Hydraulic Engineering
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Lecture – 21

Boundary Layer Theory (Contd..)

Welcome back. Last class we started solving problem 7 and finished it where we determine the shear stress and the drag force by the technique that was taught in the lecture itself.

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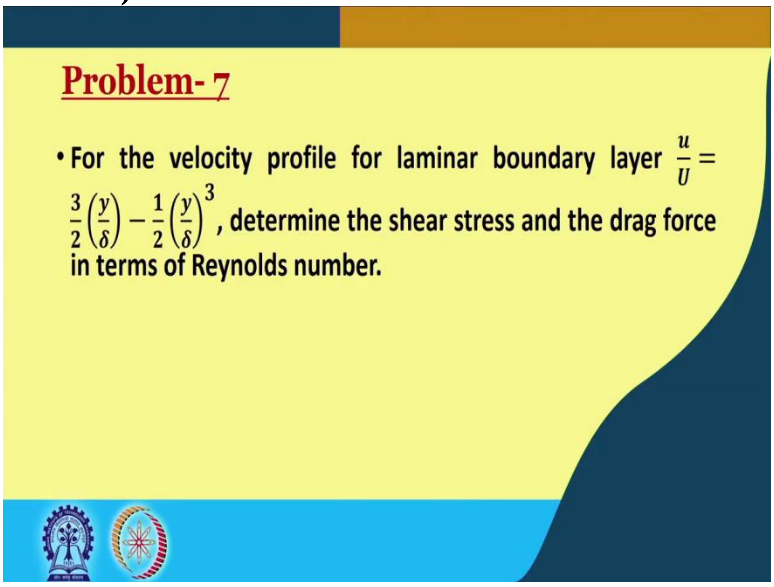

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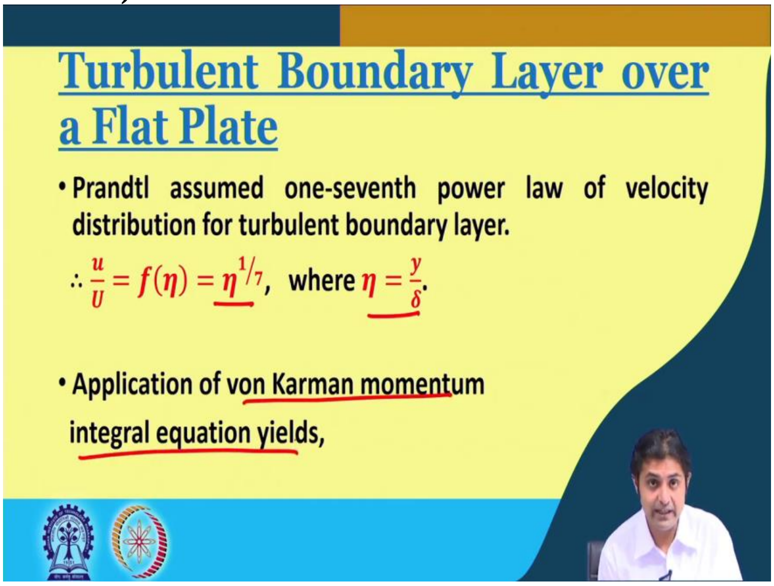

Now, we are going to proceed to apply, the von Karman momentum integral method for turbulent boundary layer over a flat plate because that particular equation was valid both for laminar and turbulent fluid flow. Actually, Prandtl assumed one-seventh power law velocity distribution for turbulent boundary layer. That was his assumption and he said, for turbulent boundary layer

$$\frac{u}{U} = f(\eta) = \eta^{1/7}$$

, where eta, the usual meaning is,

$$\eta = \frac{y}{\delta}$$

. y is the distance above the plate and delta is the boundary layer thickness. And since that von Karman momentum integral equation is applicable here too.

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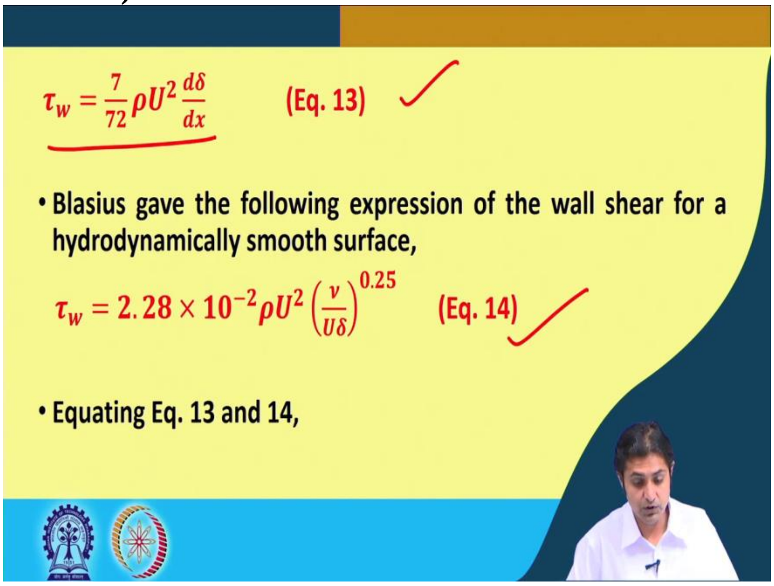

