Hydraulic Engineering
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Lecture - 52
Viscous Fluid Flow (Contd.)

Welcome back to yet another lecture of viscous fluid flow where we are deriving the Navier–Stokes equations.

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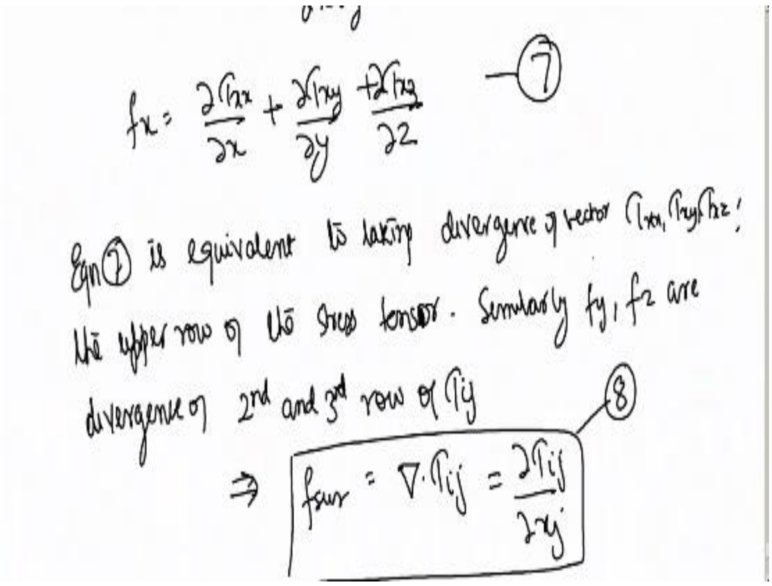

So last time we left at this particular point where we derived that the surface force is del I mean diversion of τ ij vector.

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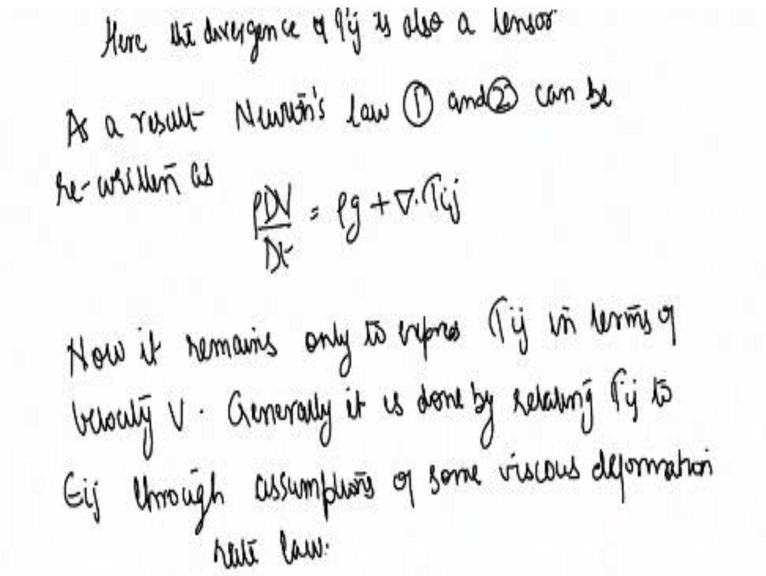

So proceeding forward from this point this it is important to note that this here the diversions of τ ij is also a tensor okay. So as a result Newton's Law given by equation number 1 and 2 can be re-written as

$$\rho \frac{DV}{Dt} = \rho g + \nabla \tau_{ij}$$

. Now it remains only to so there is another unknown term called τ ij right. So now it remains only to express this τ ij in terms of velocity V okay.

Generally, it is done by relating τ ij to strain ϵ ij okay. You remember we derived ϵ ij in terms of the velocity in our first lecture of this module. So it is relating τ ij to ϵ ij through assumptions of some viscous deformation rate law which we have already done in the beginning all right. Okay so before proceeding further we will take a small detour and talk about a topic called hydrostatic. So hydrostatic means fluid at rest correct.

