Hydraulic Engineering
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Lecture # 61
Introduction to wave mechanics (Contd.)

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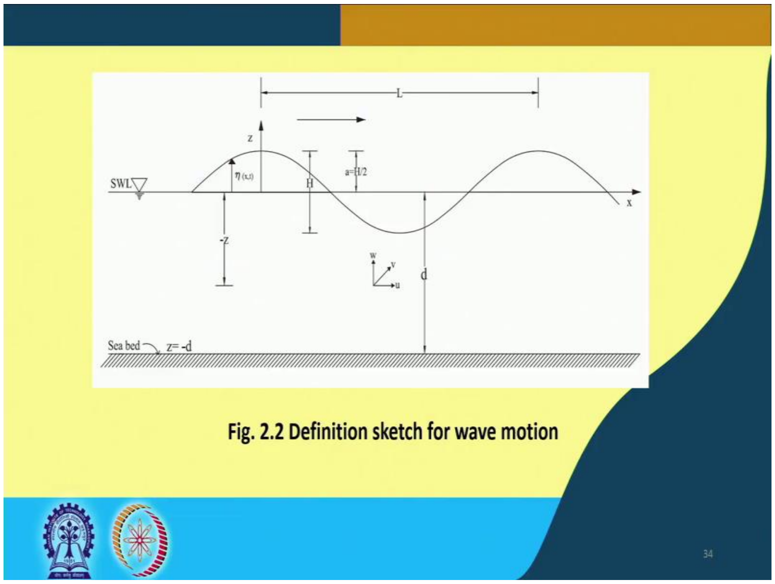

Welcome back students for this third lecture of this module, where we are deriving the velocity potential of the ocean surface waves. Last time we concluded with this particular slide indicating the definition sketch of the wave motion, where we have told you what the water depth is this figure is actually very important for understanding how this linear wave is represented. The wavelength is given as L at the top the depth is indicated wave height H is also given no sea bed is shown so we from this point onward will continue with our derivation.

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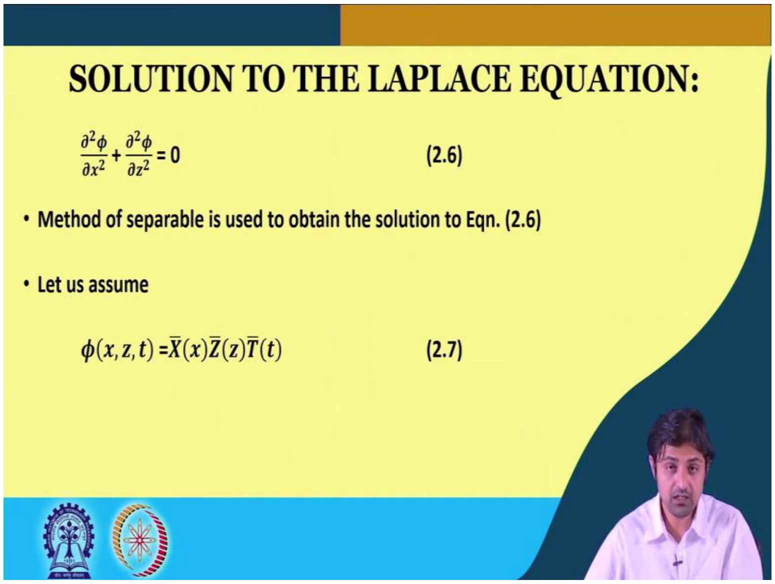

Now we have listed the boundary conditions one was w = 0 at the bottom second was we specified free surface dynamic free surface boundary condition at the top. And now we start with the solution of the Laplace equation. So, Laplace equation in terms of velocity potential in 2 dimension, as we have assumed x and z is $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$.

To obtain the solution a very simple method of separable is used to obtain the solution to this equation which is 2.6. Our let us assume, main method of separable phi of x, z, t can be written as X bar as a function of x it can be separated Z bar as z, T bar as t.

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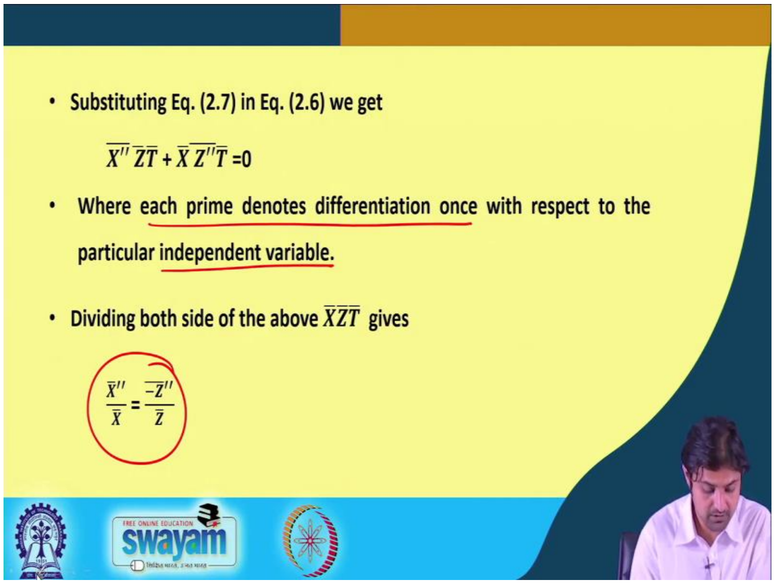

Now, if we substitute 2.7 this equation into 2.6 because due to the velocity potential Laplace equation properties, we are able to separate X, Z and T using the method of separable what do we get x double bar. So, x double prime bar Z bar and T bar + X bar Z double prime whole bar and T bar = 0. Here each prime denotes differentiation once with respect to the particular independent variable for double means double differential. Now, if we divide both sides by X bar, Z bar and T bar, we get X bar X double prime bar by X bar = - Z double prime whole bar by said this is the equation that we get.

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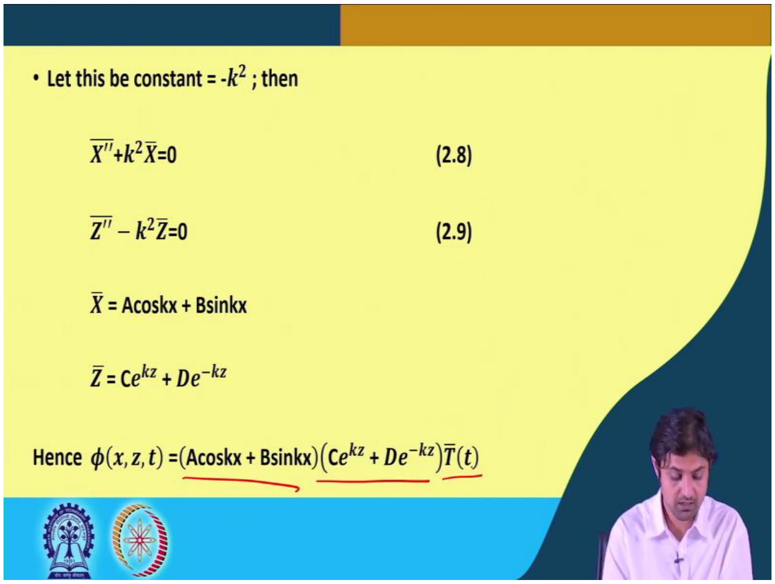

Now, see this is we let us say this is a constant - K square, then we can write 1 of the equations from this will come X double prime bar + k squared x bar = 0 and the second one will be Z double prime bar - k square z bar = 0, these 2 are very famous equation the solution for this is standard. So, x bar from 2.8 can be written as A cos k x + B sine k x. Whereas Z from equation number 2.9 solution to this type of equation is here solution to the equations of type 2.9 is this one from equation 2.9.