It yields, tau w = 7 / 72 ρ U square d delta / dx, if you follow the same procedure, this equation is more important. Now, Blasius gave the following expression for wall shear for a hydro dynamically smooth surface. He said that, tau w = 2.28 into 10 to the power - 2 ρ U square into nu / delta u raised to the power 0.25. This is another equation. Now, if equate both.

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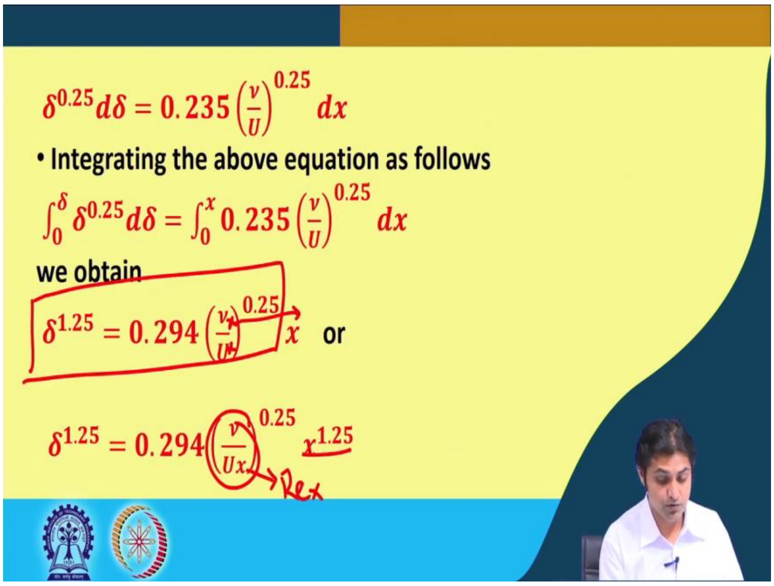

This is the equation that we get, delta as a function of delta and x. And if we integrate the above equation, we are going to get, lambda to the power 0.5 delta, sorry,

$$\int_0^\delta \delta^{0.25} d\delta = \int_0^x 0.235 \left(\frac{\nu}{U}\right)^{0.25} dx$$

and then we can obtain expression like this or if we multiply x here, and x here, as well and take it out here, we can actually write, this x becomes x to the power 1.25. And this is Reynolds number, a similar expression. So, simply what we have done is, so, this was delta to the power 1.25. So, this will become 0.25 divided by 1.25.

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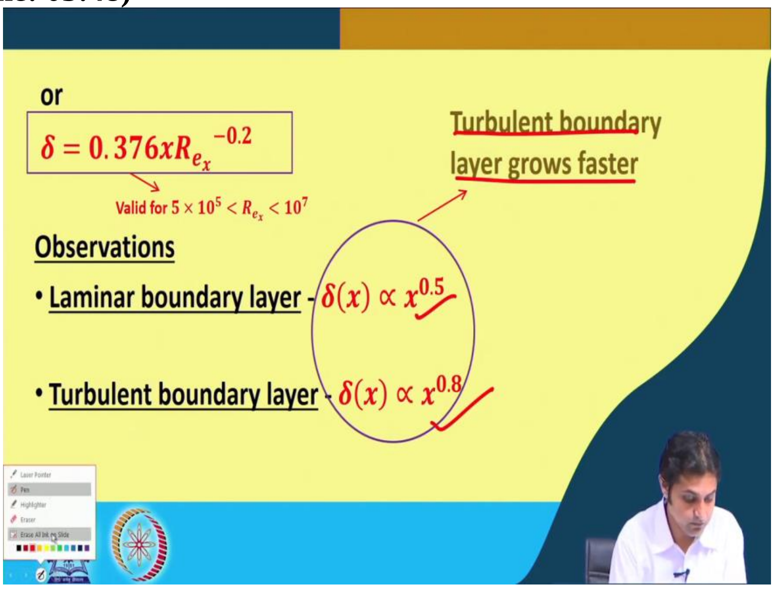

When in writing in terms 0.294 will become

$$\delta = 0.376 x R_{e_x}{}^{-0.2}$$

and this is valid for Reynolds number 5 into 10 to the power 5 between 10 to the power 7. This is the boundary layer value for a turbulent flow. Now, some observations, if you see, remember, the laminar boundary layer was delta x was a function of $x^{0.5}$. For a turbulent boundary layer is a function of $x^{0.8}$.

