**Hydraulic Engineering**
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**Lecture # 56**
**Computational fluid dynamics (Contd.)**

Welcome back students, we were discussing about the partial differential equations in the last lecture where we have seen the elliptical or PDE, a parabolic and hyperbolic PDE.

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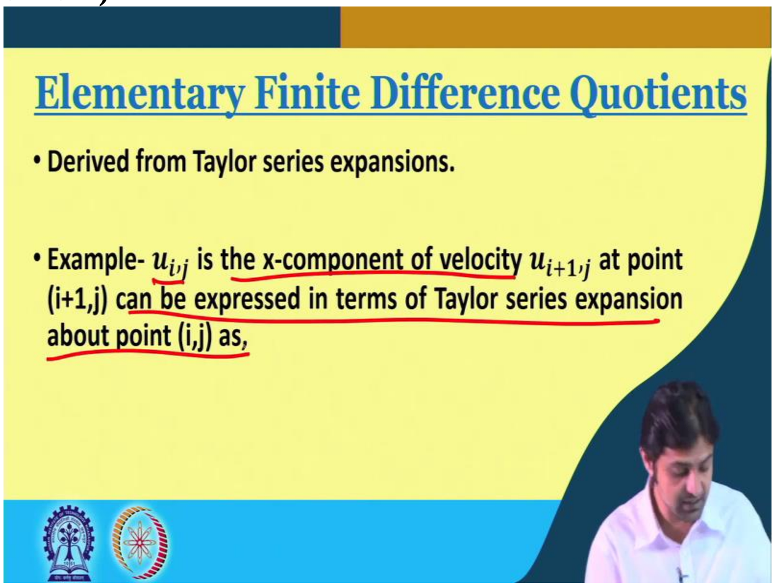


And we also saw in the last class they start the sum of the concepts of the finite differences. So, we are going to proceed forward now and talk about elementary finite difference quotients. So, derived from Taylor series expansion so, the elementary finite difference quotients are real derived from the Taylor series expansions. So, example is that u i j is the x component of velocity u i + 1 at point i + 1, j can be expressed. So, u i j is the x component of the velocity, u i + 1, j at a point i + 1, j can be expressed in terms of Taylor series expansion about point i, j as.

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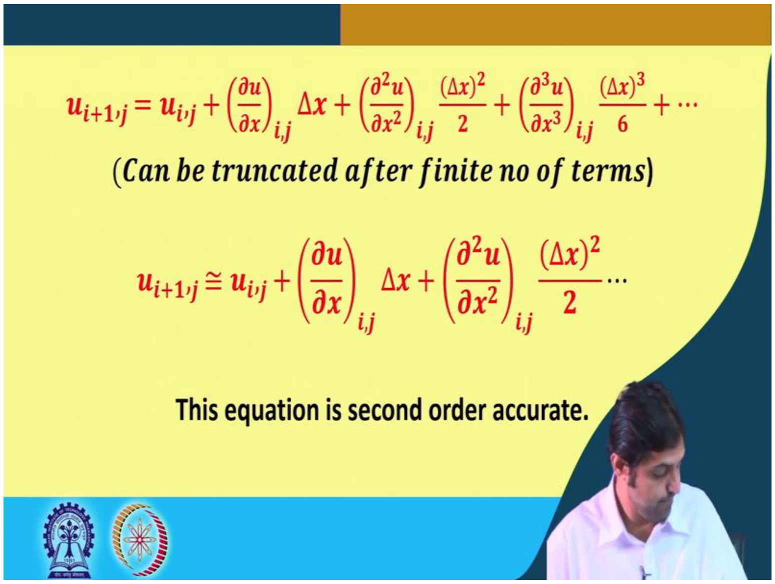


So, u i + 1, j can be written as u i, j + delta u del u del x at i, j into delta x + delta squared u by del x squared at i, j multiplied by delta x square by 2 and so on. So, this is using Taylor series expansion. So, this actually can be truncated after finite number of terms I mean we can decide 3 terms of 4 terms of 5 terms. So, we can actually we have decided to truncate it after 3 terms.

So, u i + 1, j can be written as u i, j + del u del x multiplied by delta x del squared u del x square into delta x squared by 2 and these are evaluated at i, j both of these terms. So, this equation is second order accurate because we have square term del squared u by del x squared.

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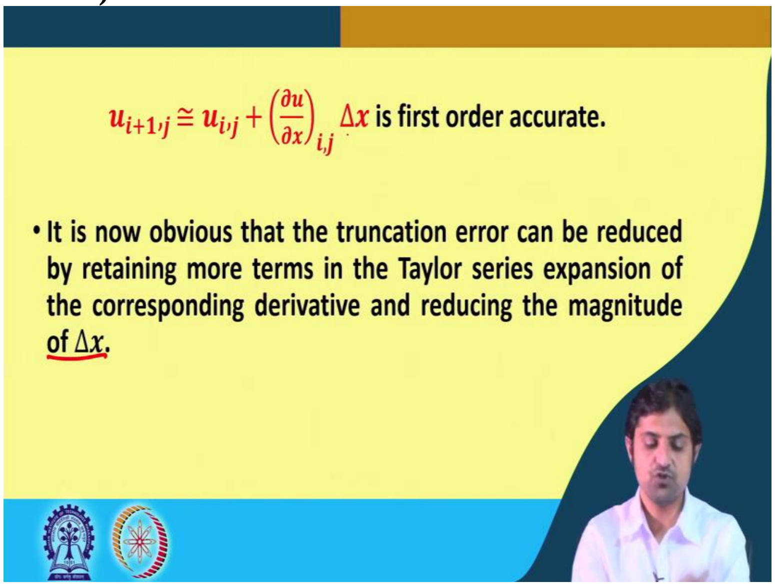

So, if you want to make it first order liquidate it accurate it can be written as u i + 1, j is u i, j + del u del x evaluated at i, j into delta x. So, this is first in first order so, its first order accurate. So, it is now obvious that the truncation error can be reduced by retaining more term in the Taylor series expansion of the corresponding derivative and reducing the magnitude of delta x. So, 1 important thing to note here is that there is a truncation error so, the full value will be the exact and perfect value.