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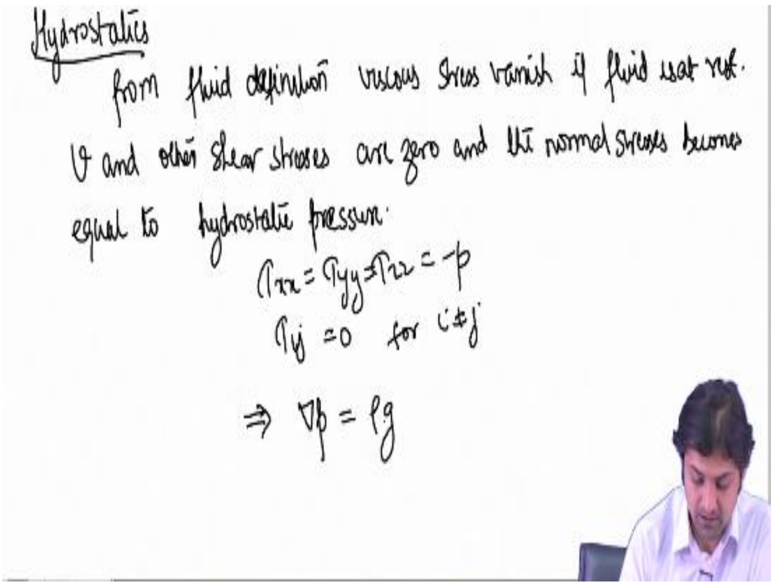

So we take a new page now and talk hydrostatics. So from fluid definition you remember the beginning definition viscous stress vanish if fluid is at rest correct. Velocity V and other shear stresses are zero and the normal stresses becomes equal to hydrostatic pressure. So τ xx becomes = τ yy = - pressure τ ij = 0 for i not = j. So this implies if we put in the equation number 9 that we have obtained we become delta p = rho g all right.

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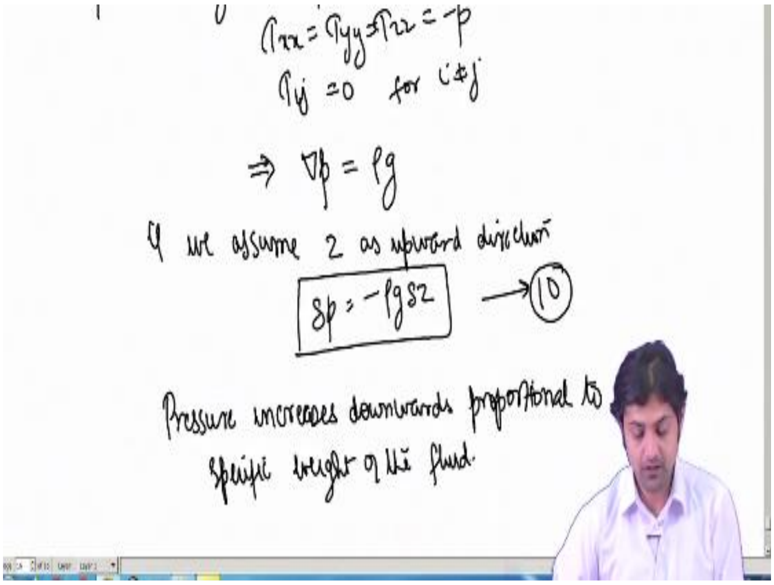

If we assume z as upward direction that means delta p will be – rho g delta z this is equation number 10. This means pressure increases downwards proportional to specific weight of the fluid and this is a very famous equation that we know from before. Now this was slight deviation, but we are going back again to the deformation for law for Newtonian fluid okay.

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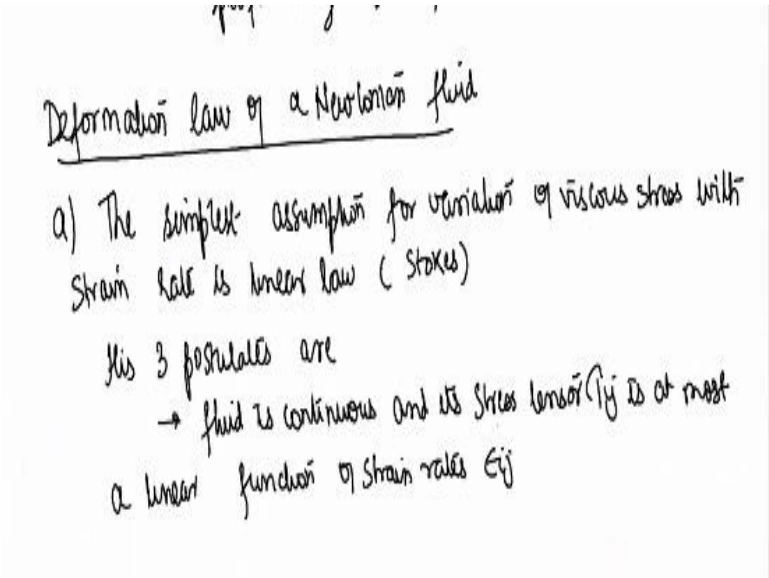

So we are going to proceed for the further derivation of the Navier–Stokes Law deformation law for a Newtonian fluid. So the idea was that we should be able to relate viscous stress to viscous strain all right. So the simplest assumption for variation of viscous stress with strain rate is linear law given by stokes. His 3 postulates are fluid is continuous and its stress tensor τ_{ij} is at most a linear function of strain rates ϵ_{ij} okay.

