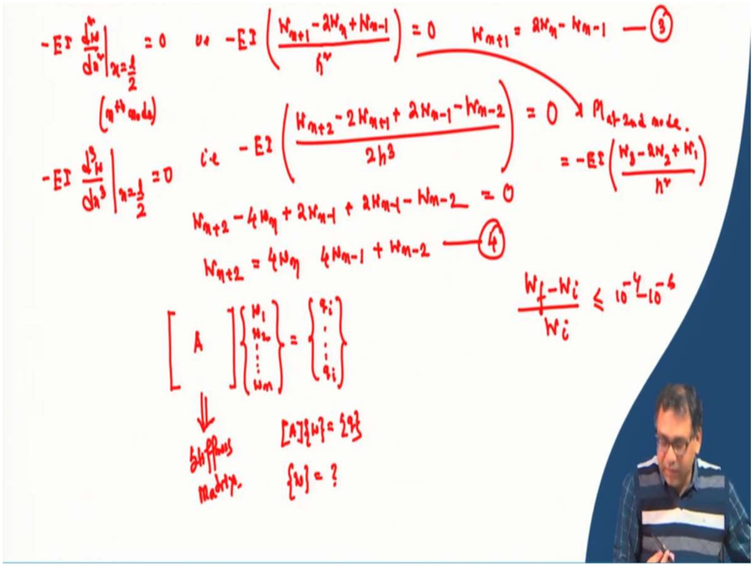

Now, if I apply the third boundary condition. So, that means bending moment $-EI\frac{d^2W}{dx^2} = 0$ at $x = \frac{l}{2}$. So, that is $-EI\left(\frac{W_{n+1}-2W_n+W_{n-1}}{h^2}\right) = 0$, I am expressing this $\frac{d^2W}{dx^2}$ and I am applying these conditions at n^{th} node because that is at $x = \frac{l}{2}$; that means this is the n^{th} node and we have to apply at the n^{th} node only and this you have to apply in the first node these two conditions first node or node number one.

So, I can write $W_{n+1} = 2W_n - W_{n-1}$. So, this is another condition so, we have four conditions this is first condition, this is second condition, this is third condition and we have the four boundary conditions that means the shear force $-EI\frac{d^3W}{dx^3}$ at $x = \frac{l}{2}$ that is also 0. So, I am applying the finite difference form so, $-EI\left(\frac{W_{n+2}-2W_{n+1}+2W_{n-1}-W_{n-2}}{2h^3}\right) = 0$.

So, finally $W_{n+2} - 2W_{n+1} + 2W_{n-1} - W_{n-2} = 0$ and now we have this condition $W_{n+1} = 2W_n - W_{n-1}$ because now we have to convert the imaginary node to real node because we have to get the equation or the relationship with the imaginary node to real node because real nodes are the nodes where I want to determine the deformation, imaginary nodes are used for these boundary conditions.

So, finally let me first write this in different form, that means this W_{n+1} is replaced by $2W_n -$

W_{n-1}. So, this will be $4W_n$ then - - + $2W_{n-1}$ then + $2W_{n-1} - W_{n-2}$. So, W_{n+2} because this is another imaginary node this will be $4W_n$ then minus this $4W_n$ and this is equal to zero usually $-4W_{n-1} + W_{n-2}$. So, this is my fourth condition. So, the final equation is $W_{n+2} - 4W_n + 4W_{n-1} - W_{n-2} = 0$.

So, all the, imaginary nodes are now converted to or we get the relationship between all imaginary nodes and real nodes. So, this is the imaginary node $2'$. So, I will get that relationship for $3'$ I am getting the relationship then for W_{n+1}, another imaginary node $n + 1$. So, this is the relationship with the real nodes here also W_{n+2} with the real nodes. So, now I want to apply all the equations.

So, now these equations we can apply at all the nodes so, from 3 to $n - 2$ directly we can apply but when we will apply at second node or the first node, then we have to use the imaginary node or imaginary node will come into this equation. So, those imaginary nodes we have to replace with the real nodes. Because, now each and every imaginary node is replaced by the real nodes because we have the relationship.

So, now you apply these basic equations in finite difference form for all the nodes from first node to n^{th} node and if in some node basically these four nodes when you apply these equations imaginary node will come. So, those imaginary nodes you replace with real node. So, finally you have the *n* number of equations.