So, this means that the turbulent boundary layer grows faster, you see, as it grows as x to the power 0.8, and this grows at x to the power 0.5.

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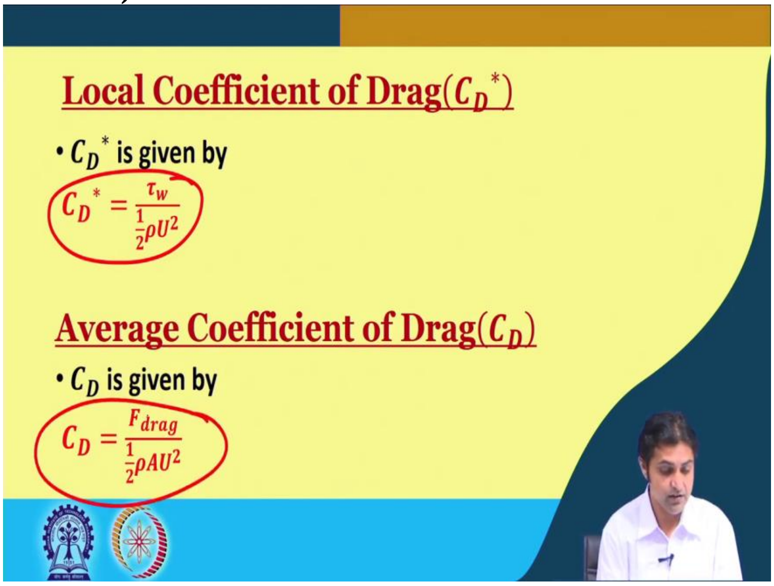

There is some, a term called local coefficient of drag C D star. So, C D star is given by, nothing, it is the ratio of tau w, the shear stress near the wall divided by 0.5 ρ U^2. Average coefficient of drag C D is C D is given by

$$C_D = \frac{F_{drag}}{\frac{1}{2}\rho A U^2}$$

. So, this is the ratio of the shear stresses sort of and this is the ratio of forces.

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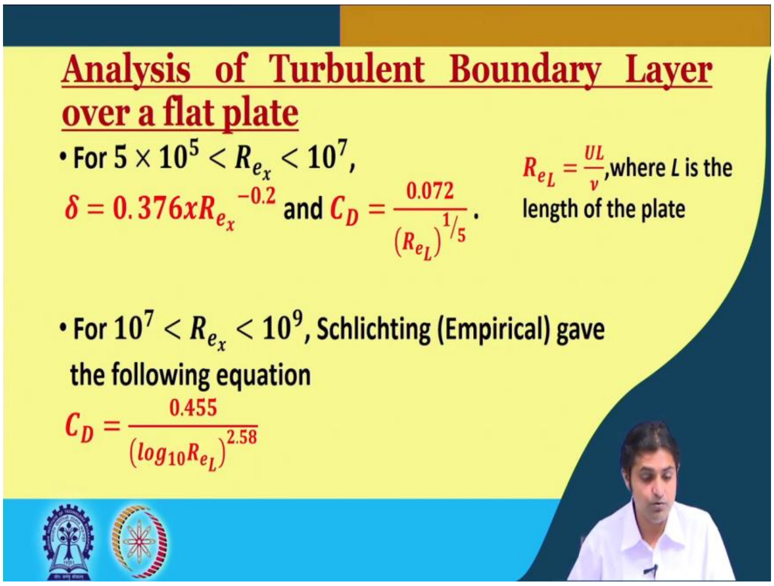

So, now the analysis gives us, that for Reynolds number between 5 into 10 to the power 5 and 10 to the power 7, this is what we got, boundary layer thickness was

$$\delta = 0.376xR_{e_x}^{-0.2}$$

and C D gave us

$$C_D = \frac{0.072}{(R_{e_L})^{1/5}}$$

. As if you put in those values we can get, C D as follows. Where, R e at length L is

$$R_{e_L} = \frac{UL}{\nu}$$

where L is the length of the plate.

And for increasing, so, if your Reynolds number is more than 10 to the power 7 Schlichting gave an empirical equation, that gave, C D as 0.455 divided by log 10 to a Reynolds number to the power 2.58. But you do not need to memorize this equation. So, now using this analysis of turbulent boundary layer over a flat plate, what we are going to do is, we are going to solve some problem.

This actually is always a good method for understanding the complex problems of the boundary layer analysis in turbulent boundary layer.