We get X bar and Z bar so, we can simply write so, you see we had the solution that we all initially assumed was phi x, z, t is X bar, Z bar and T bar so we can write instead of X bar we can write this one instead of Z bar we can read this one and T bar remains as T bar.

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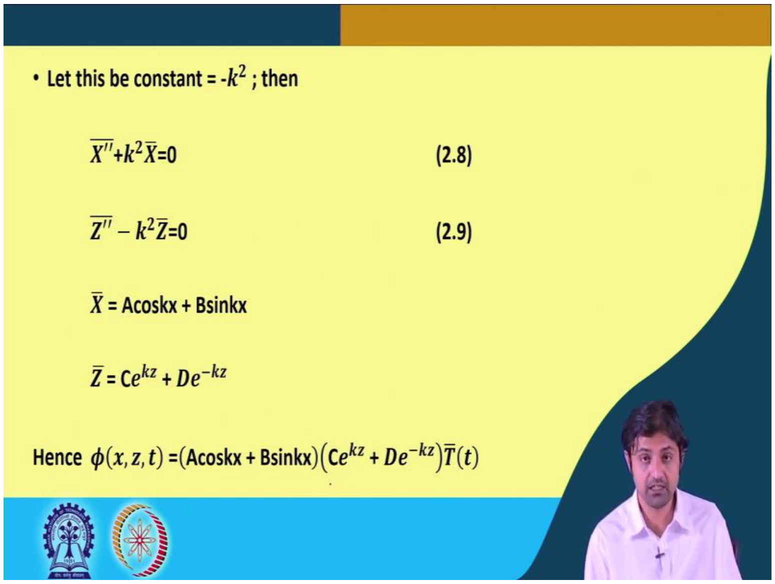

Now, the solutions to phi are simple harmonic in time, that is, because we know from previous that they are waves and periodic in time. Therefore, T bar can be simply replaced as cos t or sigma t anything it can be replaced, and therefore with each of the values of sin t or sigma t, we have 4 forms of phi one will be A cos k x multiplied by sin, the other would be A cos k x multiplied by cos. The other would be B sin k x multiplied by sin and then the third the fourth one would be B sin k x multiplied by sin.

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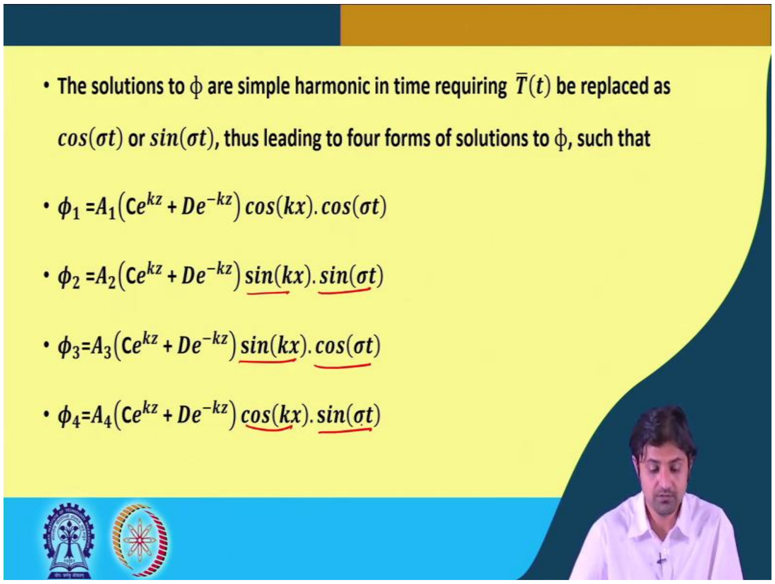

So we can write 4 different terms A1 we did not touch this, so A cos k x + B sin k x multiplied by either sin sigma t or cos sigma t. There are going to be 2 terms when we assume t as a function of sin sigma t, and they would not mean after multiplication with this and there will be 2 other term is if instead of sin we assume cos t bar T. So total we can write 4 terms, phi 1 will be A1 into C e to the power k z cos k x cos sigma t. Another thing would be A2 C e to the power k z + D e to the power - k x sin k x sin sigma t.

The third would be sine k x cos sigma t and fourth would be cos k x sin sigma t. However the total velocity potential is going to be some of 2 of those which one we will come to it later. But for now, we have assumed 4 different terms phi 1 phi 2 phi 3 phi 4, 1 would be phi 1 - phi 2 the other could be phi 3 - phi 4, but that will come to later.

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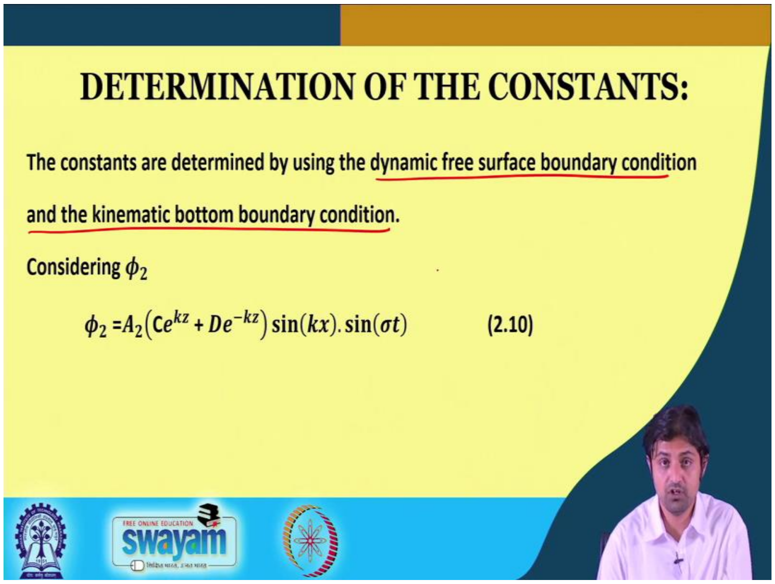

So, we have to see this these terms phi 1, phi 2, phi 3, phi 4 are constants A1, C, D, sigma is related to the time period case rate to the wave period that will come to it later. But more importantly we have to find these constants A1, C and D. So, how did it remind these constants, the constants are determined using the dynamic free surface boundary condition and the kinematic bottom boundary condition as you would remember, because, by the definition of the boundary condition, we said that, there could be infinite number of solutions to a governing equation?

