Hydraulic Engineering
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Lecture – 20
Boundary Layer Theory (Contd..)

Welcome back to fourth lecture of the boundary layer analysis and we are going to start this lecture with Problem number 5.

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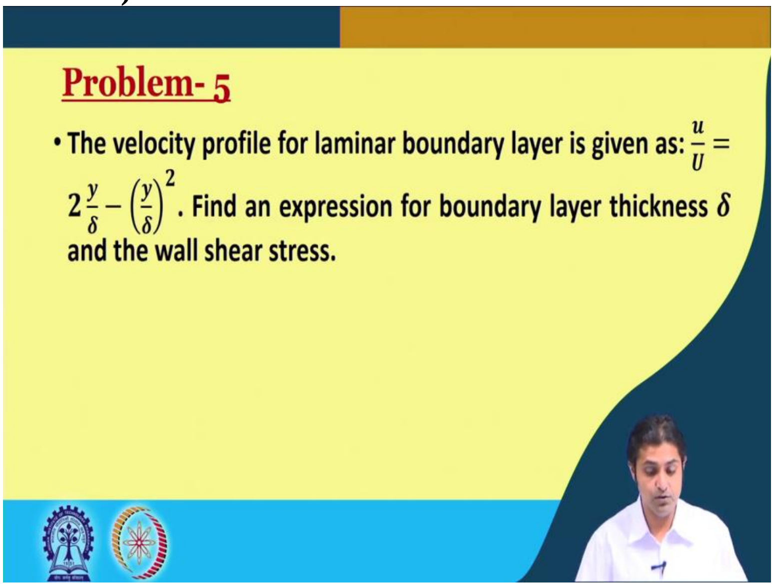

The question says that, the velocity profile for laminar boundary layer is given as u / U equal to 2 y / delta - y / delta whole square. Now, we have to find an expression for boundary layer thickness delta and the wall shear stress.

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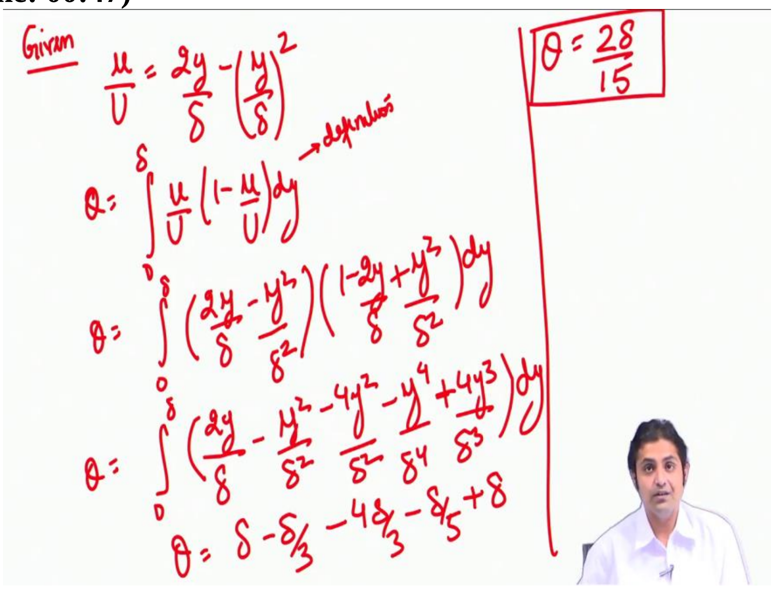

So, what is given is, u / U = 2 y / delta - y / delta whole square. Therefore, the momentum thickness can be written as, 0 to delta u / U 1 - u / U dy. This is the definition of momentum thickness. So, it is going to be, 2 y / delta – y square / delta square into 1 - 2y / delta + y square / delta square dy, whole dy or the momentum thickness is going to be, integral 0 to delta 2 y / delta - y square / delta square - 4 y square / delta square - y to the power 4 / delta to the power 4 + 4 y cube / delta cube into dy.

After integration and substituting the limits, the first term will give us, delta - delta / 3 – 4 delta / 3 - delta / 5 and the last term is give us delta. Therefore, theta is going to be 2 delta / 15. This is the momentum thickness. So, we proceed to the next page, where we are going to solve this question further. So, I am going to have a new.