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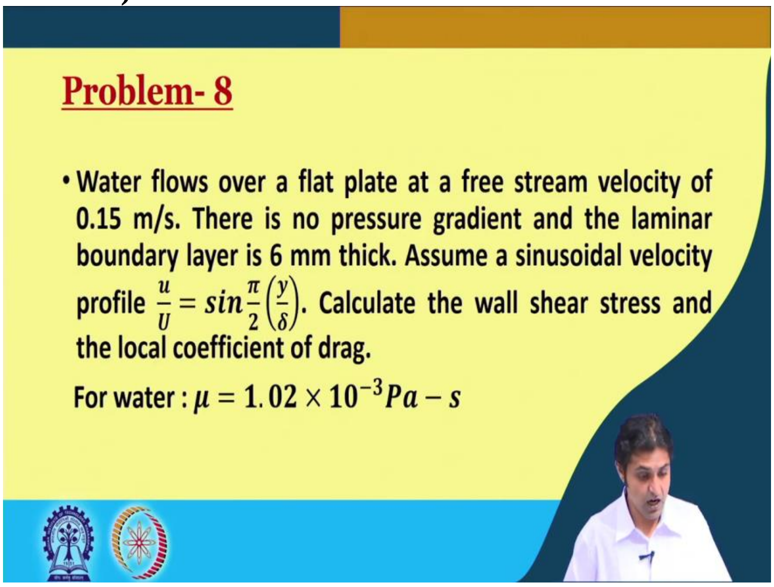

So, we start with the problem that says that the water flows over a flat plate at a free stream velocity of U 0.15 meters per second, that is, U. There is no pressure gradient and the laminar boundary layer is 6 millimeters thick. Assume sinusoidal velocity profile, like this. Calculate the wall shear stress and the local coefficient of drag. So, we have to calculate the wall shear stress and local coefficient of drag. So, how do we attack this problem?

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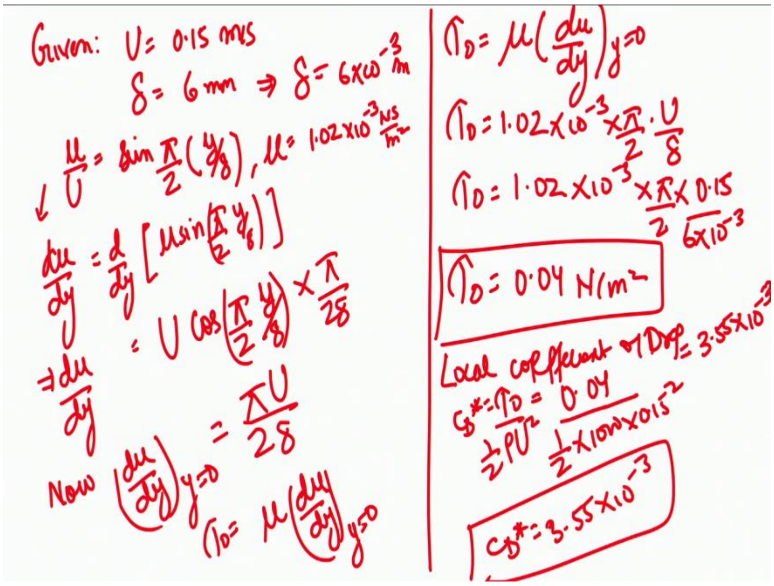

So, as always we are going to have the things that are given. So, given, U is 0.15 meters per second, delta is 6 millimeters or delta is 6 into 10 to the power - 3 meters, for example, and u / U is sin of pi / 2 into y / delta and mu here is given as 1.02 into 10 to the power - 3 Newton second per meter square. So, now, we do, du / dy, for example, that gives us d dy of u, u comes this side sin pi / 2 y / delta. So this gives us, U will come out cos pi / 2 y / delta into pi / 2 delta.

So, now, du / dy at y = 0 will be, we substitute y = 0 here. So, this becomes pi U / 2 delta. If we put y = 0 this cos term is going to be 1. And we know that tau 0 = mu du / dy at y = 0. So, tau 0 is going to be mu du / dy at y = 0 again. So, what we write is, mu is 1.02 into 10 to the power - 3 into, I will write it on the next line, so, tau 0 is going to be 1.02 into 10 to the power - 3 into pi / 2 into U / delta.

And after substituting in the value, 1.02 into 10 to the power - 3 into pi / 2 into U is 0.15 and delta is 6 into 10 to the power – 3, tau 0 comes to be 0.04 Newton per meter square, very simple to solve. So, local coefficient of drag C D star is given as, tau 0 / half ρ U square and simply putting in the value 0.04 divided by half into 1000 and U is 0.15 whole square and this comes out to be 3.55 into 10 to the power - 3 or C D star finally is 3.55 into 10 to the power – 3.

So, one very simple question, how to be, I mean, just by using the definitions of C D and the shear stress near the wall, we have found out this.