However, with the application of boundary condition, we can find one or more unique solutions to the problem that is what we are going to do. These constant will be determined from the boundary Conditions let us consider phi 2 is one velocity potential that we have written here this one. So, what does it say phi 2 is A2 into C e to the power k z + D e to the power - k z into sin k x into sign sigma t we call this equation number 2.10.

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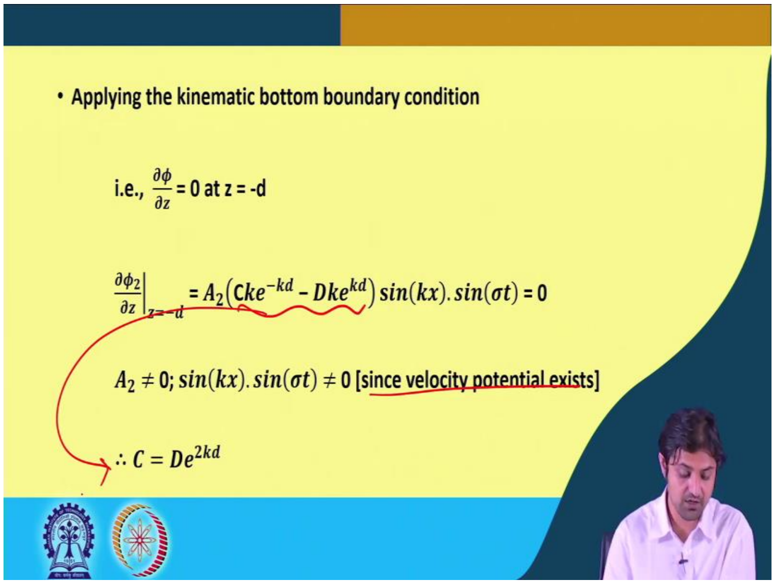

If we apply the kinematic bottom boundary condition, which means fields the velocity potential W is given by del phi del z and that will be 0 at z = - d. If you apply that del phi del z at z = - d it will be A2 into C e to the power - k d - D k e to the power k d into sin k x sin sigma t = 0 = - d. So, now there is no Z but instead of Z we have d. So, A2 is not equal to 0 otherwise there is not going to be any wave also sin k x sin sigma t will not be 0 because there will be a velocity potential if we substitute them to be either one of them to be 0 the velocity potential becomes 0.

So that means these 3 terms cannot be 0. Therefore, the only thing is that this be 0 and if we utilize this we are going to get C as D into e to the power k d. So with one boundary condition we have reduced our unknown by 1 number. C = we have related C and D.

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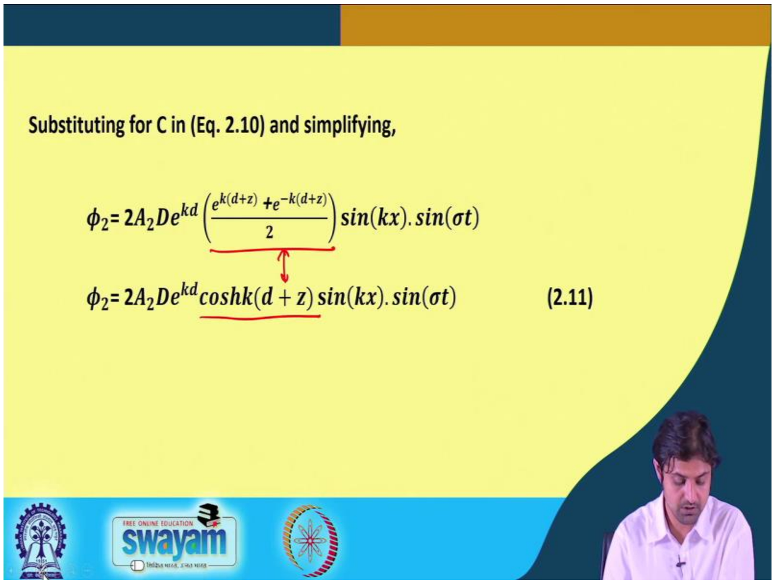

Now if we substitutes for C in equation 2.10 so instead, of CV put 2d into what the term that we have got we are going to get phi 2 = 2 A2 D into e to the power k d into sin k x sin sigma t or simply we can write this is nothing but cos h k d + z. Simple trigonometry.

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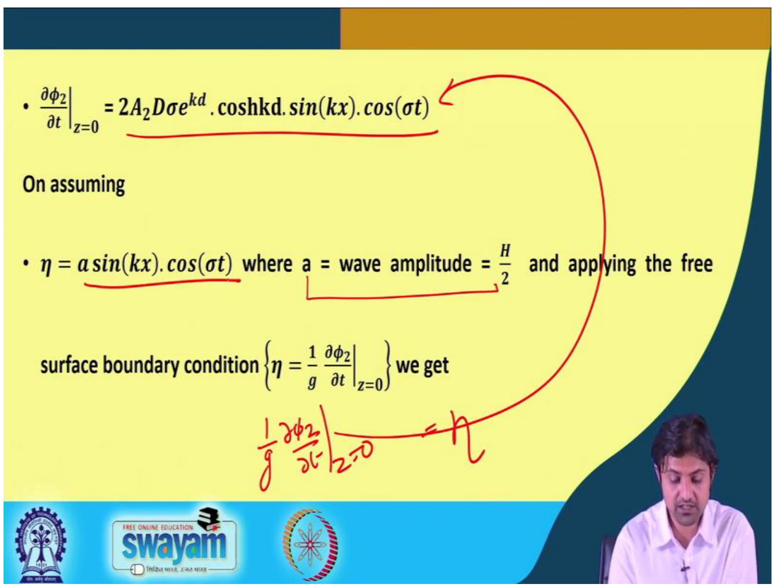

So, we have got phi 2 is 2 A2 into D e to the power k d cos h k d + z so we are left with A2 and D for now. And if we do del phi del t at Z = 0. We get 2 A2 D sigma e to the power k d cos h k d sin k x cos sigma t and on assuming if we assume that free surface elevation is A sin k x into cos sigma t. So, what we are going to do here is the wave amplitude that is H by 2 this is our assumption and by applying the free surface boundary condition that eta = one by g del phi 2 del t and Z = 0.

So, first we are applied the bottom boundary condition. On the second step we want to apply the dynamic free surface boundary condition. So, for applying the dynamic free surface boundary condition we need to calculate this term one by g del phi 2 del t at Z = 0. And this is what we get this term and this we equate to eta is the free surface which = A sin k x into cos sigma t.