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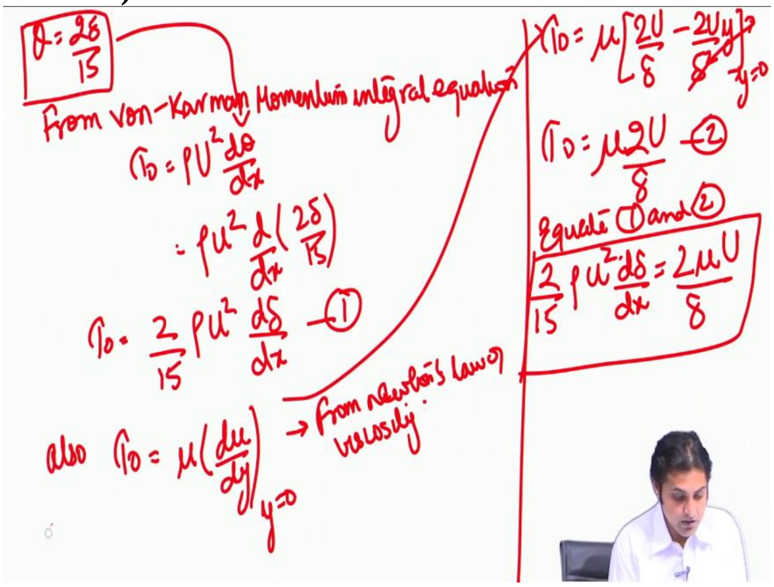

So, we have got, theta is equal 2 delta / 15. Now, from von-Karman momentum integral equation, tau 0 = rho U square d theta / dx. So, rho u square and we put the theta here. So, d dx of 2 delta / 15. Therefore, tau 0 is going to be 2 / 15 rho u square d delta / dx and this is equation number 1. Also tau 0, we know, is can be written as mu du / dy at y is = 0. And how do we get this? From Newton's law of viscosity.

So, velocity profile we already know. So, if u / U was given, in terms of y and delta. So, we can, so, from here, we can go tau 0 can be written as mu will be and now du / dy. So, it will become 2 U / delta - 2 U / delta square into y at y = 0. So, this will become mu, this term will go away. When we substitute, it will be 2, this is mu, so, 2 U / delta, this is equation number 2. Equate 1 and 2, 2 / 15 rho u square d delta / dx = 2 mu U / delta.

So, this is the equation that we get. And now to find delta what we must do? We must simply integrate it. So, we will do this integration on the next page.

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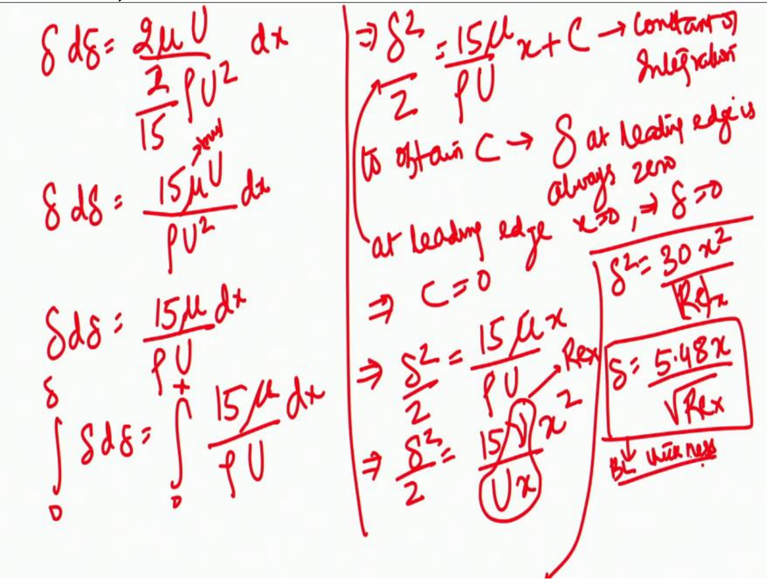

So, this integration will go like, delta d delta = 2 mu U, you just rearrange the terms, divided by 2 / 15 rho U square dx. So, this will become delta d delta is equal to, we can cancel this one, I mean, mu U, when U will get cancelled. This is mu into rho U dx, sorry, rho U square dx or d delta, U, U get cancelled, so it will be 15 mu / rho U dx and then we can integrate 0 to delta and this one will go from 0 to x.

