Advanced Foundation Engineering
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Lecture No -29
Beams on Elastic Foundation – III

So, last class I have discussed about the beams resting on elastic medium. And then I have discussed that the beam can be resting on spring or it can be resting on two-parameter model. So, now today I will discuss about the beam resting on spring.

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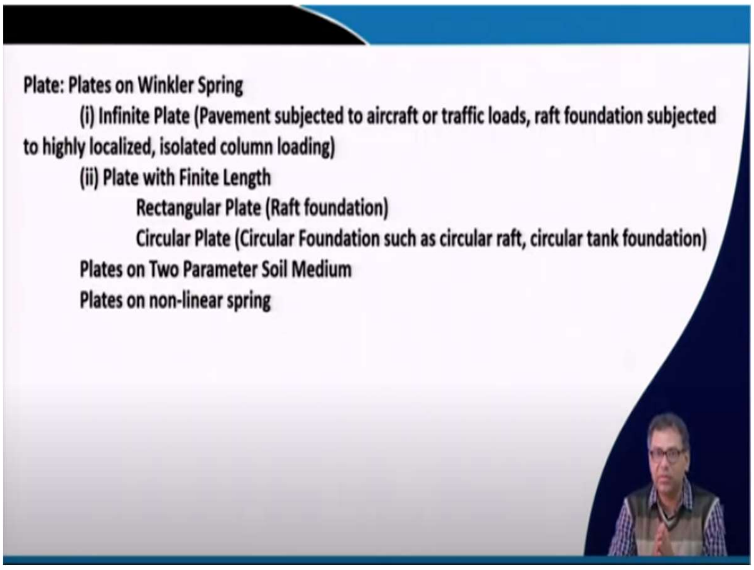

So, before I go to that part now as I mentioned that this foundation can be idealized either by beam or by plates and as the title of this topic is beams on elastic foundation or elastic medium, I will not discuss about the plate thing but you should know that the foundation not only can be idealized by beam. It can be idealized by plate also. Now in that case, the plate can be also infinite plate that means pavement subjected to aircraft or traffic loads and finite means raft foundation subject to highly localized load.

That means in a raft foundation where a particular area the loads are concentrated on that particular zone, then that can be idealized by finite plate then pavement can be idealized by infinite plate. Then the plate with finite length can be rectangular plate with finite *l* and *b*, then circular plate that can be the circular raft foundation, tank foundation. And similarly, it can be on the two-parameter model and the non-linear spring time dependent effect all these can be incorporated.

That means the circular rafts, circular tanks those can be idealized by plate also. And I have already discussed that what are the conditions that can be idealized by beams. So, these are the conditions which you can idealize by the plates.

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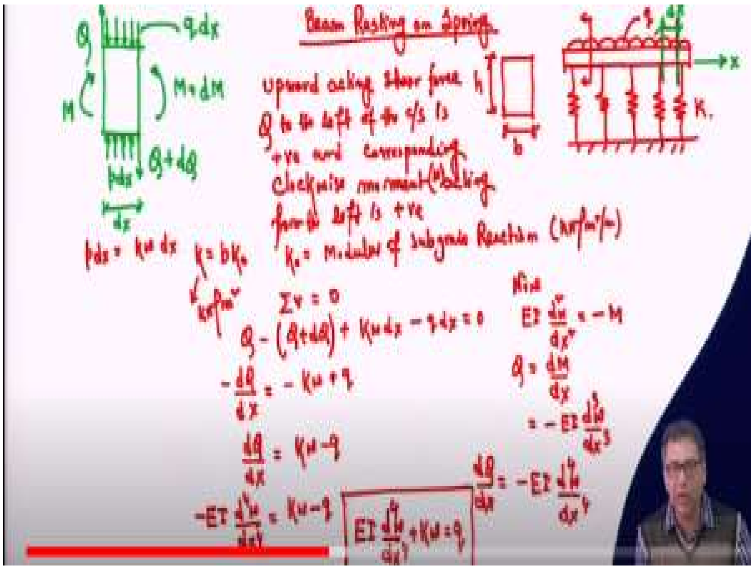

But I will discuss first, the beam resting on spring. So, this is the topic that beams resting on plate, sorry beam resting on spring. So, that means this is a beam which is resting on springs and whose subgrade modulus is k and the beam is subjected to UDL, q. Now, here this beam has a finite width. So, that means if I draw the cross section, so this will be the cross section which has a finite width b and a thickness h.