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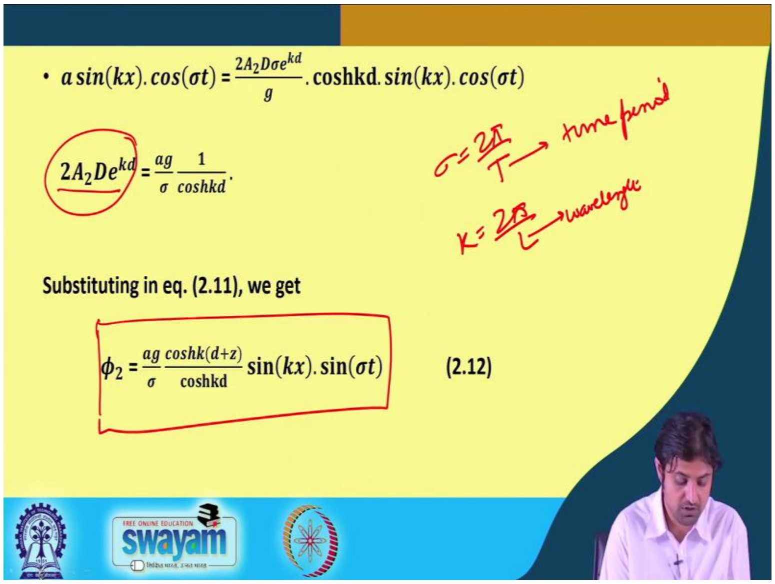

So simply what we do eta = and after solving for this we get 2 A2 D e to the power kd is ag by sigma into 1 by cos h k d. And if we substitute these values into our original phi 2 we are going to get it as phi 2 = ag by sigma cos h k d + z cos h k d into sin k x into sin sigma t. So, we have now, first what we did we solve for Laplace equation applied the 2 boundary conditions and obtain the value of the velocity potential phi 2 which is of the form ag by sigma into cos h k d + z into cos h k d sin k x into sin sigma t.

So, sigma is 2 pi by T and k is 2 pi by L this is wavelength and this is time period for you the derivation is not important what the results are, but the derivation is important in a way that it makes you appreciate how starting from the basics we can derive these equations they will ask the potential.

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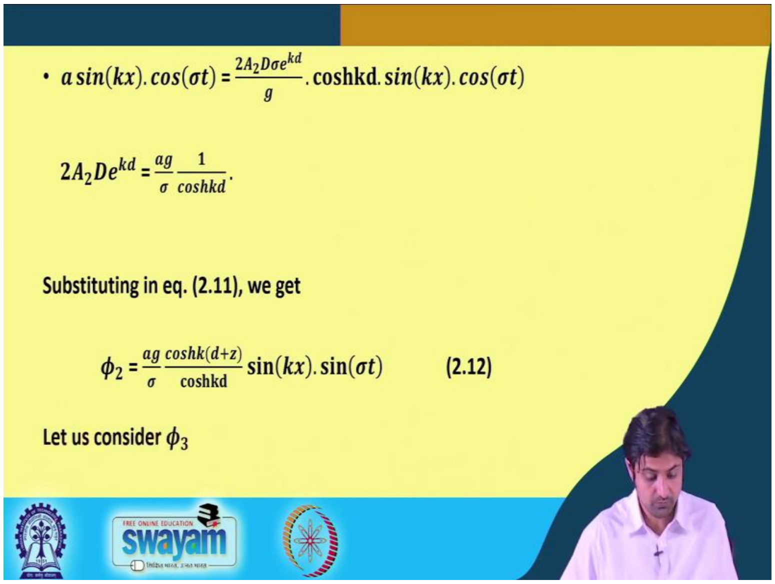

I do not expect you to remember the derivation but the steps you must know the things like the specifying the governing equation, what are the governing equations governing equation is Laplace equation for the boundary conditions we utilize the Bernoulli's equation and the continuity equation for example, so, we have obtained phi 2 here.

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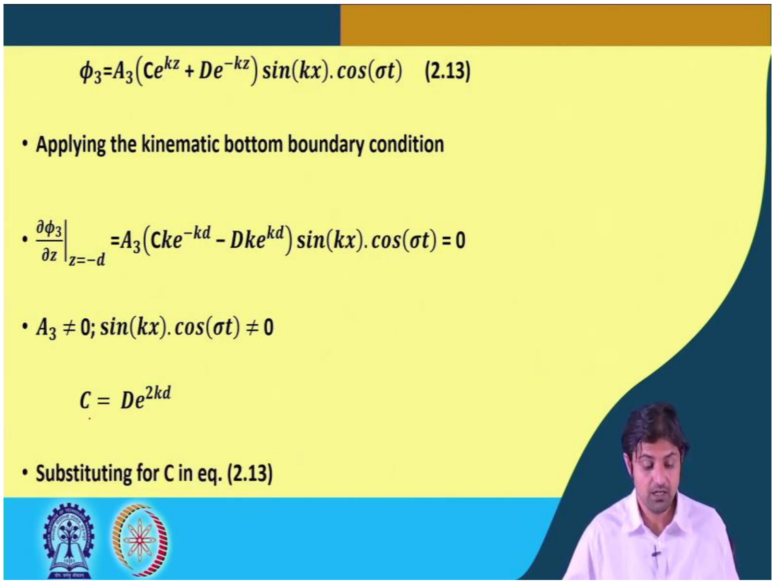

So, if we consider phi 3 and apply the same concepts that phi 3 = you know, we apply the dynamic boundary condition. This will give C equal to D into e to the power 2 k d.

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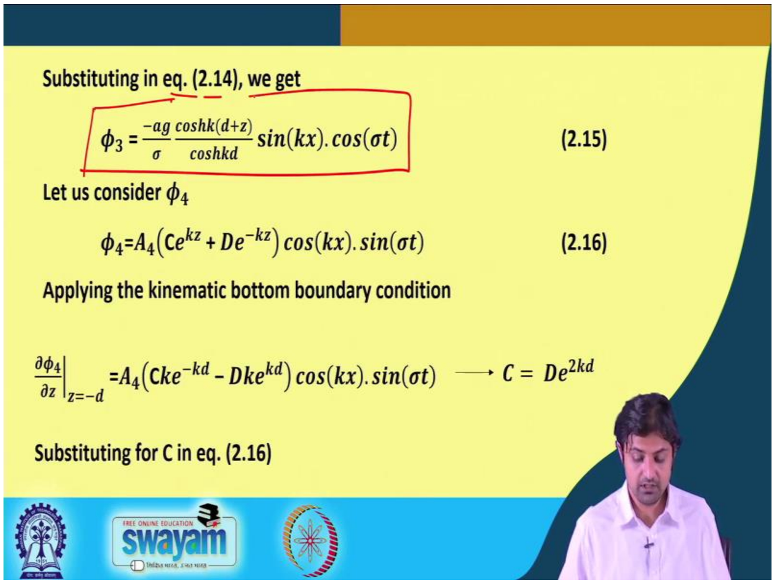

And same procedure is repeated for the dynamic free surface boundary condition and we get for phi 3 at term like this, you understand same procedure. So phi 3 we get - ag by sigma cos h k d + z similarly, we get phi 3. And same procedure we do for phi 4, for obtaining the values.