But we have decided to truncate some of those term truncate means leave out those terms. So, this error can be reduced by retaining first ways retaining more terms in the Taylor series expansion the more than number of terms the less will be there. And if you reduce the magnitude of delta x, then also that can be there because this delta x is reduced then delta x x squared x cube will be even more and more is less, therefore, the truncation of those terms we do will be almost negligible.

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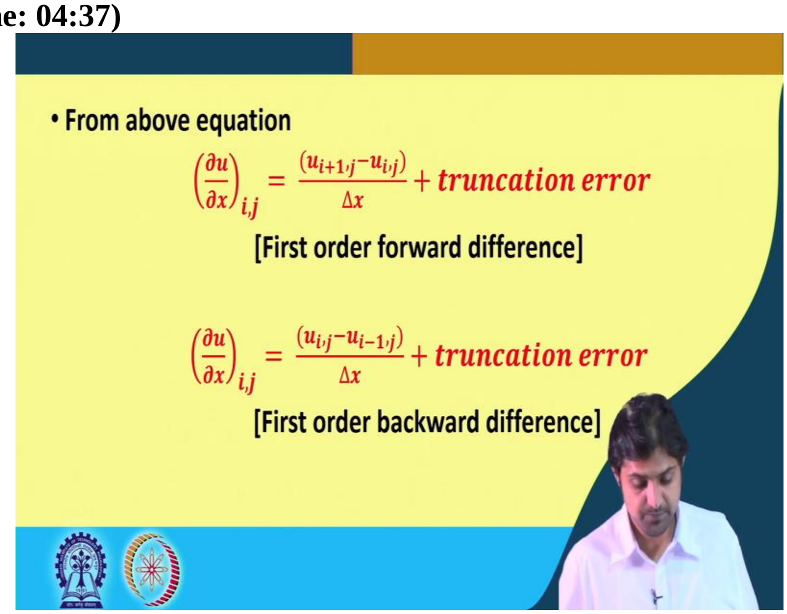


So, from the above equation, you see the first order equation and the second order equation, if we write we can evaluate del u del x at i, j which means del u of i + 1, j - U of i j divided by delta x + truncation error whatever we have left out. So, this is actually called the first order forward difference forward, because we have i + 1. So, if say this is i, j and this is i + 1 and, this is i + 1 line and this is j line, this is j - 1 and this is j + 1 and then that means, it is forward i + 1.

Similarly, we can also write del u del x i, j in the backward direction that means, u i, j - u i - 1, j by delta x + whatever the truncation error is, and this is called the first order backward difference. So, now you understand the difference between first order forward and first order fine in the first quarter backward difference. So, in the forward we have i + 1 - i and in the backward we have i - i - 1. So, I indicates the indices at the grades of the grids.

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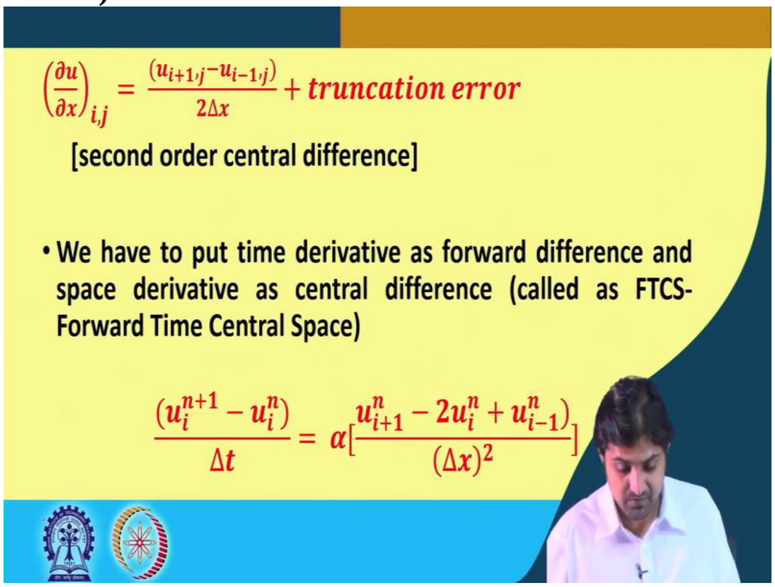


There can be another way where you can actually you know so, let us say this is i this is i - 1 this is i + 1, so, you can just do this one that. So, first write the equation between these 2 points then between these 2 points and subtract this is what we have done here, you know, so, this actually gives the difference between u i + 1 – u i - 1 you see, and then it will be divided by 2 delta x because the lens between this is delta x and this is delta x, so, this total becomes 2 delta x.

So, this is another way and this is called the second order central difference, because we are calculating what are we calculating again drawing this is i, j this is i + 1, i - 1, j this is i + 1, j. So, we are actually calculating the del y by i - 1 i + 1. So, we are actually calculating del y by del x at point i, j So, it is that there we are calculating the value of dy by dx here using the point bag here and here.