And, then you put them in a matrix and this is your W_1, W_2 to W_n, and then you have q_i to q_i. So, this is your A matrix or this is called a stiffness matrix. So, this is your $[A]\{W\}$ and this is your force vector, your load vector and this is your deflection vector, it is A matrix. So, from here you have to get the *W* equations or *W* values. So, you solve these equations so, that means it will be the inverse into the load vector and then we will get the deflection at every point.

So, once we apply these equations on *n* number of nodes so, you will get the n^{th} nodes and now all the equations can be replaced with real node deformations. So, you will get *n* number of nodes now express them in matrix and then this is the deflection vector and the load vector then you solve this equation or you can use MATLAB or you can develop your code or you can use any solver to solve these equations then you will get the deflection at any point.

So, once you get the deflection at any point then by using the finite difference method also you will get the slope or bending moment or shear force at any point. For example, now you have the deflection of all the points now you want to determine the bending moment at point two. So, at point two you express your bending moment equation. So, this is your bending moment equation.

So, this will be your bending moment equation that equations you have to express at point two. So, once we apply this equation at point two, for example I want to determine the bending moment at point two. So, that means the bending moment at second node. So, the bending moment at second node will be what? So, that means this will be equal to $-EI$ this is your second node $n = 2$. So, this will be $\frac{W_3 - 2W_2 + W_1}{h^2}$.

So, now you know the W_1, W_2, W_3 and you know the h^2 value so that you put and you will get the bending moment at second node. So, similarly you will get the shear force or the slope at any point any node if you know the deflection of all the points. So, now there is one question just how you will decide the h^2 value, so h^2 value first you take one particular value of h or you can divide it say 20 numbers.

Then, you will get the h value then you will get the deformation of all the points then you divide this number from 20 to 40 then your h will further reduce then 40 to 80 then your h will further reduce and every time you will get the deformation. Then you compare the previous deformation and the current deformation, if the difference between two deformations is very less always in the tolerance limit then you stop there and based on that you decide your h value.

For example, suppose your previous deformation, this is current or final this is your final deformation, W_f minus initial deformation, W_i divided by initial deformation, W_i this should be less than 10^{-4} to 10^{-6}. So, if it is within that tolerance limit you choose the h accordingly that means you have to increase the division and your h value will reduce and you will find after a certain increase of the division your deflection value will not change significantly or this is within a tolerance limit.

So, you decide h value at that point. So, that h you have to consider for all other calculations. So, this is the way you can divide and solve the beam problem in finite difference scheme but this is for beam with finite length similarly you can use it for infinite length also by using the boundary conditions because infinite length or semi-infinite boundary condition will change only because this equation will remain same because you need the four boundary conditions.

So, if it is the semi-infinite beam you can have the boundary condition; you say n condition there you can apply two boundary conditions and if it is infinite so, at infinity your deformation will be zero. So, you take a length which is very high where the deformation length is very large. So, at that point deformation will be zero and definitely your slope will be also zero in infinite condition so that way also you can solve it for infinite, finite any condition but this is your beam on Winkler spring.

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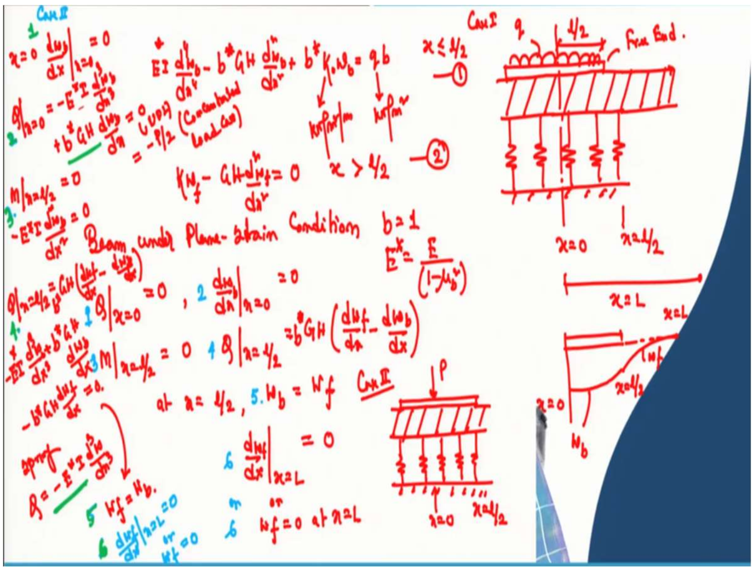

So, quickly I am giving the equations if your beam is resting on two-parameter model. So, beam on two-parameter model, your equation is $EI\frac{d^4W}{dx^4} - b^*GH\frac{d^2W}{dx^2} + b^*k_0W = qb$, if your q is in kN/m^2 and k_0 is kN/m^2/m. So, now here in this case, the beam is resting on two-parameter model.