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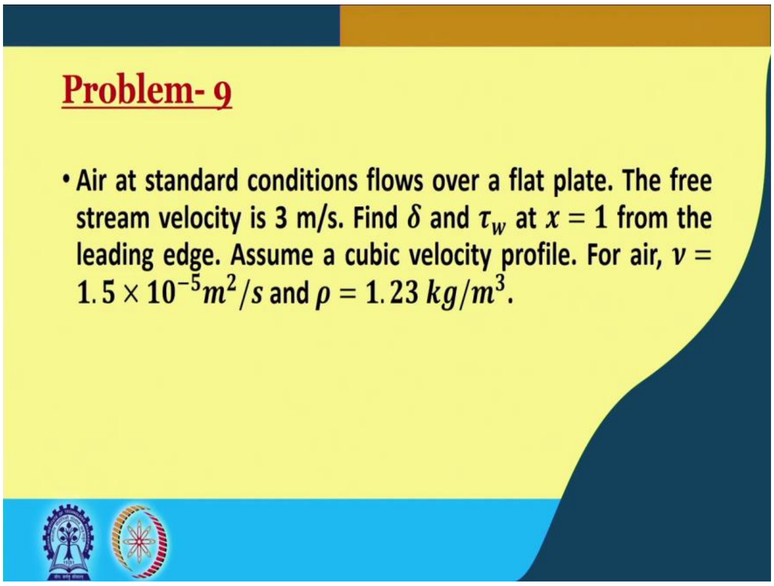

So, this was one problem. We will see another problem, which says that air at standard conditions flows over a flat plate. The free stream velocity is 3 meters per second. Find delta and tau w at x = 1 meter from the leading edge. Assume a cubic velocity profile. For air, nu = 1.5 into 10 to the power -5 meter square per second and ρ = 1.23 kilogram per meter cube.

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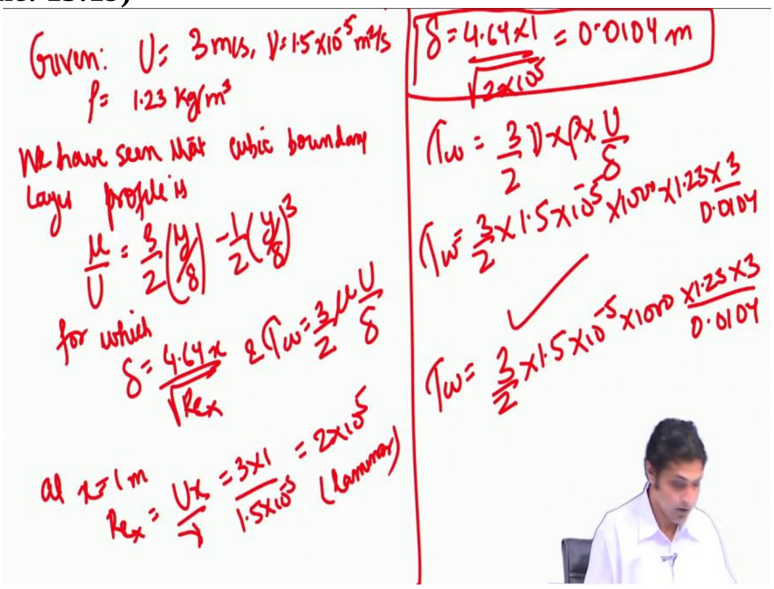
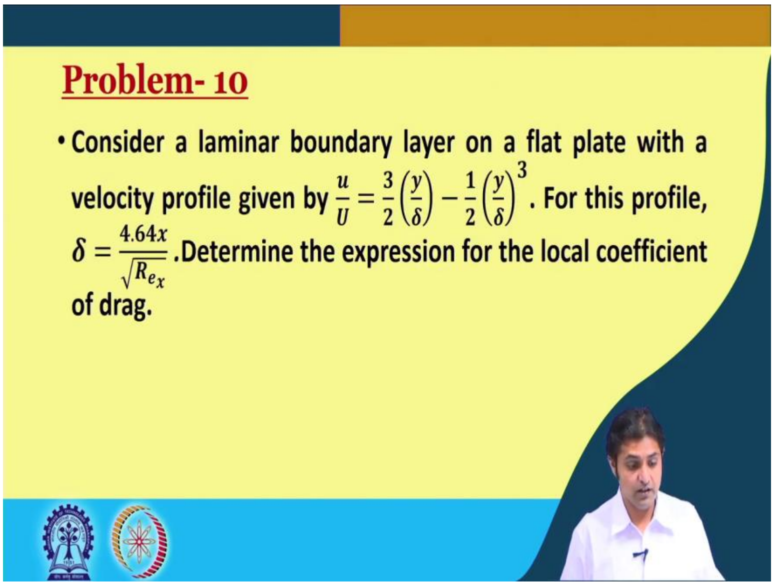


So, what we are going to do? We are going to given, U = 3 meters per second, nu = 1.5 into 10 to the power - 5 meters square per second and ρ is given as 1.23 kilogram per meter cube. So, we have seen that cubic boundary layer profile is u / U = 3 / 2, from our laminar analysis, - half y / delta cube. For which boundary layer thickness is 4.64 x under root R e of x and tau w is 3 / 2 mu U / delta. So, at x = 1 meter, Reynolds number at x is going to be U x / nu, so, 3 into 1 divided by 1.5 into 10 to the power - 5 and this is going to be 2 into 10 to the power, so, basically laminar.