So, 15 mu divided by rho U dx or after integrating, it will become delta square / 2 = 15 mu divided by rho u, sorry, this is capital U, x + C. This is the C is constant of integration. But how do we obtain C? We know to obtain C, we know that boundary layer thickness at leading edge is always 0. So, at leading edge, x is 0 and here delta is also 0. So, if you substitute this in this equation, we get C = 0, Therefore, we can simply say boundary layer thickness, delta square / 2 = 15 mu times x / rho U.

Or just to simplify, what I am going to do? This left hand side will remain the same. So, if you divide mu / rho, mu / rho this becomes nu. And this U, multiply it with x numerator and denominator as well, and this one is written as Reynolds number at a distance x. So, I just conclude here, then delta square can be written as, we take 2 this side, that will be, 30 x square divided by R e x, or in other words, under root, so, 30 when done under root it becomes 5.48 at x divided by under root R e x. So, this is our first answer, boundary layer thickness. Now, the second part, I will erase all ink.

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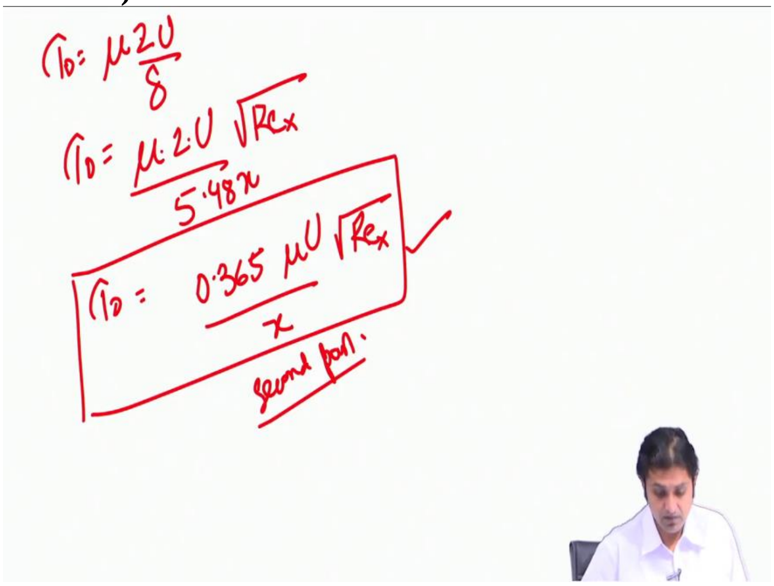

So, for second, we know, that tau = mu 2 U / delta right, that is, what it came to. Now, it is a matter of just a substitution. So, tau 0 is mu 2 U delta is 5.48 x under root R e x. So, tau 0 comes to be 0.365 mu U / x under root R e x. So, after that is found you can express it in any form that you want. So, this is the second part. It was a long question, but actually clearly indicated how we can use the von-Karman momentum integral equation for finding the boundary layer thickness.

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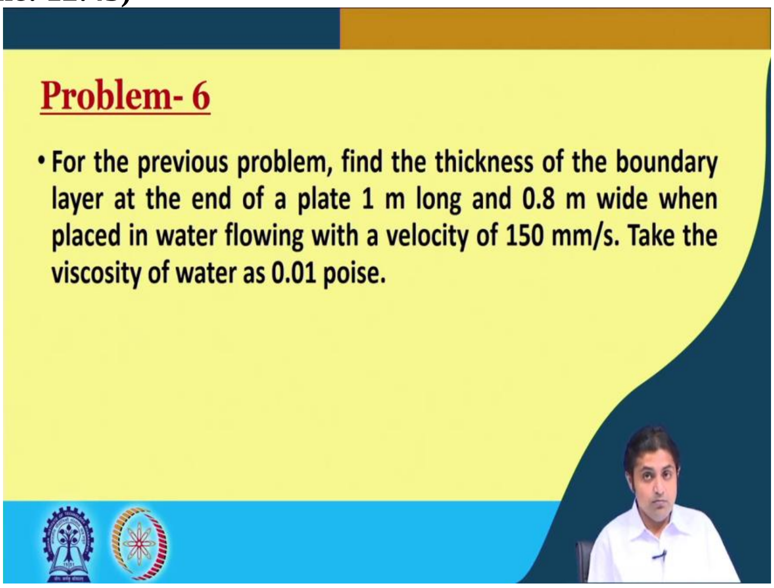

So, now, the last problem in this set is that, for the previous problem find the thickness of the boundary layer at the end of plate 1 meter long and 0.8 meter wide when placed in water flowing with a velocity of 150 millimeters per second. So, the question, theoretically we have done. Now, we need to put in some values. And it says, take the viscosity of water as 0.01 poise. So, to solve this we will go to the white screen again.