That is why it is called the send second order central difference this is central in the backward you see we are calculating it i using the 1 there - the 1 at the back here we are calculating it i using the 1 in front - the current 1. So we have to put time derivative as forward difference and space derivative of central difference. Called the FTCS forward time central space. So what we do we have to put time derivative as forward difference or time as forward difference, because time cannot be in the back?

So we will always have to be forward and for the space we use central difference that is the most standard practice and this is called as FTCS forward time central space. So, you are expected to know these terminologies what FTCS is what is forward difference what is backward difference what is central difference in for time derivative which is used for space which is most commonly used. So, for space we said that was a central difference for time which is the forward difference. So, terms like that are quite important.

So, we write u i in this is for the time, you see this n so for so we ride u i n + 1 – u i n is alpha u i + 1 and So, these i + 1 and i and i - 1 are all calculated at the previous time or at the current time. Suppose we are at let us say we are at we are at time t = 1 seconds we have the solutions we want to calculate 40 = 2. So, velocity at t this is at t = 1 second, this is also this here. This is also at t = 1. So, you see n n n. So, these all things we know from the previous time step what we need to know is u i at n + 1.

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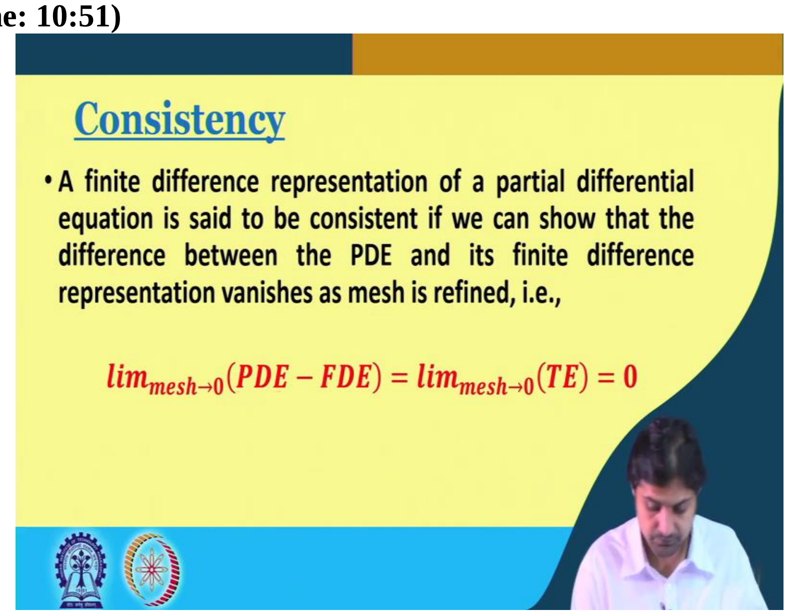


So, there is a term called consistency now, so what actually is consistency. So, a finite difference representation of a partial differential equation is said to be consistent, if we can show that the difference between the PDE and its finite difference representation vanishes as measures defined.

So, this again in a different word, if this representation of I mean CVV said that we are going to represent partial differential equation in an algebraic form correct.

So, this algebraic form is currently finite difference method we are using the method of finite difference to represent it into algebraic form. So, this solution is consistent if we are able to show that the real solution that is the PDE solution. The difference of that with the 1 that is obtained using the current method that is a finite difference method will go away will be 0 or it starts to go to 0 as we have gone delta x is refined as measures refined as max goes to 0. ,

As the measures refined this is then you remember we had this limit delta x goes to 0 in your limit analysis. So, limit off mesh goes to 0, the partial differential equation - the finite difference equation goes to 0, if mesh the mesh size goes to 0 if we are able to show that that means our solution of the partial differential equation using the finite difference representation is consistent.

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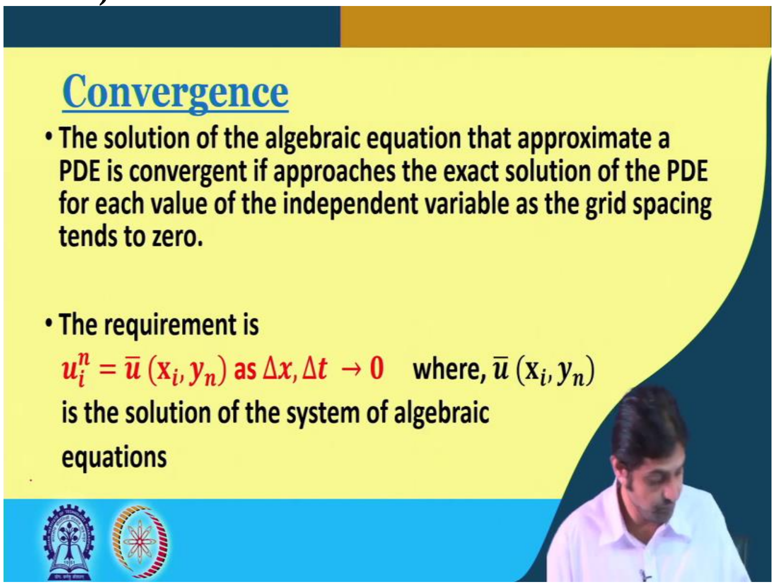


There is another term called convergence. The solution of the algebraic equation that approximate a partial differential equation same thing, is convergent If it approaches the exact solution of the PDE for each value of the independent variable as grid spacing tends to 0. So, the solution of this if the solution is turned to be convergent, if this solution approaches exact solution of the PDE for each value of the independent variable as the grid spacing turns to 0, not the combination of different variables for each.