And then we have to go for some sign convention also and if I take one small element say, on a small segment over the beam. So, this is a small segment that I am taking. So, if I draw this small segment suppose this is the small segment I am taking. And that segment thickness is dx. Because this is in x direction, so that thickness is dx. So, that means this thickness is basically dx. So, this is dx over which UDL, q is acting, so total load acting on the top surface of the beam due to this UDL will be $q \times dx$. So, that is qdx.

So, that is the UDL which is acting. So, let me draw in a better way, so this is the UDL acting that is $q \times dx$ or I should write this as $q \times dx$. And there will be a reaction that will act at the bottom of the beam. This spring will give me the reaction. So, there will be a reaction that will be at bottom of the beam and that reaction will be $p \times dx$.

The reaction is p and the total reaction force is $p \times dx$ and in this segment there will be a moment. This is the left side where, the shear force Q and moment, M are acting. On the right side, there will be a shear force $Q + dQ$ and there will be a moment $+dM$. So, these are the forces or the free body diagram of the small segment. That means the left side shear force and moment are Q and M, respectively, whereas, the right side shear force is $Q + dQ$ and the moment is $M + dM$ and these are the reaction forces.

Another is the external force acting on the small segment. Now the sign convention that you have to consider is that the upward acting shear force Q to the left side of the cross section or left of the segment is positive and the corresponding clockwise moment acting on the left side is also positive. So, what does it mean that the left side upward acting shear force is positive and corresponding bending moment which is clockwise bending moment acting on the left side is positive.

And then we will consider the anticlockwise moment acting on the right side is positive. Clockwise moment acting on the left side is positive and anticlockwise moment acting on the right side is positive, remember that. So, now if I take this qdx this q is basically the stresses acting on a Winkler spring. So, the stresses acting on a Winkler spring is equal to $k \times w$.

So, this $qdx = k \times w \times dx$. And now if you see that unit of q is kN/m, so that means if I multiply it will be kN. And then this k unit is kN/m^3. Now if I multiply with w it will be kN/m^2, if I multiply with dx it will be kN/m. So, that means this unit is not matching. So, actually, this k is basically $b \times k_0$.

So, $k = b \times k_0$ where k_0 is the modulus of subgrade reaction whose unit is kN/m^2/m. So, k unit is basically kN/m^2 because k_0 is multiplied with the b. So, remember that this k is k_0 × width of the beam, b. So, now if I take the vertical component of all the forces 0 then I can write $Q - (Q + dQ)$ acting in the upward direction.

Then $+kwdx - qdx = 0$. So, if I simplify that I will get $-\frac{dQ}{dx} = -kw + q$. And then I can write $\frac{dQ}{dx} = kw - q$. So, now, $EI\frac{d^2w}{dx^2} = -M$ and $Q = \frac{dM}{dx}$, where M is the bending moment and Q is the shear force. The Q is the shear force and M is moment, so this shear force is $\frac{dM}{dx}$.

So, I can write that $Q = \frac{dM}{dx} = -EI\frac{d^3w}{dx^3}$. And there is a $\frac{dQ}{dx}$ term, so $\frac{dQ}{dx}$ will be finally $-EI\frac{d^4w}{dx^4}$. So, now you will put this here. So, this will be $-EI\frac{d^4w}{dx^4} = kw - q$. So, further if I simplify this the equation is $EI\frac{d^4w}{dx^4} + kw = q$. So, this is the basic differential equation for a beam resting on spring.

So, this is the basic differential equation for a beam. Now first I will explain that how you will get the closed form solution and then I will explain how you can solve these equations numerically also. I will explain how you can solve these equations by using finite difference technique. But here I will give you the idea because this is the introduction of beams on elastic foundation.

I will not go in detail. Suppose if you want to get all this detailed information then those things are being discussed in another course that is soil structure interaction. So, you can go to the soil structure interaction part or course and you will get detailed description of beams, plates and all the nonlinear models, time dependent model everything. But here it is the introduction so I will just give a brief discussion on these beams on elastic foundation.