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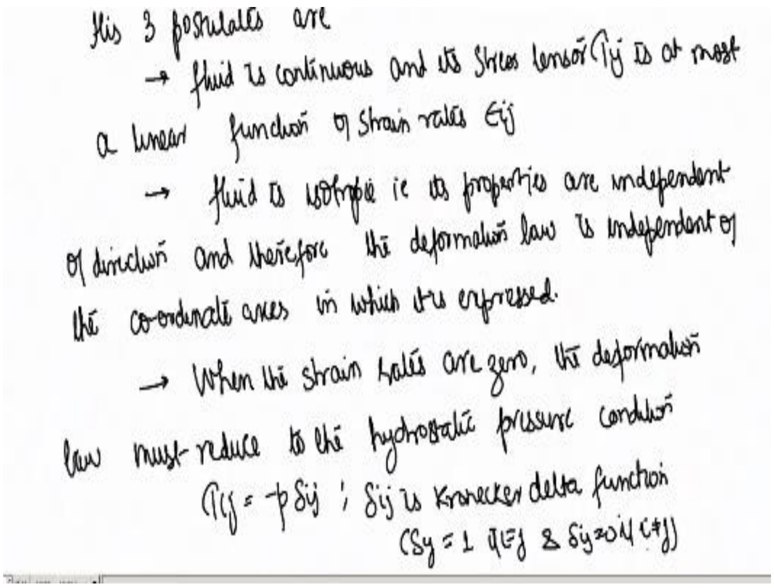

The second postulate is fluid is isotropic that is its properties are independent of direction and therefore the deformation law is independent of the co-ordinates axes in which it is expressed. Third is when the strain rates are zero the deformation law must reduce to the hydrostatic pressure condition. T ij = -p delta ij okay. Here delta ij is Kronecker delta function that is delta y = 1 if I = j and delta ij = 0 if i is not = j all right.

So this was his 3 postulates that stokes said. So this is the last postulate is valid for the hydrostatic pressure condition all right.

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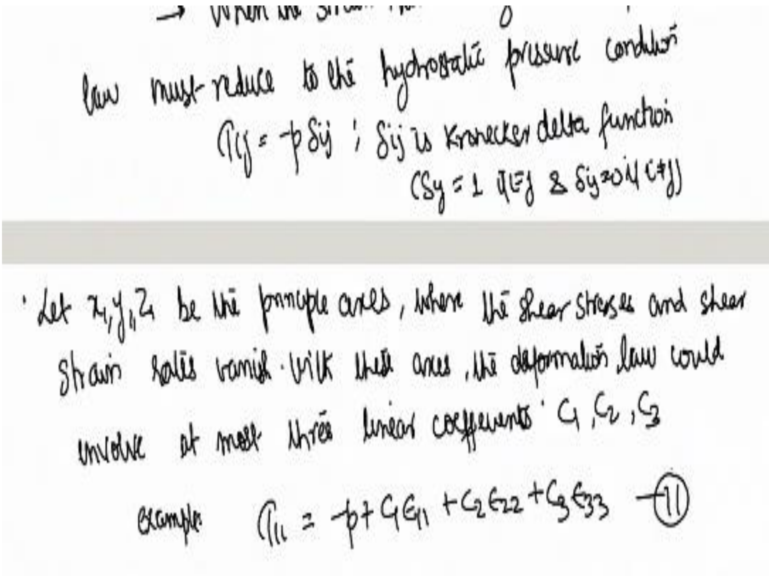

So we proceed and say so his postulates are finished. So we say let x, y, z be the principle axes where the shear stresses and shear strain rates vanish. Let us assume an axes x, y and y so instead of x1, y1 and z1 be the principle axes where the shear stress and shear strain rates vanish. With these axes the deformation law could involve at most three linear coefficients C1, C2 and C3.