Now, delta is going to be 4.64 into 1 divided by under root 2 into 10 to the power 5 and this gives us 0.0104 meter. This is the delta. And tau w will give us 3 / 2 nu into ρ into U / delta. So, this is going to be 3 / 2 into 1.5 into 10 to the power - 5 into 1000, nu is directly given, into 1.23 into 3 divided by 0.0104. So, this is the way tau w can be found out, just do some multiplication. I will write it again, 3 / 2 into 1.5 into 10 to the power - 5 into 1000 into 1.23 into 3 divided by 0.0104. So this is some small problems. That we are doing using the. **(Refer Slide Time: 17:26)**



So, now, this gives us another, we will solve another problem. If we consider a laminar boundary layer on a flat plate with a velocity profile given by the cubic velocity profile, that is, what we have seen for this profile, we already know. Determine the expression for local coefficient of drag. So, how to do that?

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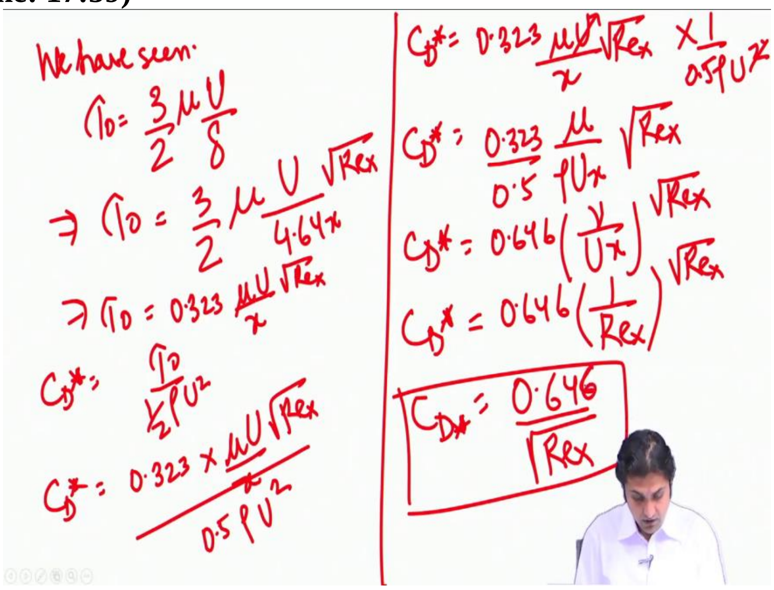

So, we know everything here before So, we will get directly to the point here. For given profile we have already derived that tau 0 is 3 / 2 times mu times U, sorry. Sorry, I will write it again. We have seen that tau 0 = 3 / 2 mu U / delta implies tau 0 = 3 / 2 mu will remain mu, U will remain U but delta is the equation we had derived 4.64 x under root R e of x. So, this is our tau 0, or if we just simplify it more, this 3 / 2 and 4.64, it will become 0.323 into mu U / x under root R e x.

Now, what is the local coefficient of drag? Tau 0 / half ρ U square. So, C D star tau 0 is 0.323. So, basically this is a sort of a derivation of steps which some results which I showed you directly. So, now this is the derivation in terms of a question. And I basically skipped it so that they could solve it as a question. So, mu U / x under root R e x divided / 0.5 ρ U square or C D star = 0.323 U / x under root R e x into 0.5 ρ U square.

So, you can actually cut this U and U here one so C D star can be written as 0.323 divided by 0.5 mu / ρ U x under root R e x. So, C D star is equal to, this will be 0.646 and now, this mu / ρ is nu and this is U under root R e x or 0.646 and this is Reynolds number. So, C D star finally can be written as 0.646. I will take the eraser write it properly 0.646 divided by under root, so, this is the drag coefficient 0.646. So, this is the solution to one of the other questions.