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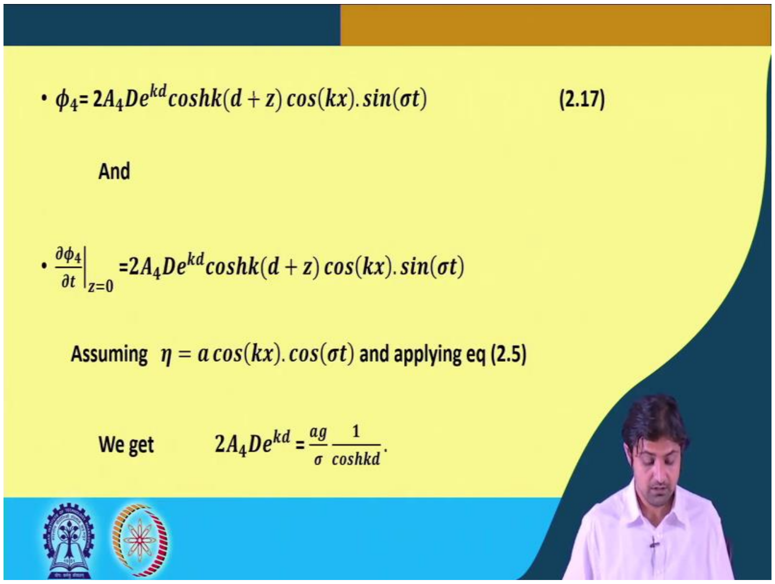

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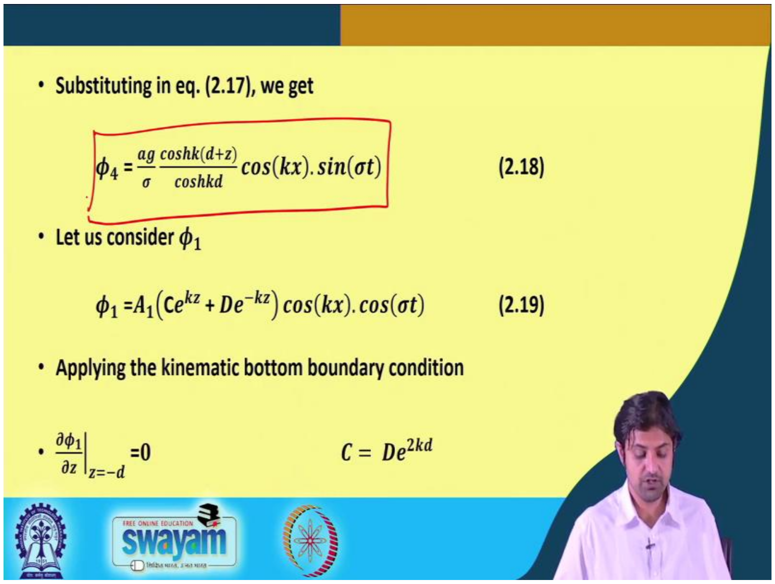

And what we get phi 4 is ag by sigma a similar result we get and we repeat the same procedure again for phi 1.

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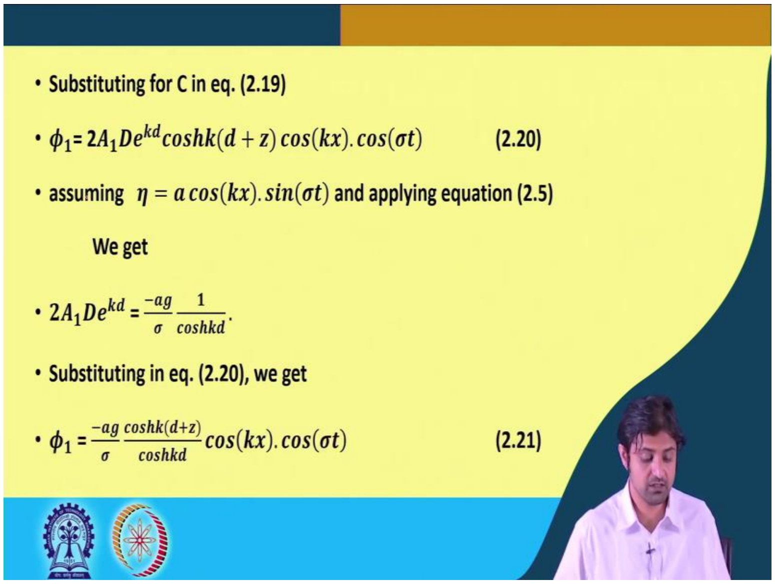

Applying the kinematic bottom boundary condition first and then applying the dynamic. See I am skipping because it is essentially the same process.

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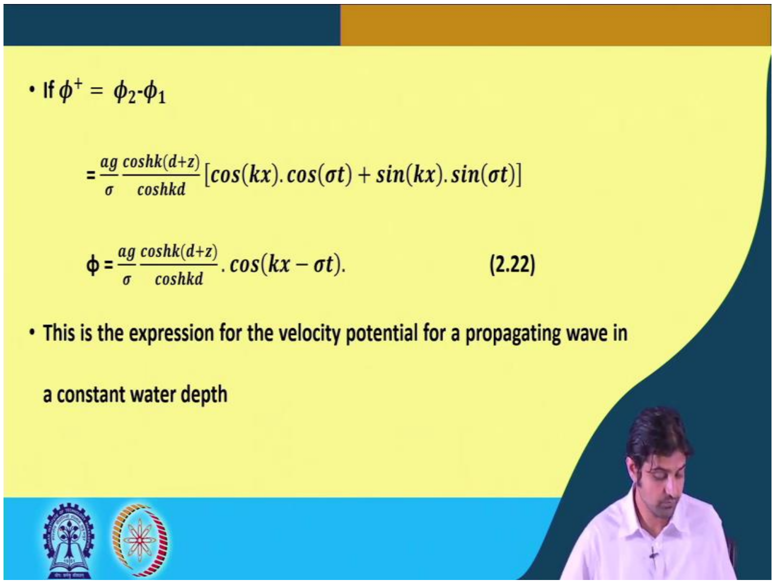

More important is this term so phi 1 we get is - ag by sigma cos h k d + z if you remember, phi 2 and phi 4 was positive with a positive sign phi 4 and phi 1 was with the negative sign after the derivation. Now you remember I said that the total velocity potential will be the summation of the two terms. So our velocity potential is going to be phi 2 - phi 1, or also in terms of phi 3 and phi 4 also so this becomes, so if you add to velocity potential, this was you know, phi 2.

And, this one with a negative sign was phi 1 so, we get this so, cos k x cos sigma t + sin k x sin sigma t can be return as cos k x - sigma t using trigonometry. So, the final velocity potential with the value I mean the formula for which you are supposed to remember is this 1 ag by sigma cos h k into d + z divided by cos h k d into cos k x - sigma t this is the final velocity potential. So, now if you see we have determined phi is the amplitude of the wave.

Sigma is 2 pi by t k is 2 pi by L everything you know. So, as a function of x we have found out the velocity potential here as I said this is the expression for the velocity potential for a propagating wave in a constant water depth. So, this is at one particular constant water depth, this is the velocity potential which we have derived from hydraulic assumptions and conditions.