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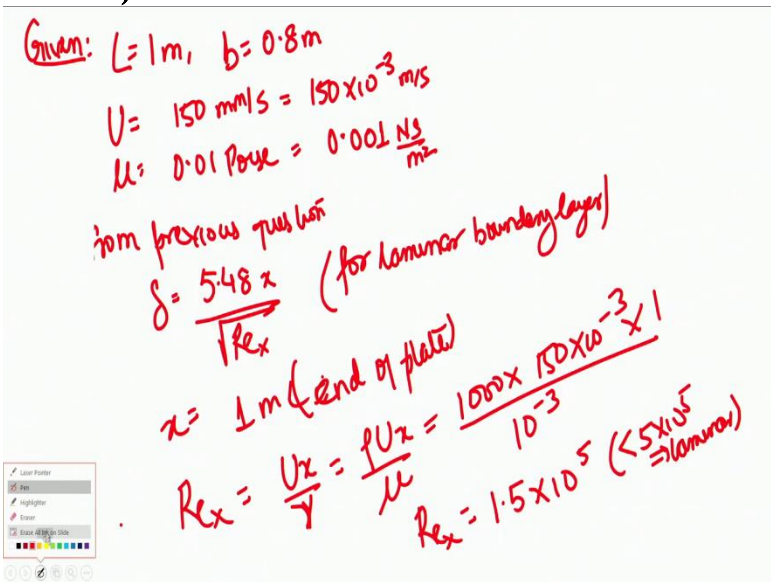

As always, we will write given. So, length is given as 1 meter, b is given as 0.8 meter, U is 150 millimeters per second or we can write 150 into 10 to the power -3 meters per second, mu is given as 0.01 poise or in SI unit, we can write, 0.001 N second per meter square. So, what we have found out from the previous question?

Delta is 5.48 x divided by under root R e at x, Reynolds number for laminar boundary layer. Because why laminar? Because we have used tau = mu du dy, that is, for the laminar boundary layer, At end of the plate x = 1 meter, end of plate. So, Reynolds number at this position will be U x / nu or rho U x / mu. So, 1000 is the density of the water, 150 into 10 to the power - 3 into x is 1 meter divided by mu is 10 to the power - 3.

So, Reynolds number at x is going to be 1.5 into 10 to the power 5. And of course, it is less than 5 into 10 to the power 5 means, laminar. So, our assumption was okay. So, I will, before proceeding.

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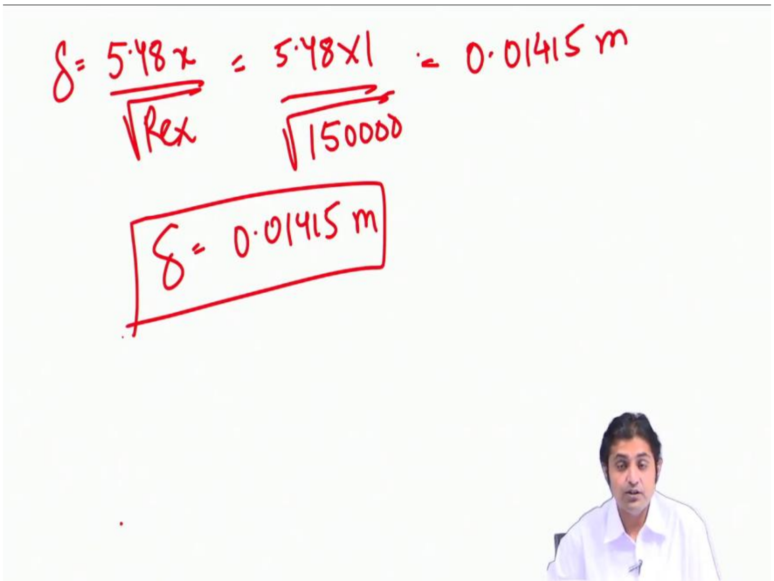

Alright. So, delta by formula is 5.48 x divided by under root R e x. So, 5.48 into 1 divided by under root 150000 and that comes out to be 0.01415 meters. So, the real values, I mean, the numerical values come 0.01415 meters. So, this is the boundary layer thickness. Here, we have calculated the boundary layer thickness in terms of the numerical values.