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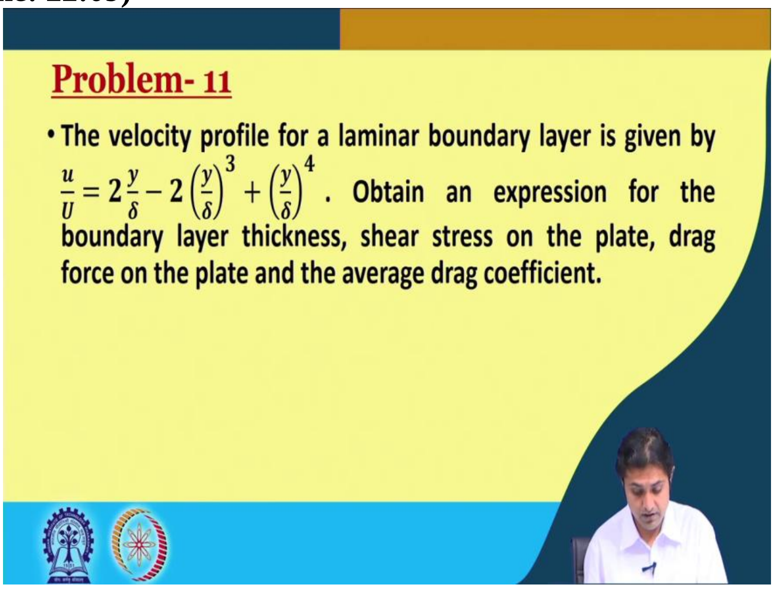

One more problem. So, the velocity profile for a laminar boundary layer is given by, so it is not a cubic but it is a one more term. Obtains an expression for the boundary layer thickness, shear stress on the plate, drag force on the plate and the average drag coefficient? So, this is a question where you have to find almost everything. So, you have to closely follow the procedure here. I will try to lay down the procedure and write the results, but you are expected to solve this problem at your own, at your home.

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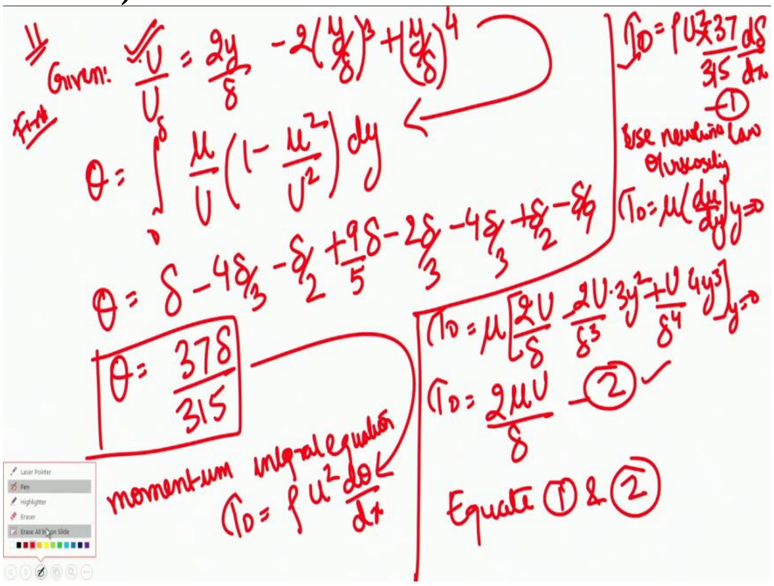

So, we have been given. So, problem number 11 is, we have been given, u / U is 2 y / delta - 2 y / delta whole cube + y / delta to the power 4. So, first thing that we calculate the momentum thickness because that is required in von Karman momentum integral equation. So, it is 0 to delta u / U into 1 - u square / U square dy. So, this profile you substitute in this equation.

And after substituting you integrate in with respect to y, y vary from 0 to delta and what you are going to obtain, I will just write down. There are many terms so one of the term, first term will be delta - 4 delta / 3 - delta / 2 + 9 / 5 delta - 2 delta / 3 - 4 delta / 3 + delta / 2 - delta / 9. So, the momentum thickness that you are going to get is 37 delta / 315. Now, this is the first. Second, you have to use momentum integral equation. This is the first step.

Secondly, what you do is, in any type of such type of equation? Use momentum integral equation. And what do you do? Tau 0 is ρ u square d theta dx and we substitute theta here from this one and what you do is, so tau 0 can be written as ρ U square into 37 / 315 d delta / dx. So, this is tau 0. This is equation number 1. Second, is you use Newton's law of viscosity.

Use Newton's law of, this is the standard way to solve all the sum, if you follow this everything is easy, is equal to mu du / dy at y = 0. So, u velocity profile we already know from here. So, this point and we do this tau 0 will be, so, after differentiating what we get is 2

U / delta - 2 U / delta cube into 3 y square + U / delta to the power 4 into 4 y cube and you substitute y = 0. Therefore, you get tau 0 = 2 mu U / delta.

This is number 2. So, this is 1 and this is 2. And what is the best way? You equate equation number 1 and equation number 2. So, we will go to another page.