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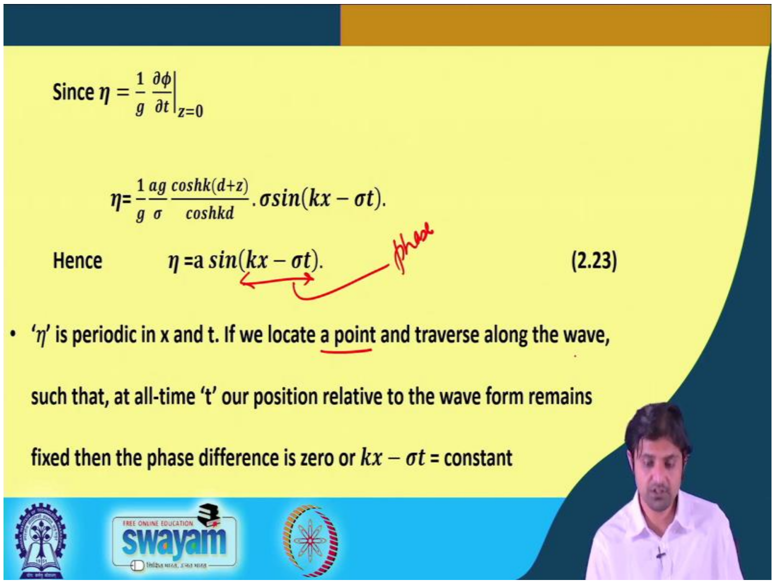

Now, because we have got the velocity potential we can finally write the term for eta, we are going to check so, eta can be return as so, 1 by g ag by sigma, so, del phi del t at z = 0 will come out to be this one. So, ag can be return as so, if you just g and g will cancel sigma and sigma will cancel. So, at z = 0, this will also get canceled. So, simply we left with n = a sine k x - sigma t.

So, now this eta is periodic in x and t, if we locate a point and traverse along the wave says that at all time t, our position relative to the waveform remains fixed. So, basically this term k x - sigma from your earlier mechanics class you would know this is what this is the phase k x - sigma t. So, with this phase we do x an experiment what we do is if we locate a point and traverse along the waves such that at any time t our position relative to the waveform remains fixed.

Then this means that k x - sigma t will be 0 phase difference is going to be 0 which means that k x - sigma t is going to be constant. Because if phase will be constant phase difference is going to be 0.

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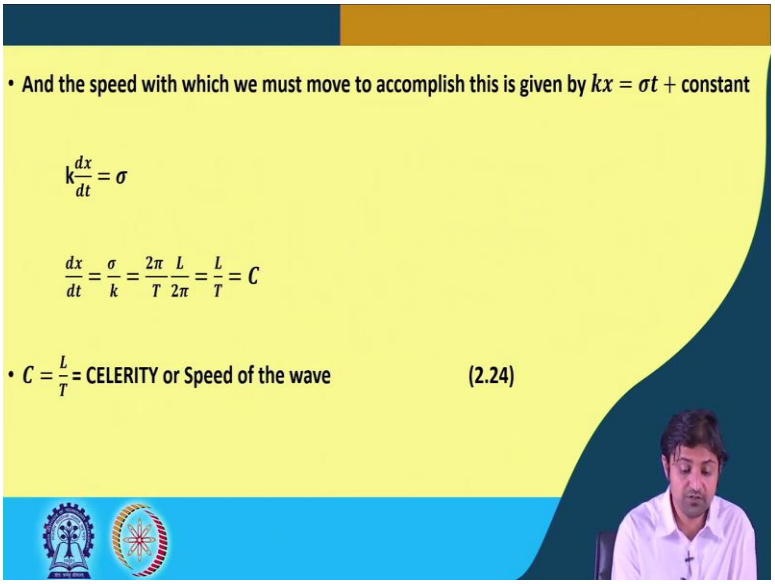

And the speed with which we must move to accomplish this will be given by k x - sigma t = constant or k x = sigma t + constant. On differentiating we can get this equation we differentiate we get k dx dt = we differentiate with respect to time = sigma or dx by dt can be written as sigma by k, and sigma was nothing but 2 pi by T and k was nothing by 2 pi by L. So, it is become L by T and that is the C and C is the wave celerity.

So, to determine the velocity of the wave, this is the so, this is how the wave celerity that is length wave length by time period is the celerity and this is the basis of finding the way of celerity that if locate a point and traverse along the wave such that at all time t our position relative to the waveform remains fixed when this will happen when we are going to move with the speed of the wave. And that our speed and wave speed will be the same in that case and that is what we have found out that C = L by T is the celerity of the wave with which we must be moving.

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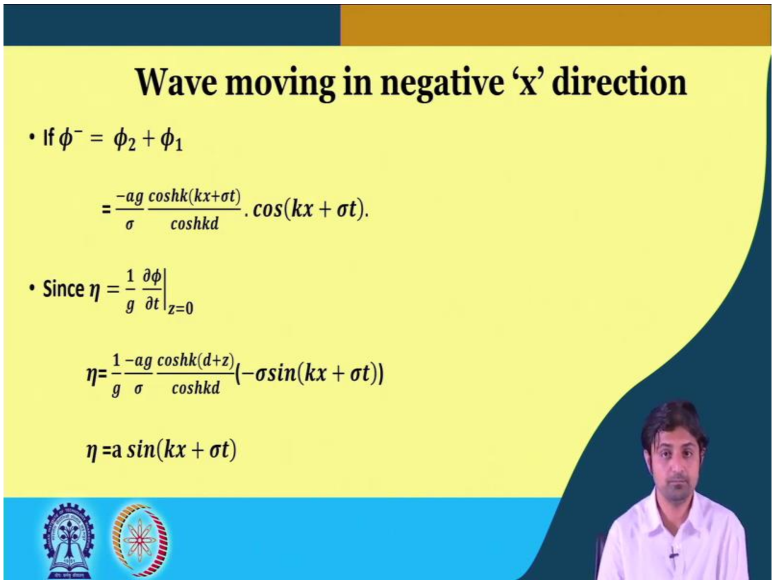

Now, if there is a wave that is moving in x direction, then we can simply write the velocity potential phi - = phi 2 + phi 1 not - phi 1 and then we will get nothing but with - sin - ag by sigma cos h k x sigma k x + sigma t into cos h k d. So, what you do is you do x = so, instead of k x + - sigma t becomes cos k x + sigma t. So, this is a different waveform. So, in this case, since a eta is

1 by g we will also get a similar you know, if there was it will remember it was k x - sigma t it becomes k x + sigma t. So, the phase is changing when the wave is traveling in the negative direction.

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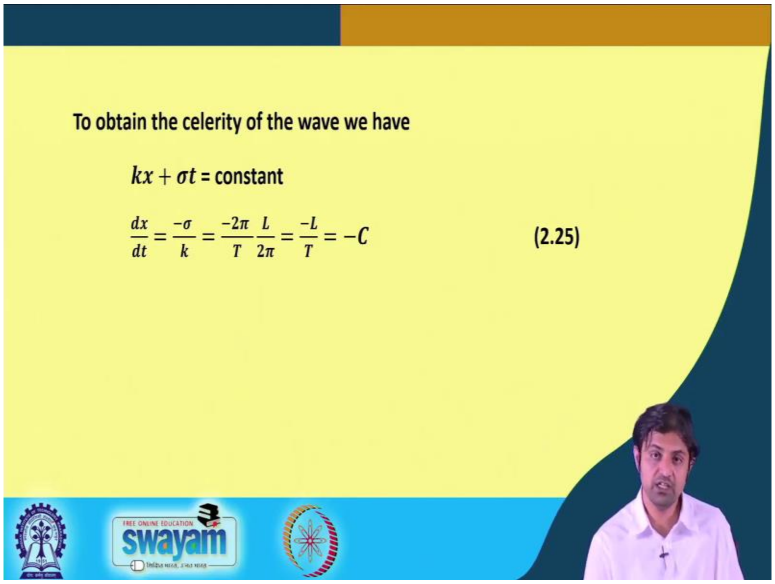

Now, to obtain the celerity of the wave in this case we have k x + sigma t = constant same procedure is repeated. So, it will become - c sonic wave negative means negative celerity in the negative x direction.