And individual for each x and y for example, x and y are the independent variables are for each of these x and y, this should converge to the real original solution of the PDE then that means the solution is convergent. So, the requirement is u i at n = u bar x i, y n as delta x delta t goes to 0, where u bar x i y and this dissolution of the system of algebraic equations.

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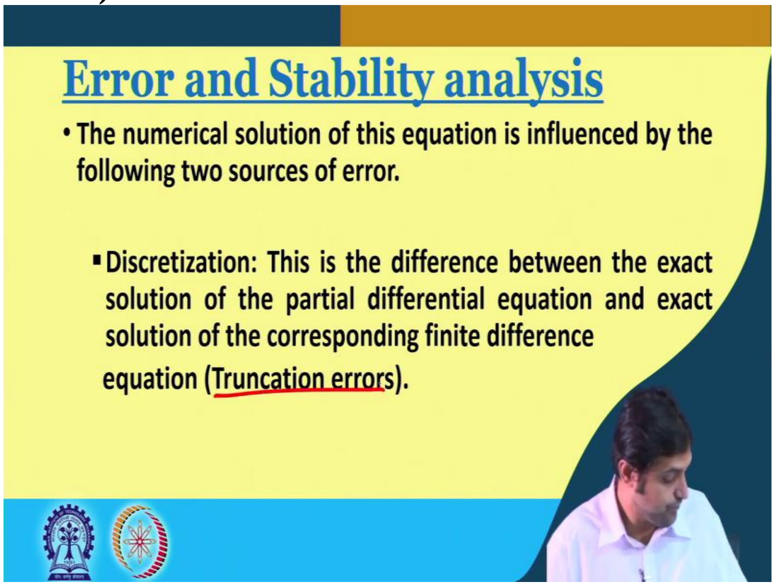


Now, there is something called the error and the stability analysis the numerical solution which we have obtained using the finite differences scheme they are influenced by 2 sources of error 1 is discretization the 1 I mean the errors are introduced you to 2 methods 1 of which is discretization this is the difference between the exact solution of the partial differential equation and exact solution of the corresponding finite difference equation truncation errors.

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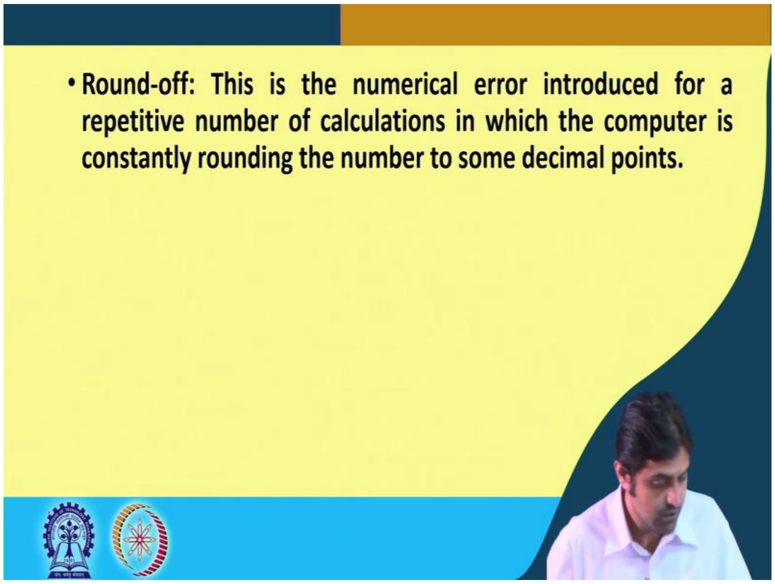


Second there is round off error. So, this is a numerical error introduced for a repetitive number of calculations in which the computer is constantly rounding the number to some decimal point. Suppose the accuracy of the computer is let us say 3 decimal units. And if our solution is 0 point, I mean 1.117985 or let us say assume the value of phi is 3.14 it goes on  so, when the computer is let us say a third of the accurate up to third order, the computer will keep on rounding the values.

Therefore, there will be 1 error introduced due to the rounding off. So, 1 due to discretization second is due to the rounding off. So, this are the 2 errors that we had talked about.

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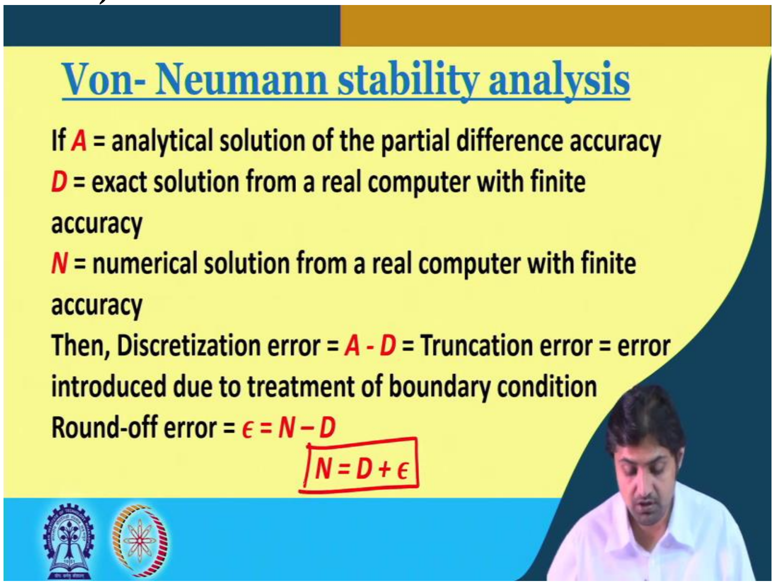

Now, we can proceed to another concept in the computational fluid dynamics module that is von Neumann stability analysis. So what is the stability analysis? We say if A is the analytical solution of a partial difference accuracy and D is the exact solution from a real computer with finite accuracy. A is the analytical solution of partial difference accuracy and D is the exact solution from a real computer with finite accuracy and N is the numerical Samuel simulation from a real computer with finite accuracy.