So, this is the basic equation and then I will try to get the closed form solution and then I will discuss how you will get the solution by numerical technique.

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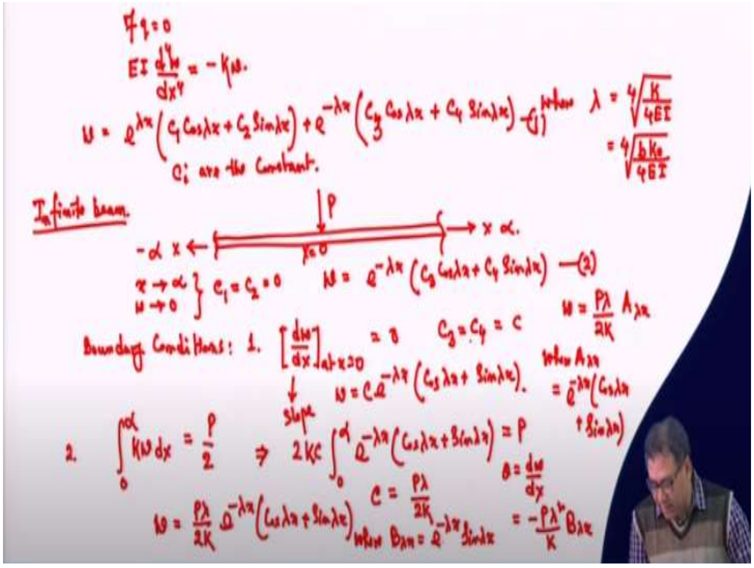

So, once I get this basic differential equation now if my q = 0 then this equation will be $EI\,\frac{d^4w}{dx^4} = -kw$. Now we can get a closed form solution of this equation, particularly this equation where external load is taken as 0, because that loading effect will be incorporated now by using the boundary condition. But right now, q is considered as 0 and I will get a closed form solution.

This type of equation is well known so that means we have to take $w = e^{mx}$. So, if I take $w = e^{mx}$ then I will get the imaginary part and the real part and finally I will get the solution. I am giving the direct solution of this particular equation. I am not going to detailed derivation of this particular solution. This derivation is available in any book or you can also do this derivation.

Because it is derivation, you know that you have to consider $w = e^{mx}$, so I am giving that mx and then you will get the solution of this particular equation. So, the closed form solution of this particular equation is $w = e^{\lambda x}(C_1 \cos \lambda x + C_2 \sin \lambda x) + e^{-\lambda x}(C_3 \cos \lambda x + C_4 \sin \lambda x)$.

So, your $\lambda = \sqrt[4]{\frac{k}{4EI}}$. What is EI? As I mentioned in the previous lecture EI is the flexural rigidity of the beam. So, that means here E is the elastic modulus of the beam material and I is the moment of inertia. So, in this particular case the I value will be $\frac{bh^3}{12}$, which is moment of inertia where b is the width of the foundation, h is the thickness of the foundation.

So, this is your *I* value and *E* value. So, *EI* is called the rigidity of the beam or specifically I should say flexural rigidity of the beam. So, now this is the 4th root or you can write this as $\sqrt[4]{\frac{bk_0}{4EI}}$. So, if you have the k_0 then you have to multiply by *b* to get the λ value and then C_1, C_2, C_3, C_4 are the constants. So, these four C_i are the constants.

So, this is the basic solution for this particular equation where $q = 0$ and C_i means C_1, C_2, C_3, C_4 and these constants we have to determine by our boundary conditions. So, first I will go as I discussed that there are three types of beams infinite, semi-finite and the finite beam. So, first I will discuss about the infinite beam then I will give an idea about the semi-infinite beam then I will give you an idea about the finite beam.

Because this semi-infinite and finite beam concept will be used when I will discuss laterally loaded piles. Because in such conditions piles will be idealized by semi-infinite or finite length beam and then we will determine the settlement of the pile due to the lateral load. So, that is why giving that concept is very important in this particular section. So, later on you can use that concept to determine the settlement of the laterally loaded pile.