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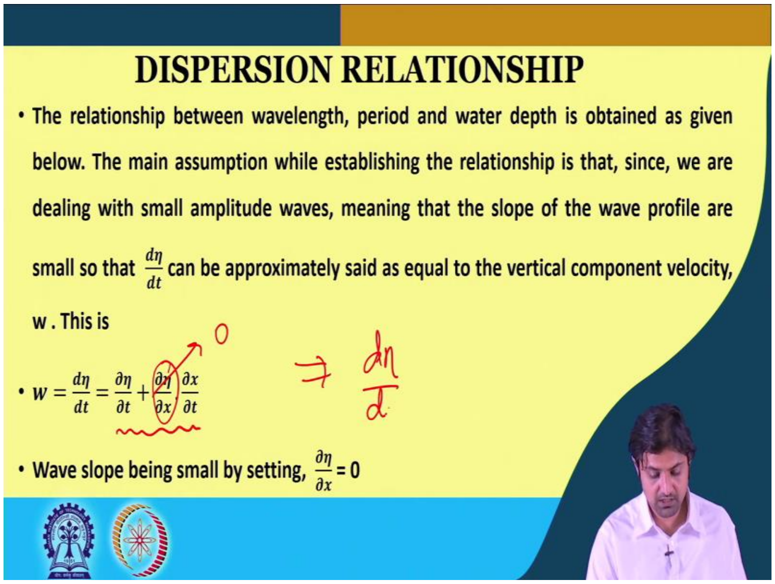

Now, one important thing after this derivation of the velocity potential is something called a dispersion relationship that is one of the core concept of the wave mechanics. So, the relationship between wavelength with period and water depth is obtained as given below for the dispersion relationship, the main assumption while establishing the relationship is that since we are dealing with small amplitude waves, meaning that the slope of wave profile are so, small that del eta by d eta by dt can be approximately said equal to the vertical component of the velocity w.

So, w can be written as d eta dt or in differential form del eta del t + w is what del eta by del t because that is the vertical velocity correct d eta dt. So, that if you apply the total you know, differentiation by you know, total differentiation, it will be del eta dt + del eta by del x to del x dt. So, wave slope being small means del eta by del x = 0 which implies d eta by dt = del eta by del t.

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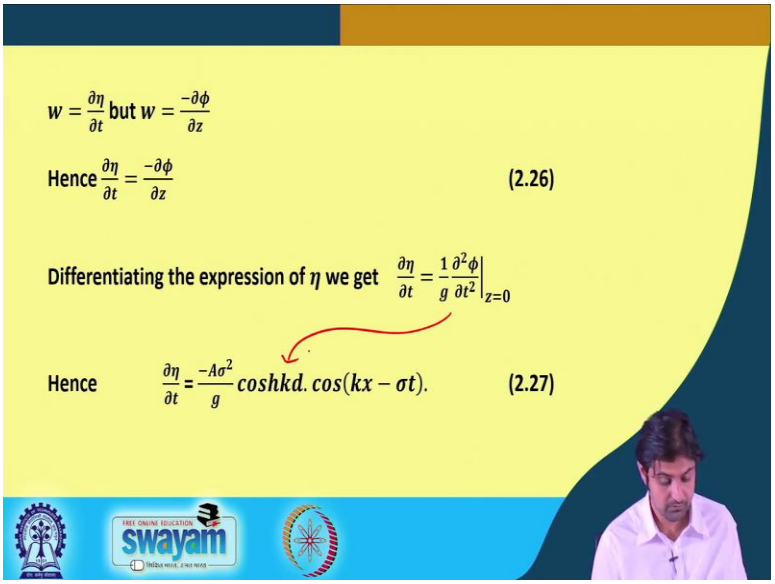

W = del eta del t but w also equal to - del phi del z in form of velocity potential hence del eta by del t equal to - del phi del z we know the velocity potential we know the equation of a eta, if we do the differentiation of a eta we get del t = one. So, if you do the differentiating the expression of eta, what do we get? Del eta by del t = 1 by g you remember, that term headset = 0 1 by g del square phi by del t square.

Eta was 1 by g del phi del t at z = 0. So, if you differentiate this you get del eta del t = one by g del t square phi by del t square that = 0 this is what it is written here so, we substitute del eta by del t we get from here.

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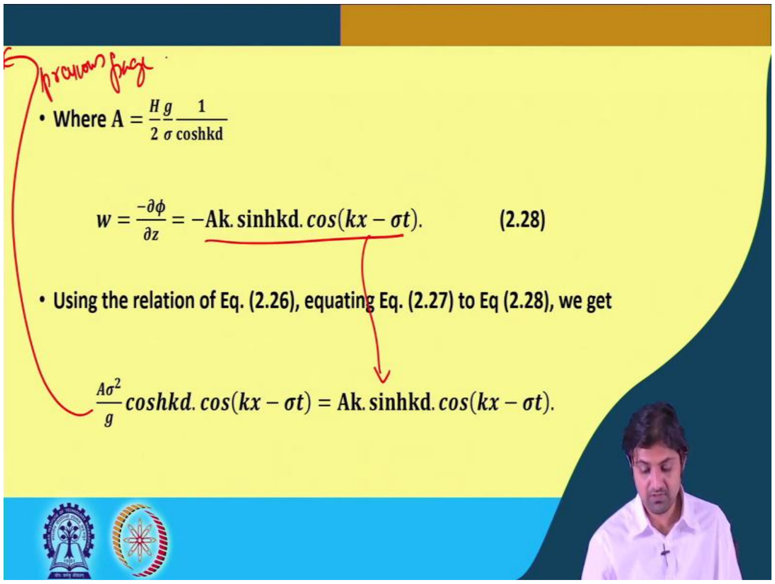

Where = be write capital A is where A is here is H by 2 into g by sigma divided one by cos h k d and w is also - del pi del z it that is - Ak sin h kd into cos k x into sigma t. And we equate this both these term A square sigma from the previous equation with this one here from the previous page because both are same.