So, A is analytical solution D is exact solution exact means perfect solution and N is the numerical simulation, if we use these 3 terms. Then the discretization error is A - D = truncation error = the error introduced you to treatment of boundary condition whereas the round off error epsilon is N - D. Now you understand these round off error and the discretization error discretization error is a - d.

So discretization A - D. So, analysts the difference of the analytical solution of the partial diverse difference accuracy - exact solution from a real computer with finite accuracy, whereas, the round off error is N - D. So, this is von Neumann stability analysis and we can write N = D + E D + epsilon.

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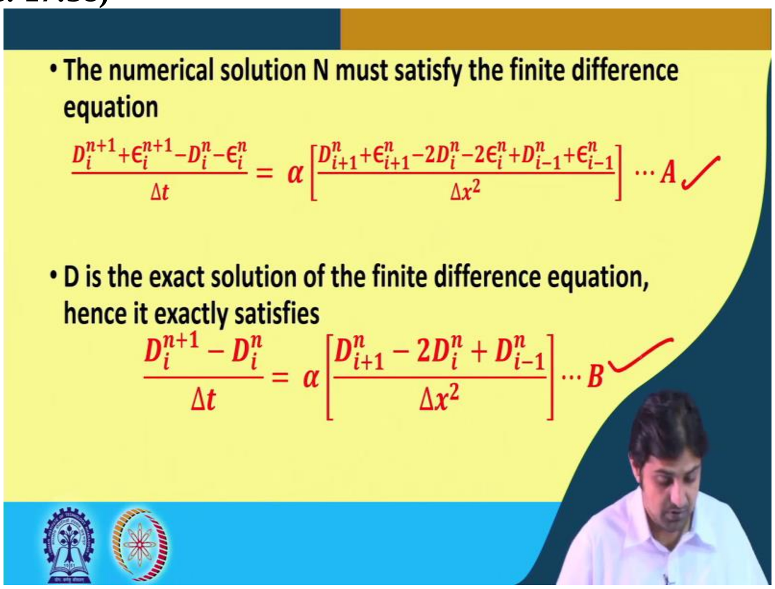


So, the numerical solution and N satisfy the finite difference equation like this, and D is the exact solution of the finite difference equation and it exactly satisfies this one A and B, A was the finite difference in the first page that we saw A. You see this is A these the exact solution of the finite difference equation and it is exactly satisfies this, so and will satisfy both.

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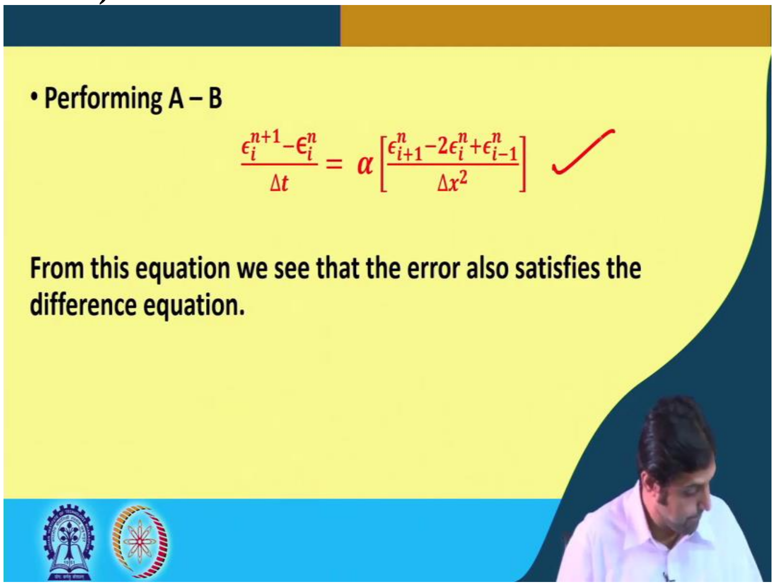


So, if you perform A - B, so, epsilon of n + 1 - epsilon i of n delta t will be written. So, from this equation we can see that the error also satisfies the difference equation.

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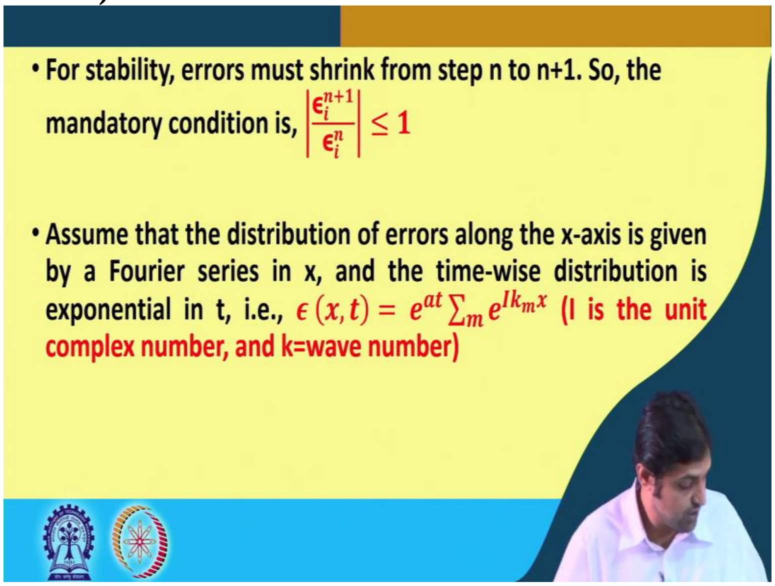