So, first, I will go for the infinite beam. So, suppose this is infinite beam. This is the *x* direction and both are the *x* directions. It is infinite and a concentrated load is acting. And this is infinite, this is also infinite. Positive and negative is not an important thing but both are infinite so, this is *P*. If *x* tends to infinite then *w* will tend to 0 because infinite distance there will be no influence of this *P*, so that your deformation will be 0.

And if you look at this equation, this is the equation that is given but the previous equation it is not for infinite or finite. It is a general equation. Now in this equation *w* can be 0 if *x* tends to infinity is only possible if C_1 and C_2 are 0. Otherwise this is not possible because this is $e^{\lambda x}$, $+\lambda x$. So, if you have some finite value or any value of C_1 and C_2 in this term then it will be never be 0, if *x* is infinity.

And this can be only possible if the C_3, C_4 are present. Because here is your $e^{-\lambda x}$. So, there is a possibility that if your x tends to infinity then w will tend to 0. But in the first term it is not possible. So, we have to vanish the first term and this is only possible if C_1 and C_2 are 0. So, this condition is only possible if $C_1 = C_2 = 0$. So, that means this equation is general as I mentioned.

So, you can use it for finite beams also. So, that means for an infinite beam this equation will be reduced to $w = e^{-\lambda x}(C_3 \cos \lambda x + C_4 \sin \lambda x)$. So, that means this will reduce to two constants. Again, these two constants we have to determine by using boundary conditions. So, what are those boundary conditions? So, the first boundary condition is due to symmetry because you can say the point where the load is acting you can say this is a zero point or starting point and it is symmetric.

Now because as the beam is infinite, so at whatever location you place the P load, it will be the center point of the beam and that means it is symmetric. So, because of your symmetric conditions, slope will be 0 at the point of application of load. So, I can write slope $\frac{dw}{dx}$ at $x = 0$ because this is $x = 0$ where the load is acting. So, now if I put the slope and if I make $\frac{dw}{dx}$ and then if you differentiate this equation $\frac{dw}{dx}$ and put $x = 0$ then I will get $C_3 = C_4$. So, you differentiate this equation or equation 2.

So, if you differentiate equation 2 and put $x = 0$ and that differentiation I mean that slope is also 0 then you will get $C_3 = C_4$. And that I can write as C. So, after application of first boundary condition this equation will be $w = Ce^{-\lambda x}(\cos \lambda x + \sin \lambda x)$. Now I will apply the second boundary condition. The second boundary condition is that there will be a reaction which is given by the spring and so that reaction is $k \times w$.

That $k \times w$ is the reaction and for a small section of dx and total reaction force is $\times\, w \times dx$. Now, we have to integrate from 0 to infinity. So, if I integrate that total reaction from 0 to infinity then I will get the force that is acting from the top. But that I will not get P that I will get $\frac{P}{2}$ because there is other half also, I have taken 0 to infinity that is why I will get $\frac{P}{2}$. So, if you take the other half then that other half will also contribute $\frac{P}{2}$.

This left half is taking the $\frac{P}{2}$ and the other half is taking $\frac{P}{2}$. So, that is why I am taking $\frac{P}{2}$. That means here if you put these equations *w* here, so this will be $2kC\int_0^\infty e^{-\lambda x}(\cos\lambda x+\sin\lambda x)=P$. So, after integrating this equation and then putting the limits I will get the constant $C=\frac{P\lambda}{2k}$. So, after getting these things I will get $\frac{P\lambda}{2k}$.

So, finally, the equation will be $w=\frac{P\lambda}{2k}e^{-\lambda x}(\cos\lambda x+\sin\lambda x)$. So, this is the equation of *w* and I can write this *w* as $\frac{P\lambda}{2k}A_{\lambda x}$, where, $A_{\lambda x}=e^{-\lambda x}(\cos\lambda x+\sin\lambda x)$. So, this is $A_{\lambda x}$. Similarly, I will get the slope also that means I am just giving how I will get the slope? So, slope is nothing but $\frac{dw}{dx}$.