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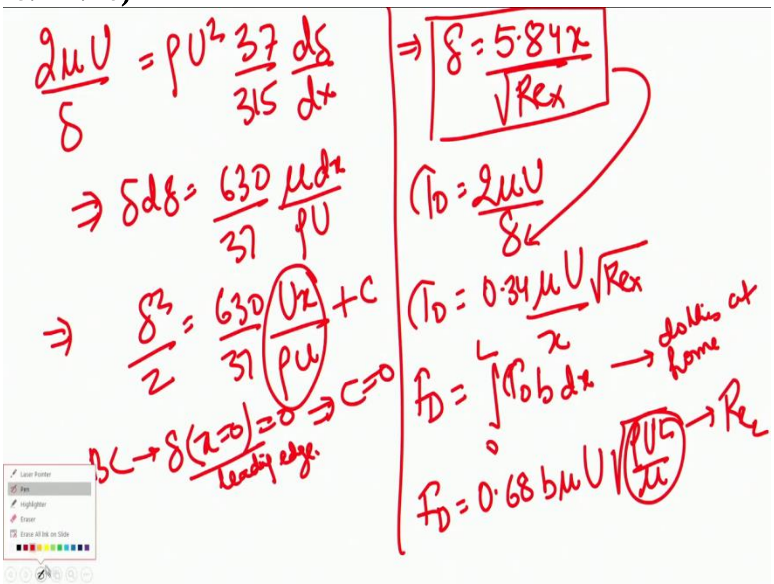

So, after equating, you get, 2 mu U / delta = ρ U square into 37 divided / 315 d delta / dx and if you integrate it, so it will become something like delta d delta will, before integration 630 / 37 into mu dx / ρ into U, or delta square / 2 = 630 / 37 U x / ρ u + C. The boundary condition here is, that the boundary condition at the leading edge is going to be the boundary layer is going to be 0, which gives C = 0.

So, this is leading edge. This is the boundary condition. So, implies if you put here in terms of, because this is going to be Reynolds number, 1 / Reynolds number. So, this is going to give you delta = 5.84 x / under root R e x. So, this is how you get delta. Now, you can get, obtain, 2 mu U / delta. So, delta is known from here and everything else is known. So, you can obtain tau 0 as 0.34 mu U / x under root R e x. Now, the drag force is simply nothing but 0 to L tau 0 b dx.

And if you substitute in terms of x, you are going to find out, I think you do this at home, you substitute all these values tau 0 in terms of x, also use Reynolds number you put at U and x and everything and then you can get 0.68 b mu U under root ρ U L / mu. This actually is

Reynolds number at L. But anyways, this is the thing. And lastly, is the remaining is the drag coefficient, you know.

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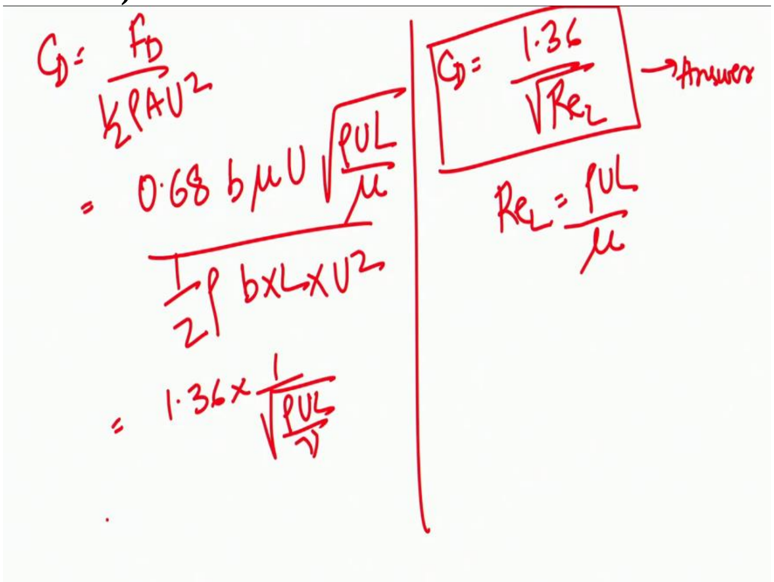

C D is given by F D divided by half ρ A U square. So, F D you know, everything is known. So, I will just show in a little more detail, it is 0.68 b mu U under root ρ U L / mu divided by half ρ b is, the area is b into length into U square and this will give us 1.36 into 1 / ρ U L / nu. Or finally, C D is 1.36 / under root R e into L. Where, R e Reynolds number at length L is ρ U L / mu.

So, this is how you have seen for any velocity profile, we are able to find out almost all the parameters that we know. So, what we do is we will stop this lecture here and resume with a problem, and then in the next class. And then we will go to the final chapter of the boundary layer analysis. It might require 1 or 2 more lectures. Thank you so much.