So example is so if you want to write τ 11 it will be – p + c this we called equation number 11.

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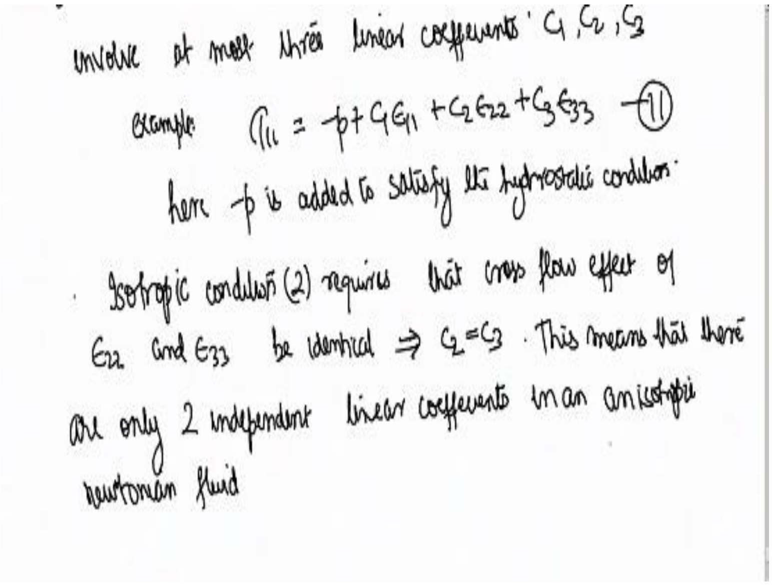

Here –p is added to satisfy the hydrostatic condition okay. Now the second postulate about the isotropic condition 2 requires that cross flow effect of ϵ 22 and ϵ 33 be identical which implies C2 = C3. This means that there are only 2 independent linear coefficients in an anisotropic Newtonian.

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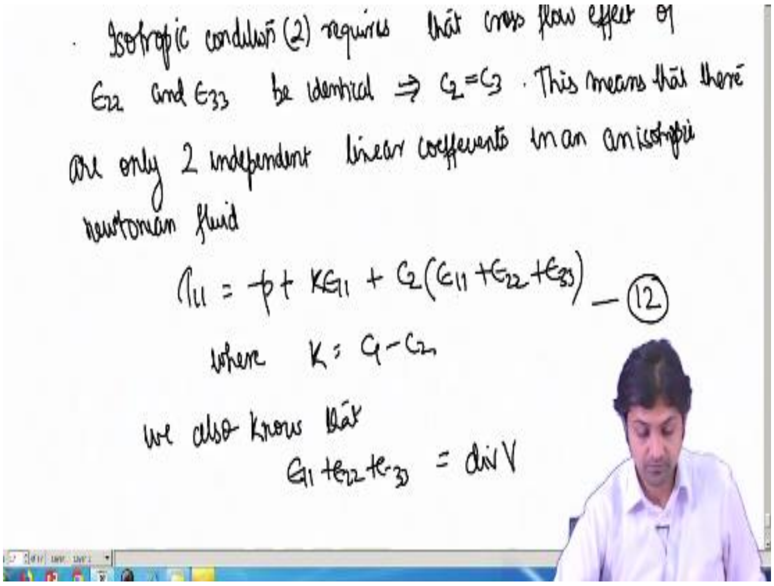

All right where K = C1 – C2 we this is equation number 12. We also know that ϵ 11 + ϵ 22 + ϵ 33 is we have seen this (()) (18:09) derivation of continuity equation divergence of V.

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we also know that

$$\epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \text{div } V$$

Lets transform eqn (12) to some arbitrary axes x, y, z where shear stresses are not zero. This will give us a general deformation law.

→ With respect to principal axes x_1, y_1, z_1, let the x axes have direction cosines l_1, m_1, n_1. Let y axes have cosines l_2, m_2 and n_2 and z axes have cosines l_3, m_3, n_3.

So the next step is so we have done this on an axes I mean this, this, this terms which we this, this terms we have written we have done it on principle axes where the shear stressor and shear strain are zero. So there exist only normal stresses and normal strain, but now we have to transform it into an arbitrary axes where all the stresses and strain will exist. So let us transform equation number 12 to some arbitrary axes x, y, z where shear stresses are not zero.