For stability the errors must shrink from step n to n + 1. So, the mandatory condition is epsilon i of n + 1 divided by epsilon of i has time n must be less than 1 because it is supposed to decrease assume that the distribution of errors along the x axis is given by a Fourier series in x and the time was distribution is exponential in t, then we can write epsilon x, t is e to the power at So, this is the equation using which it is given L is the unit complex number i capital i and k is the wave number. This is very complex.

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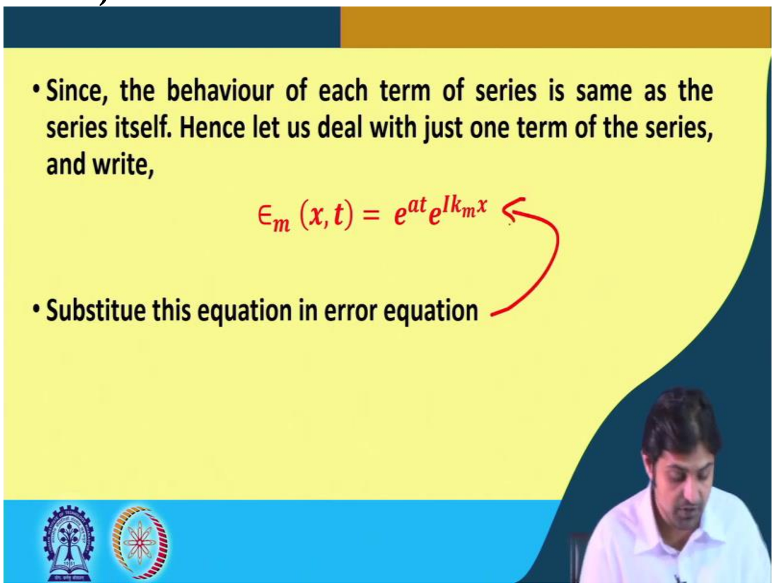


Since the behavior of each term of series is same as the series itself. Hence, let us deal with just 1 term of the series and so, we do see there are a lot of term in this series e at is the sum e Ik mx So, we just you know use only 1 term and we write E of m x, t e to the power at e to the power I k m x. And if you substitute this equation in the error equation, you see that error equation that we had got this was the error equation and if we substitute this.

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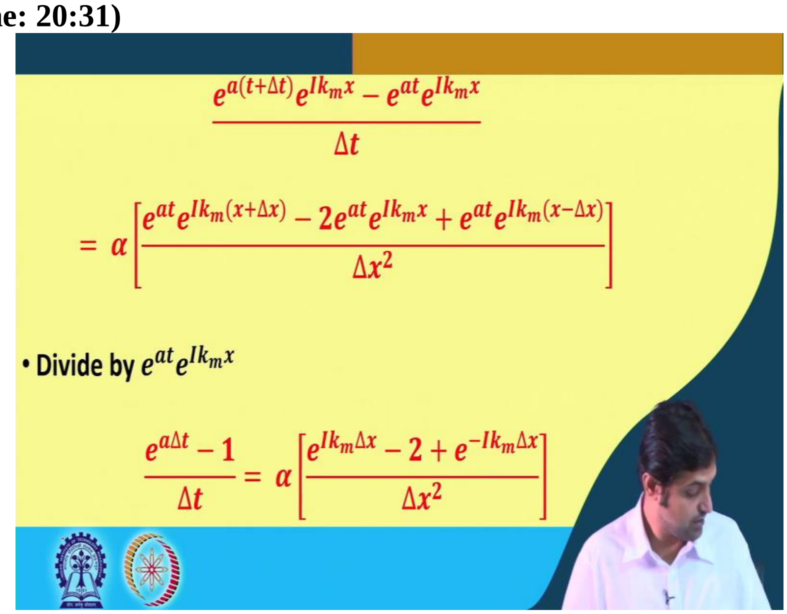

In the error equation, we are going to get it a complex equation like this. Now, if we divide this equation by e to the power a t to the power i k m x we are going to get on the left hand side we are going to get e to the power a delta t because T and T will get cancel = a more simplified solution e to the power i k m delta x because x and x + delta x so, this will remain delta x here also delta x and y this will completely go away this will you know so, on dividing by e to the power alpha t at e to the power i k m of x.