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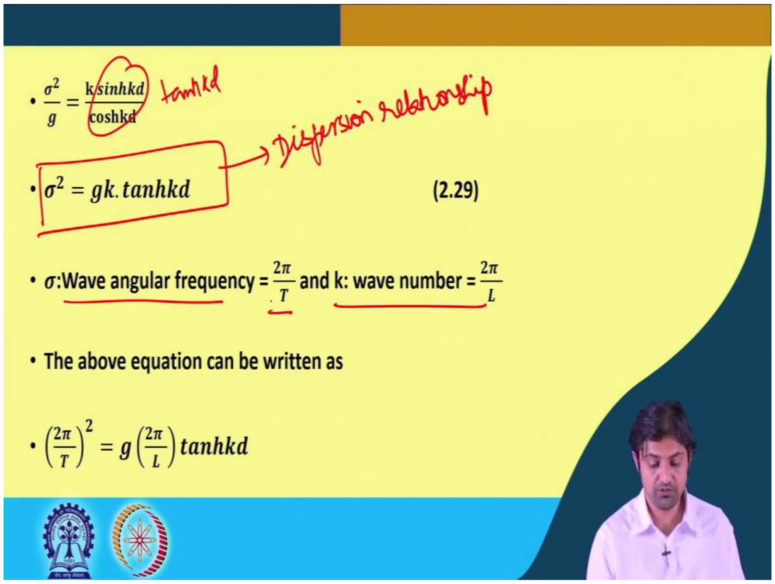

And on canceling the common terms what we get is sigma square by g = k sin h kd divided by cos h k d and this is nothing but tan h kd. So, sigma square can be written as g k into tan h k d. Here sigma is the wave angular frequency equal to 2 pi by T and k = wave number which = 2 pi by L as we have seen before and this above equation can be written as 2 pi by t whole square = g into 2 pi by L into tan h kd. So, actually this is the famous dispersion relationship.

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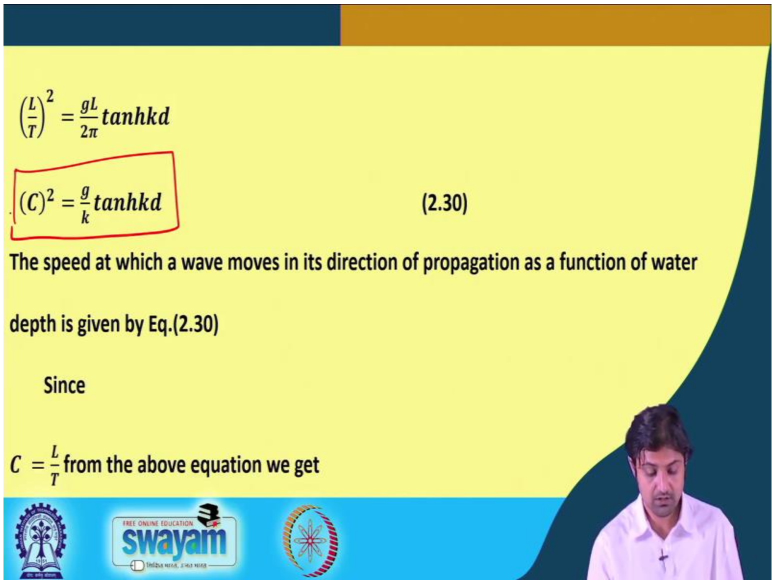

So, but is substituting since sigma and k in terms we get L by T whole square = gL by 2 pi into tan h k d or C squared =, because it can be written as C square = g by k into tan h k d. So, the speed at which a wave moves in the direction of propagation as a function of what a depth can be given by equation. Earlier we obtained the velocity potential at constant depth but now we have found out the speed of the wave as a function of water depth.

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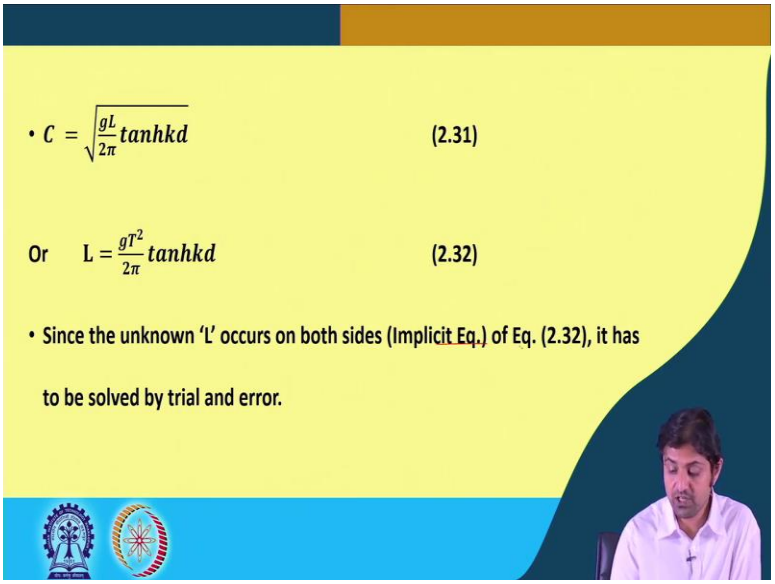

Since C = L by T, we get C = we just writing it in a different form. So C can be written as gL by 2 pi tan h kd, or L = gT square by 2 pi tan h k d. Since the unknown L occurs on both side of equation 2.32 it has to be solved by trial and error because k is nothing but 2 pi by L. So, if you this, you want to, so, this is the equation dispersion famous dispersion equation, sigma squared =

g k tan h k d which you must remember and doing some no manipulation here and there we can also write L = gT square by 2 pi tan h k d.

And this is also a different form of dispersion relationship and because this L appears on both sides of this equation 2.32 this equation if needs to be solved will be solved by trial and error method.

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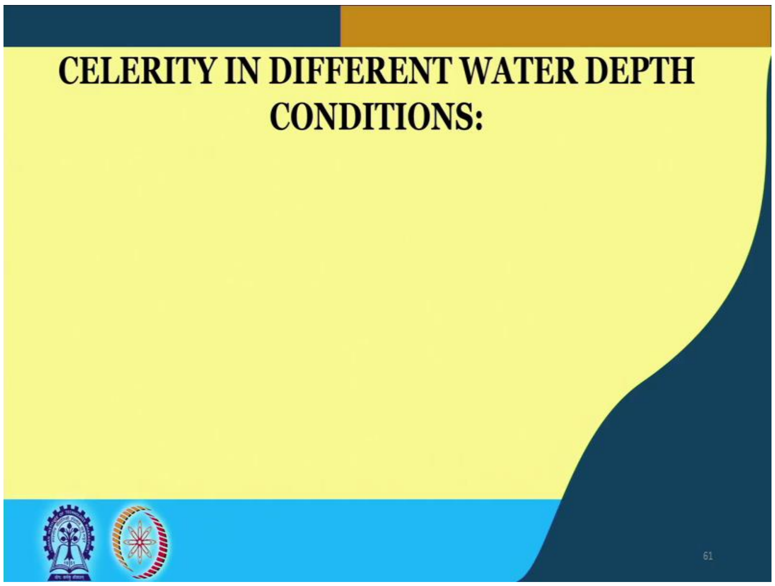

So, I will stop at this point in this lecture and when we start the next lecture, we are going to study the celerity in different water depth conditions. So, thank you so much for listening in this lecture and I will see you in the next class.