So earlier I mean the axes the shear we have assumed an axes where the shear stresses and shear strains used to vanish, but now since we have transformed into some arbitrary axes where these might not be zero okay. So if we are able to do that this will give us a general deformation law all right. So that is our next step to do. So with respect to principle axes x, y, z let the axes so with respect to x1, y1 and z1 let the current x axes.

So with respect to the previous principle axes x1, y1, z1 let the current x axis have direction cosines l1, m1 and n1. Let y axes have cosines l2, m2 and n2 okay with respect to the previous principle axes and z axes have cosines l3, m3 and n3 okay.

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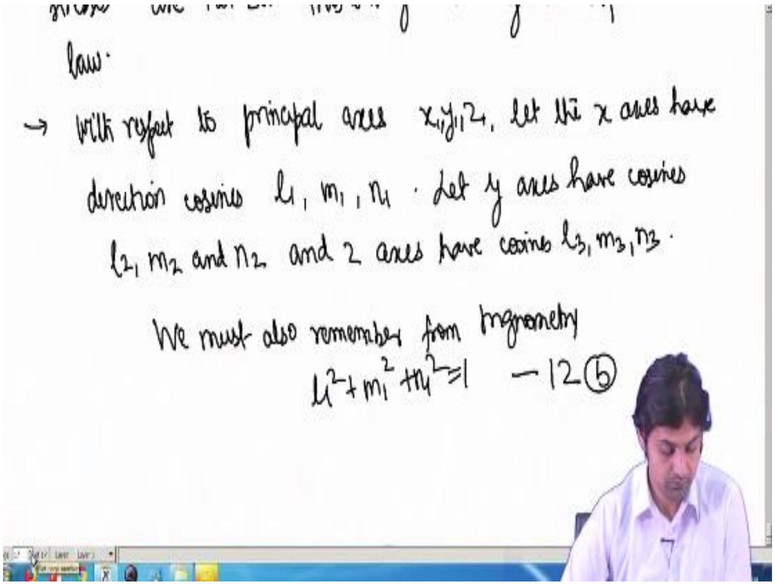

We must also remember from trigonometry l1 square + m1 square + n1 square = 1 and so on okay and this equation we called equation number 12 b all right.

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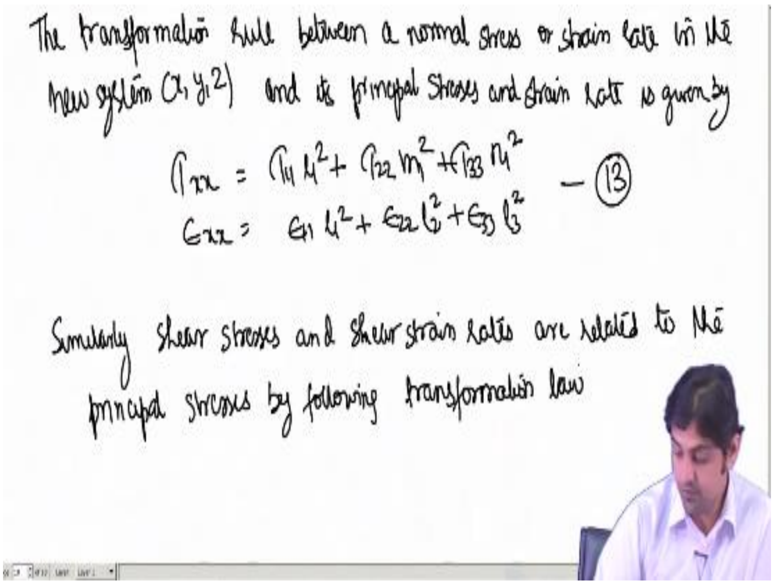

So based on this the transformation rule between a normal stress or strain rate in the new system which is the new system x, y and z axes which is arbitrary and its principal stresses and strain principal stresses and strain are τ xx, τ yy, τ zz, ϵ xx, ϵ yy and ϵ zz is given by τ xx = τ 11 l1 square + τ 22, m1 square + τ 33 n1 square and ϵ xx is given by same l1 square ϵ 22, l2 square, l3 square okay.

So similarly so this one equation number 13 relates the principle stresses. So similarly we can also relate shear stresses and shear strain rates are related to the principal stresses by following transformation law.