So, I have the *w* equation. I will do the $\frac{dw}{dx}$ and then you will get the slope. And this slope is coming out to be $-\frac{P\lambda^2}{k}B_{\lambda x}$ where $B_{\lambda x}=e^{-\lambda x}\sin\lambda x$. So, this is $A_{\lambda x}$, this is $B_{\lambda x}$ and equation for slope is given. Similarly, I am giving the expression of moment and the shear force.

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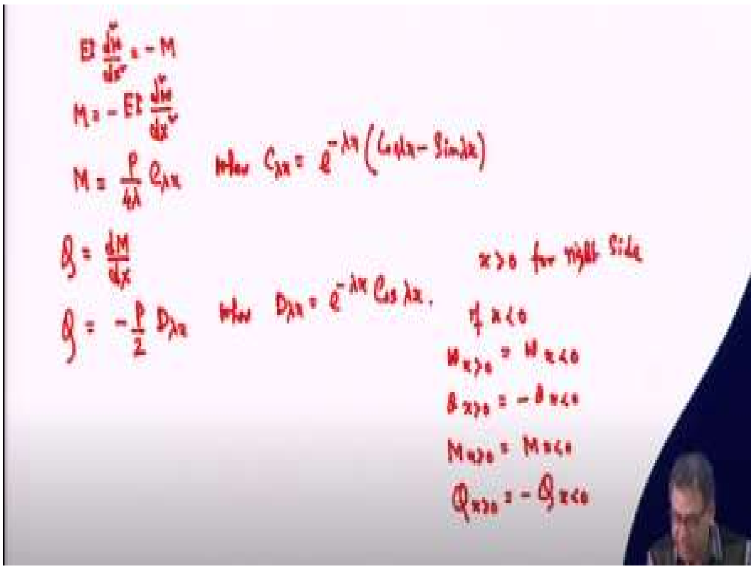

So, you know the moment is $EI\frac{d^2w}{dx^2}=-M$. So, $M=-EI\frac{d^2w}{dx^2}$. So, now if I do the $\frac{d^2w}{dx^2}$ and then put those values and simplify it then I will get the equation of *M*. That is $\frac{P}{4\lambda}C_{\lambda x}$ where $C_{\lambda x}=$

$e^{-\lambda x}(\cos\lambda x - \sin\lambda x)$. Similarly, I will get the shear force which is basically $\frac{dM}{dx}$.

So, once you do the $\frac{dM}{dx}$ then you will get shear force. So, shear force will be $-\frac{P}{2}D_{\lambda x}$ where $D_{\lambda x} = e^{-\lambda x}\cos\lambda x$. So, these are the equations when the infinite beam is subjected to concentrated load. And remember that all the equations that I have given are applicable if your $x > 0$ that means for the right side. That means for the right side these equations are given.

For the left side, if your $x < 0$ then all these equations will remain the same, only the sign will change. How? That means $w_{x>0} = w_{x<0}$. That means for deflection there will be no change. But for slope $\theta_{x>0} = -\theta_{x<0}$. Similarly, for the moment $M_{x>0} = M_{x<0}$.

For shear force $Q_{x>0} = -Q_{x<0}$. That means if it is the right side or the left side, so suppose if you want to determine the deflection of a point which is away from the loading point. Because due to the loading there will be deflection at other points also. So, I can determine the deflection of that particular point. If I want to determine the deflection of the loading point itself then I have to put $x = 0$.

So, if I put $x = 0$ means $\sin 0$ is 0 and $\cos 0$ is 1, so that means at $x = 0$, $w = \frac{P\lambda}{2k}$. Similar way I can determine the deflection or bending moment or slope at any point away from the loading point. Now if your point of interest is on the left side of the loading point, then these equations you can use directly.

But if your point of interest is on the right side of the loading point, then you can use these equations directly. If your point of interest is on the left side of the loading point, then your deflection will be the same, bending moment will be the same.

But in case of shear force and the slope your sign will change. That means whatever values you are getting for the right side that means $x = x > 0$ you have to put minus for left side for the same distance from the loading point. Clear? So, this you have to remember. So, in the next class, I will discuss other few loading conditions like I will give the equation for moment, I will give the equation for the UDL and then I will go for the semi-infinite beam and finite beam.

I will give the idea how you will get the bending moment or the other quantities that is settlement by using this infinite beam concept. Thank you.