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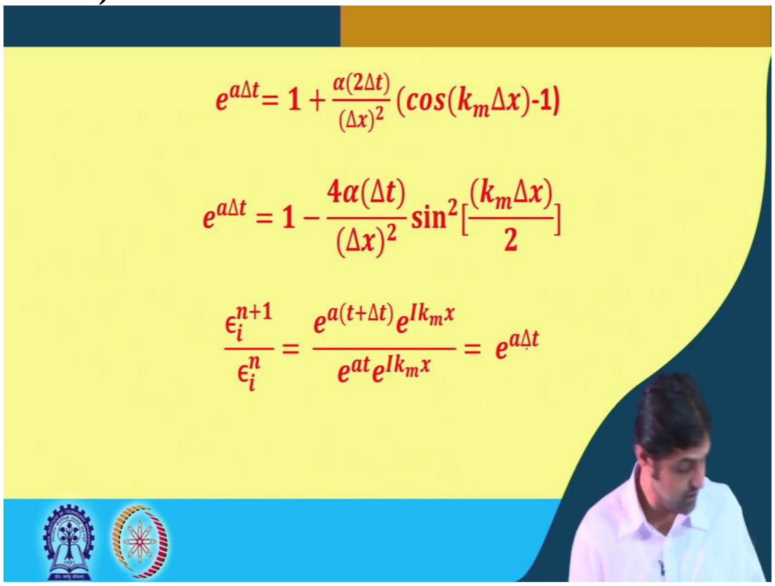


we can therefore  using this equation from here we can write e to the power a l 40 can be written as 1 + alpha 2 delta t divided by delta x squared. So, you see this term here or we can write it in terms also of sine square by 2 term so this becomes 1 - 4 alpha delta t by delta x squared sine square km delta is by 2 the derivation is not in your scope, but we will finally look at the answer the result that we are going to get therefore, we can write this e to the power n + 1 see we have got e to the power a delta t.

Therefore, this error we can actually write e to the power alpha t + delta t into e to the power i k m x divided by e to the power a t e to the power i k m x = e to the power a delta t.

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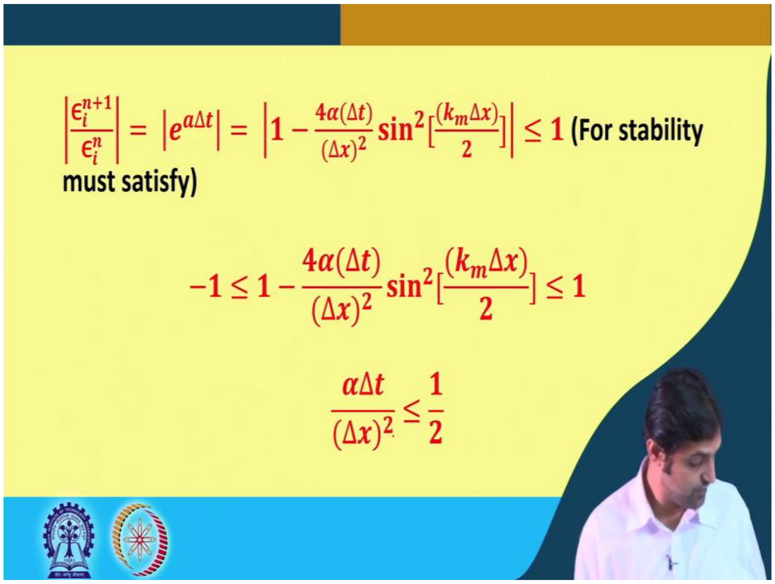


So, this equation must be satisfied for stability mod of 1 - 4 alpha delta t by delta x squared sin squared k m delta x by 2 must be less than 1 or in other words this equation I think if you can remember this equation it will be quite good. So, just remember this final equation, not the entire derivation, but you should be able to follow what has been done from the lectures, so, this means what does this mean is, so see this sign squared k m delta x by 2.

You know so, the condition can be written to be alpha delta t by delta x squared should be less than or equal to half, because this will always be less than 1. So, this is the final solution that we get for stability very important, so, you should remember this 1 as well on this 1 as well. So, the final solution here alpha delta t by delta x squared less than or equal to half.

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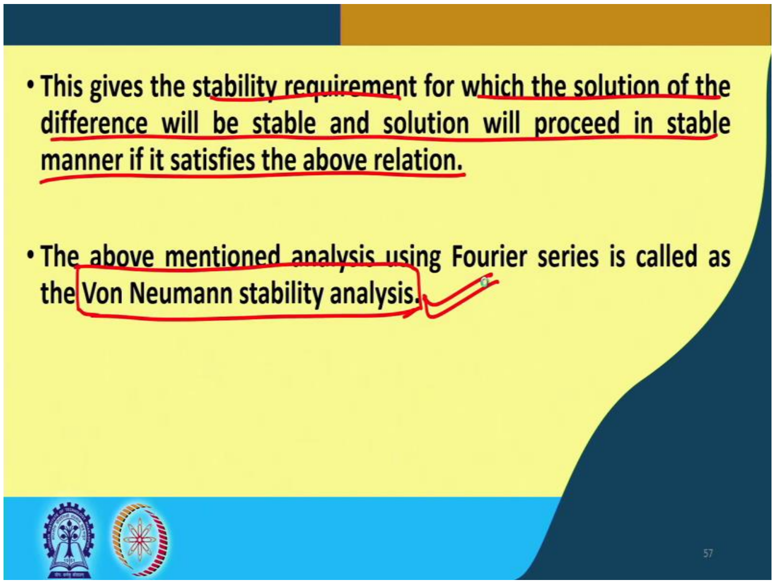


This gives the stability requirement for which dissolution of the difference will be stable and solution will proceed in stable manner if it satisfies the above relation. So, which relation this relation that is the final solution. So, the above mentioned analysis using Fourier series is called von Neumann stability analysis. So, you must be in principle knowing what is Von Neumann stability analysis.