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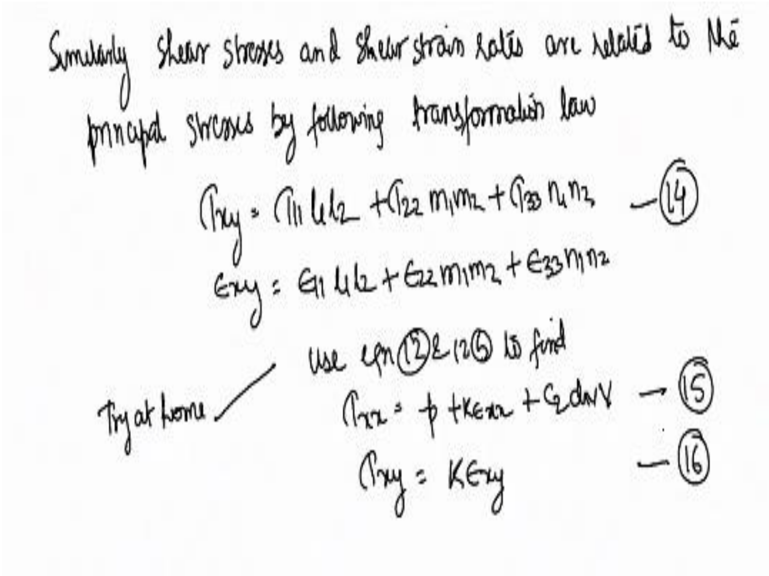

That means τ xy can be written as τ 11 it is related to the principle stresses l1, l2, τ 22, m1, m2 + τ 33, n1 n2. Similarly, ϵ xy is given as ϵ 11 l1, l2 + ϵ 22, m1 m2 + ϵ 33 n1, n2 and this equation we call as equation number 14 all right. So there are some things that you can try to find which I expect you to do at your home, but if you have some problems we can discuss that in the forum.

So two things try at home is use equation 12 and 12 b to find the τ xx that we have written – p + k ϵ xx + c2 divergence of V try to prove that okay and secondly τ xy = k ϵ xy all right. So this is equation number 15 and this is equation number 16 all right.

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So equations 15 and 16 are desired general deformation law okay all right. So if by comparing 16 with shear flow between parallel plates we say k is 2 mu this we have done we have seen before in our topic of laminar and turbulent flows so k = and there mu is viscosity and this is equation number 17 it is just by comparison.

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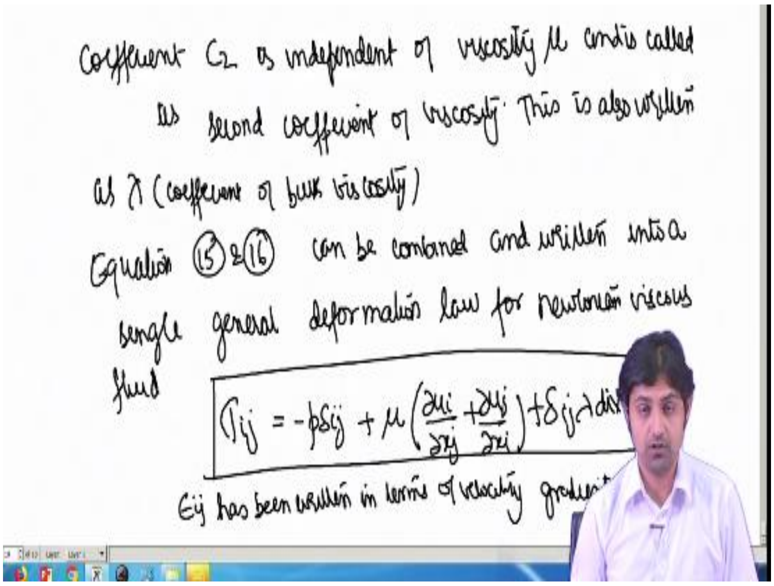

Coefficient C2 that we have mentioned before is independent of viscosity mu and is called as second coefficient of viscosity. This is also written as lambda coefficient of bulk viscosity all right. So equation 15 and 16 can be combined and written into a single general deformation law for Newtonian viscous fluid.

$$\tau_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\lambda divV$$

all right.

ϵ ij if you remember has been written in terms of velocity gradient. You remember our lecture 1. So in lecture 1 we have derived this ϵ ij and I mean ϵ ij the shear strain rates and we wrote in terms of velocity gradient. So I think this is the nice and appropriate point to stop. In the next lecture we will talk briefly about the thermodynamic pressure and the mechanical pressure Stokes hypothesis.

And then we finally derive the final version of Navier–Stokes equation. So that is enough for today and I will see you in the last lecture of this module which will be pretty brief one. Thank you so much, have a nice day.