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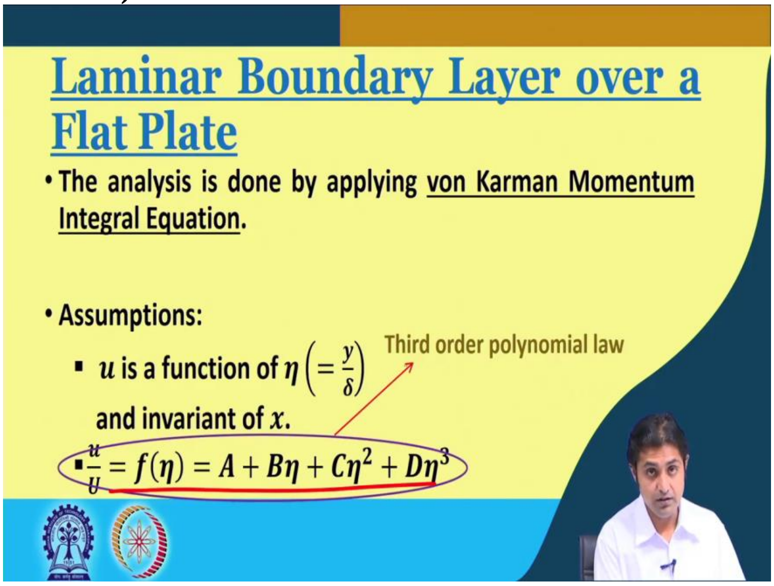

So, we will now proceed to the other topic. So, we last, when we concluded before solving the problems what we said? We said was, that the equation 1, I mean, that equation is valid both for laminar and turbulent boundary layer. So, first we study laminar boundary layer over a flat plate. So, now, this is analysis is done by applying von-Karman momentum integral equation, as it is common for both laminar and turbulent flow.

The assumption says that, u is a function of eta. Eta is y / delta. y is the distance above the plate and delta is the boundary layer thickness and is not a function of x. So, u is not a function of x but only a function of y or to be precise y / delta. So, what we do is, we assume that, u / U is a function of eta and this is written as A + B eta + C eta square + D eta cube. So, how do we approach this problem? How many terms are there? 4 terms. How many unknowns? ABCD.

But fortunately, we know, that while doing those equations, I mean, von-Karman momentum integration equation, we had 4 boundary conditions, if you remember. I will just briefly, take you and show that. So, you see. So, what were those boundary conditions? Very simple.

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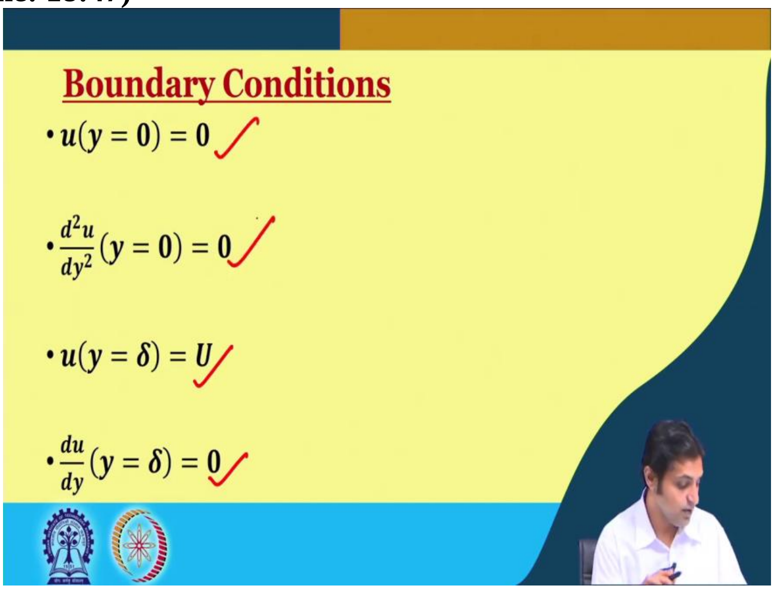

Boundary conditions. So, these were the 4 boundary conditions, if you remember. u at y = 0 is 0. Del square u / del y square at y = 0 is also 0. And u at y = delta = U and then del u / del y = at the boundary layer, the velocity gradient at the top of the boundary layer is 0. There is a free stream velocity, these two are 0. So, now, we apply this condition for also for laminar cases. What we get? I mean, this function is a third order polynomial law.