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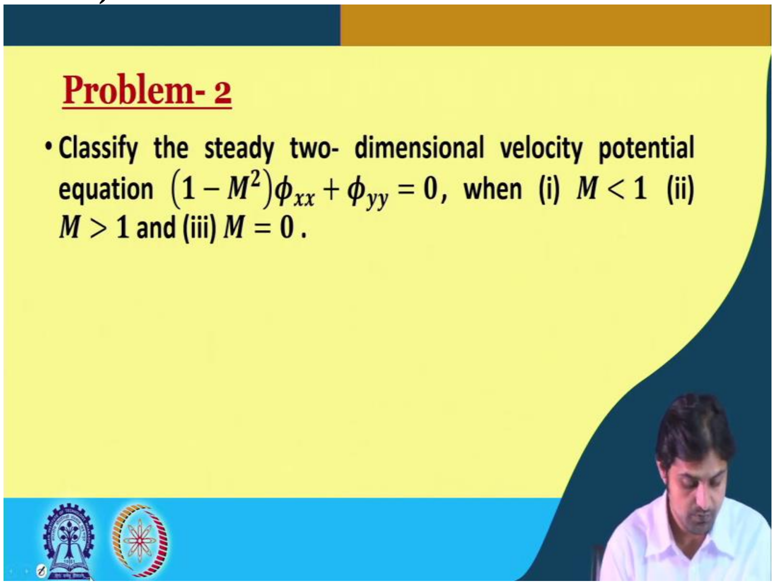


So, using this we are actually going to you know until the concepts which we have learned and now, we are going to solve this 1 question using this now so, the question is classify the steady 2 dimensional velocity potential equation, which is 1 - m squared phi xx + phi y y = 0 when m is less than 1 and secondly, when m is greater than 1 and third when m = 0, so, like always we are going to using.

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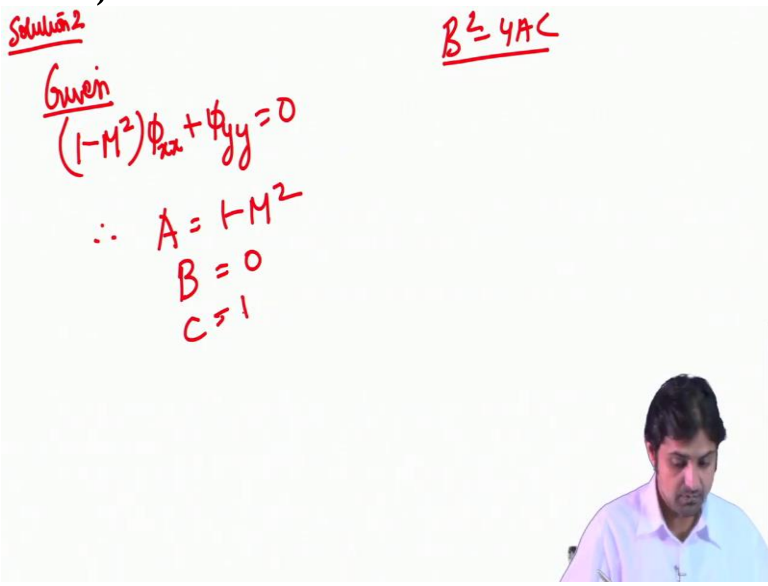

White screen for this so, solution to so, first we write down what is given to us 1 - m squared into phi x x + phi y y = 0. So, therefore, a = 1 - m squared. So, you remember that we said discriminant of B squared - 4 AC we have to calculate you remember that part here they are no B, but C = 1, so I will let me take you to that because that is quite important thing to you know.

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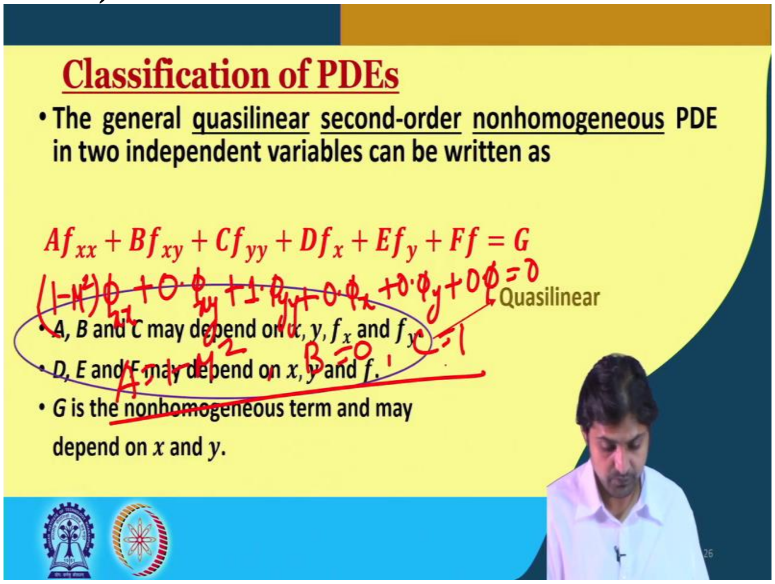


See A f x x + B f x y + C f y y + D f x + E f y + F f = G. So, let me just quickly just y here. So, you see our equation, it is write down here. So it is 1 - m squared f x x instead of f x x we write f phi xx + there is no phi x y. So, 0 in 2 phi x y + C is 1 into 5 y y we have + there is no only a phi x so we just write 0 into phi x + 0 into phi + 0 into phi = 0 that is the so when we write down what we say A is 1 - m squared and B is 0, and C is 1, because only these 3 terms are being used in the discriminant.

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Because so, you see, the classification depends on b squared - 4 AC good the C, we have to classify the study to diminish the velocity potential. So, I think it is would be better if we start doing this problem in our next class, because otherwise, this is a slightly long problem and needs a little bit more detail to solve. So I think I will end the class today here and we start with the solution of this question in our next lecture. Thank you so much.