So, there is a difference is that you will get the deformation within the beam region as well as outside the beam region also. So, now if this is your symmetry this is equal to $\frac{l}{2}$ so now this equation is valid if your $x \leq \frac{l}{2}$. And beyond the loaded region your equation will be $-GH\frac{d^2W}{dx^2} + kW$, which is the equation of a Pasternak model because your $q = 0$ because we are not applying any load.

So, this equation is valid if $x \geq \frac{l}{2}$. So, now your beam is on two-parameter model. So, there will be basically two equations one is within the beam another is outside the beam and the boundary conditions what are the boundary conditions. So, the beam is under plane-strain condition. So, your $b = 1$ and $E^* = \frac{E}{1-\mu_b^2}$. So, here remember that in all the equations this will be your E^*.

So, I have written only E but this all will be the E^*. So, this is E^*. So, what are the boundary conditions? In such case, so now we have two equations, equation number one and equation number two, so in equation number one the boundary condition is that the shear force at $x = 0$ and $x = \frac{l}{2}$ is zero because we are applying the UDL and it is a free end beam and $\frac{dw}{dx}$ at $x = 0$ is 0.

Then moment at $x = \frac{l}{2}$ is zero and shear force at $x = \frac{l}{2}$ but here is the difference, it is not zero it is for two-parameter model, this is the equation $\frac{dw_f}{dx} - \frac{dw_b}{dx}$. So, what is the difference between this f and b, w_f is the deformation beyond the beam region and w_b is the deformation within the beam regions so I should write this is b within the beam region and this is f beyond the beam region.

So, this is the boundary condition for equation number one and another boundary condition that at $x = \frac{l}{2}$ your $w_b = w_f$. So, these five boundary conditions you have to use and additional boundary condition you need to solve equation 2 because I have one equation, the $w_b = w_f$ another boundary condition that $\frac{dw}{dx}$ at here I should write this is b and this is f at $x = L$.

So, that means L is the length up to which you have considered your domain. So, you have considered your domain because this is your beam and then you have considered additional part beyond the beam so that you will get the deformation then how will I decide this L again you have to take different values of L and you have to run the program and you will find after certain length your deformation at the centre will not change significantly or within the tolerance limit.

So, you choose your domain according that means you change the domain L value and you will find after certain L value if you increase the L value further then your deformation of the centre will not change significantly. So, you choose that L value as your domain. So, at the end of that domain your slope is equal to zero because your deformation will be something like this.

Because, this is up to the beam and then beyond the beam so, your deformation will be something like this, so this will be the deformation. So, that means at $x = L$, the slope is zero and this is your $x = 0$ and this is your $x = \frac{l}{2}$. So, these deformations this is your w_b and this one w_f after the loaded or beam region is w_f and within the beam region w_b so, this is the difference now you use the boundary conditions that are given.

So, this is for free end condition when UDL applied is q. Now, you solve these equations by using finite difference scheme the way I have discussed for the beam on Winkler model or spring now you can use different boundary conditions for different loading conditions also. So, how we can do that? Suppose we can use different boundary conditions also for different loading conditions.

For example; that our next loading condition is for the UDL are given the boundary conditions suppose you have a beam which is resting on two-parameter model and it is subjected to a concentrated load at the centre. So, how we can use the boundary condition? So, one thing I want to say that in previous condition you can use these boundary conditions or sometimes you can use $w_f = 0$ at $x = L$ that is also another boundary condition you can use.

But if it is for beam under concentrated load P and it is a free end beam because if your n condition changes you have to change the boundary condition that I have discussed for semi-infinite beam the same way you can apply this here also. Now if it is the case this is your $x = 0$ this is $x = \frac{l}{2}$ and then how I can write this boundary condition. So, boundary condition is that $\frac{dw_b}{dx} = 0$ at $x = 0$.