So what we get? So, if you apply u at y = 0 is 0, which will give us A = 0. So, if you put, if y = 0 eta is 0. So, A+ 0+ 0+ 0 equals to 00. If you apply second boundary condition, del square u / del square y. So, you differentiate this f of eta twice, so, this will become 0, this will become 0 on twice differentiation. This will give us 2C + 6 D eta = 0. So, when eta = 0, this means, C = 0.

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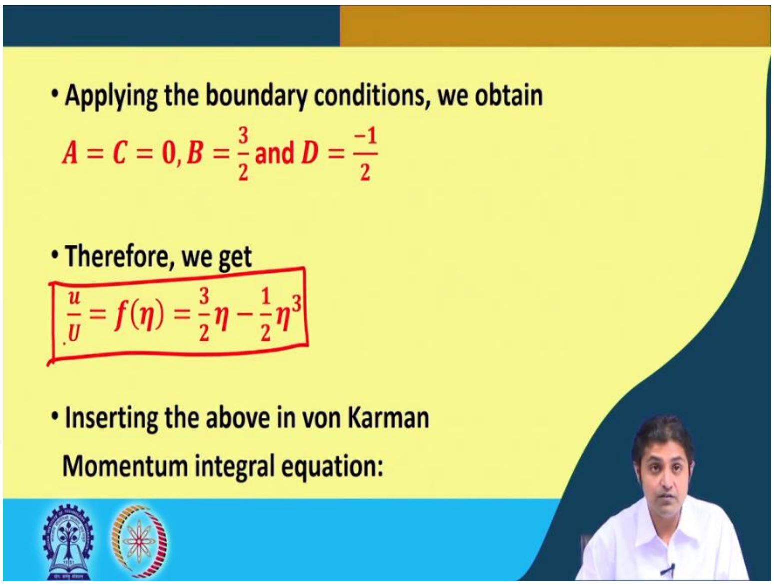

So, the first 2 conditions are going to give us A = C = 0 and substituting third and fourth. So, fourth one, the third one where we have to differentiate once, will give us this condition, you can try this at home, and the fourth condition will give us D is equal to minus 1 /2. So, those 4 boundary conditions are used to solve for these value of A B C and D. Therefore, we can simply write that, u, I mean, u / U which is a free stream velocity is given by

$$\frac{u}{U} = f(\eta) = \frac{3}{2}\eta - \frac{1}{2}\eta^3$$

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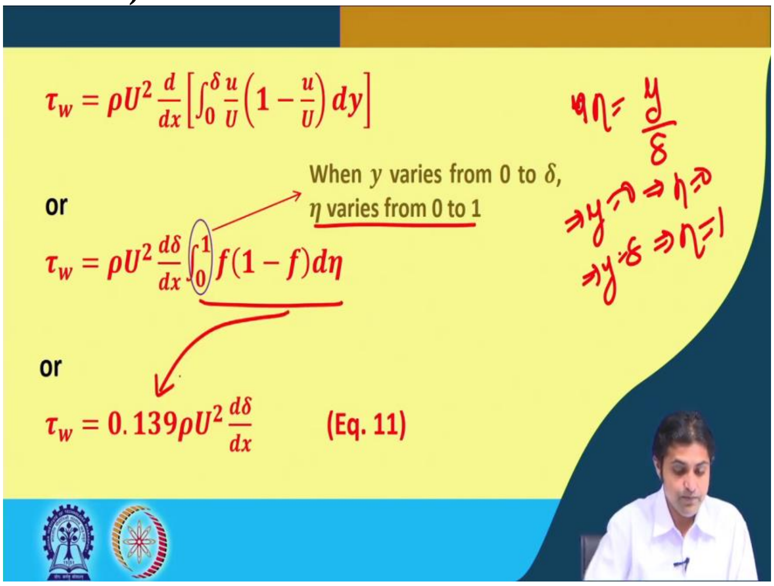

Now, if inserting the above in von Karman momentum integral equation. If we insert this, what do we get? So, tau w is written as rho U square d dx of, this is momentum thickness, theta, or in other words, you see, this is f, 1 - f d eta. Because if, eta = y / delta, so, d eta = dy

/ delta. So, now the limit will change because when y varies from 0 to delta eta can vary from 0 to 1. So, eta is y / delta. When y = 0 implies eta = 0 and when y = delta implies eta = 1.