Now your shear force at $x = 0$ will not be zero here. So, your shear force will be $\frac{P}{2}$. So, now again here the shear force equations are given $Q = \frac{P}{2}$ at $x = 0$. So, shear force equation is here $-E^*I\frac{d^3w}{dx^3}$, this is the equation but this is valid for if the beam is resting on Winkler spring, but if beam is resting on two-parameter model, then in your equation you have to add this $b^*GH\frac{dw_b}{dx}$.

Because, this second contribution is due to the contribution of the shear layer. So, that means the shear force, if it is spring then your shear force expression is simple $-EI\frac{d^3w}{dx^3}$. This is for spring but for two-parameter model it is $E^*I\frac{d^3w}{dx^3} + b^*GH\frac{dw_b}{dx}$ and for this UDL case it is zero. So, you have to write zero for UDL case but it will be equal to $-\frac{P}{2}$ for your concentrated load case.

So, this is your force then as your $\frac{dw_b}{dx} = 0$ at $x = 0$. So, ultimately this part will cancel but this is for centre only not for the edge because for the moment that means here also moment M at $x = \frac{l}{2}$ is 0. So, for the moment, this is same as the Winkler model, that means $-E^*I\frac{d^3w_b}{dx^3} = 0$ and the fourth boundary condition that Q at $x = \frac{l}{2}$ that is equal to the expression, that is $GH\left(\frac{dw_f}{dx} - \frac{dw_b}{dx}\right)$.

So, now expression is $-E^*I\frac{d^3w}{dx^3} + b^*GH\frac{dw_b}{dx}$. So, that means, this is b^*, that will be b^* here because this is b^* remember that this is b^*, so this is also b^* so, now we are taking this value $-E^*I\frac{d^3w}{dx^3} + b^*GH\frac{dw_b}{dx} - b^*GH\frac{dw_f}{dx} = 0$ and this condition $w_f = w_b$. So, that means what is the difference between this case this is my case 1 and this is my case 2.

Because more or less all the four boundary conditions are same what are those four boundary conditions that is when the beam is under plane-strain condition the slope at the centre is 0 here also slope at the centre is 0, then shear force at the centre is 0, here shear force at the centre is not 0 it is $-\frac{P}{2}$ for UDL it is 0, bending moment at the end is 0 here also bending moment at the end is 0.

And, here your shear force at the end we can express in this form here also you can express in this form only the difference is at the centre in case of UDL shear force is 0 but in case of concentrated load shear force is not 0. So, that means I can write the four boundary conditions, this is one boundary condition this is again, this is third and this is fourth and this is the fifth and this is the sixth.

So, these two boundary conditions we have to apply in this case also if I want to determine the deflection beyond the loaded region that means for beyond the loaded region also $\frac{dw_f}{dx} = 0$ at $x = l$. So, now for this case what are the boundary conditions this is the first boundary condition second boundary condition, then third boundary condition and fourth boundary condition and this is the fifth and this is the sixth or this is the sixth.

So, these are the six boundary conditions 1, 2, 3, 4, 5, 6 four for the equation number 1 and two more for equation number 2 and for second case because this is for case two where this is 1, 2, 3 4, and then 5 and 6 these are the boundary conditions. So, if I write in different colour this is 1st, 2nd, 3rd, 4th, 5th, 6th. So, this blue colour are the boundary conditions blue numbering of the boundary condition for case 1 and the green number are the boundary condition for case 2.

So, only difference in UDL the shear force at the centre is zero but in case of concentrated load at the centres shear force is $-\frac{P}{2}$ because we are taking half portion that is why it is $-\frac{P}{2}$ and one thing that in case of beam resting or spring your shear force equation is simple $E^*I\frac{d^3w}{dx^3}$ but here a second term the contribution due to shear layer you have to incorporate that is the second term this term.

This term is incorporated due to the shear layer for this spring this is the term where this second term is not there. So, these are the different boundary conditions and under different loading conditions and the end conditions. So, you have to use, these boundary conditions to solve the beam problem by using the finite difference scheme. So, I have finished these beams on elastic foundation part.

So, next class I will start the new topic that is the design of shallow foundation. Here design means that we will find the proportion or the dimension of the foundation by using the field test data mainly and by using the lab test data also how we can find the dimension of a shallow foundation, thank you.