So, this limit will change, as shown. Or in other terms, what we can write is, so, on integration and converting it into form of delta y / delta, this will give us tau w 0.139 rho U square remains, rho U square. So, this value comes out to be 0.139, you can see that here.

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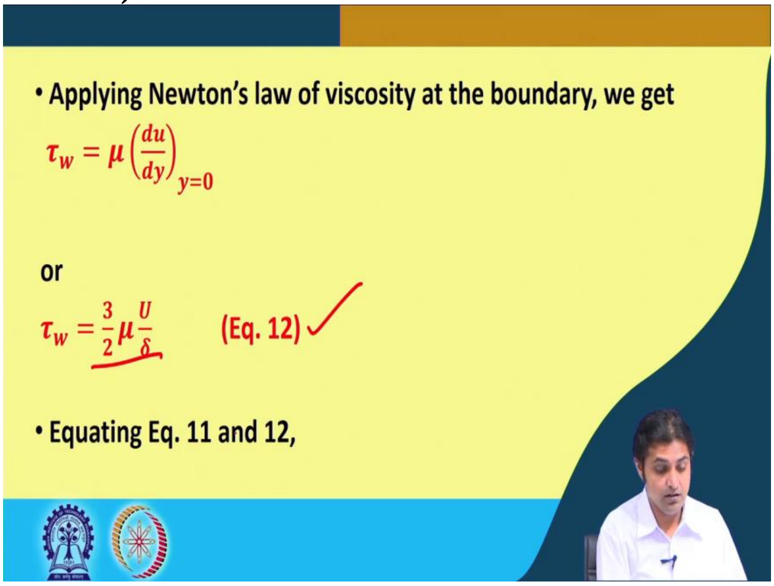

Now, if we are apply the Newton's law of viscosity at the boundary. Why can we apply the Newton's law of viscosity? Because it is a laminar flow. We did the same principle when we were solving the numerical problems. Or, if you see, du / dy at y = 0. So, you see, so, du / dy at y = 0, it is going to give us 3 / 2 mu into U / delta.

So, what do we do next? One way we have obtain Newton law viscosity and the other equation was, equation number 11 here, because what we started with was the velocity profile here, u / U = A+ B eta + C eta square. And then differentiating that, we got this and we are going to equate equation 11 and equation 12.

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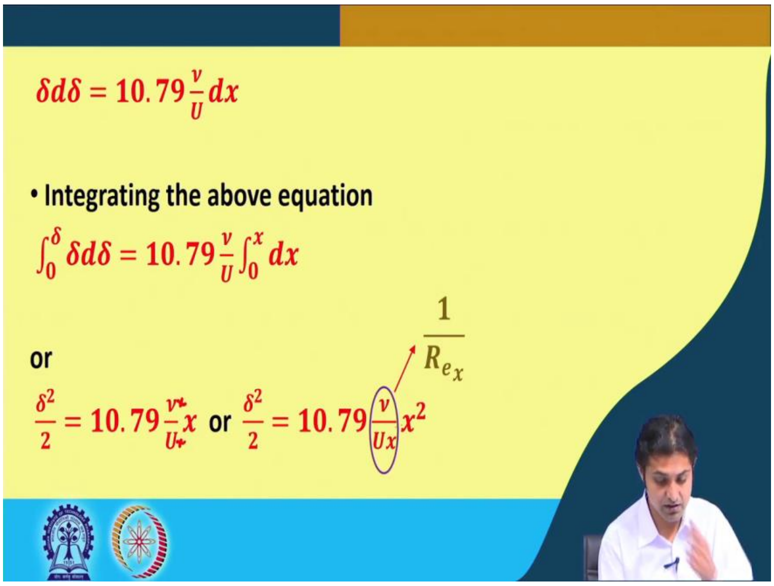

This will give us, delta d delta is equal to 10.79 v / U dx. And if we integrate this above equation, we can get the boundary layer thickness. Actually, the last problem that was their problem number 5 and 6 was exactly the same problem that we solved the way, I mean, the question was different, but the way of solving was the same as this derivation. So, we can write, delta square = 10.79 nu / U x or delta square = 10.79 nu / U x.

So, what we did was, we multiplied x and x up stair, I mean, denominator here also here also. So, this becomes x square and this is Reynolds number. This is what exactly we did.

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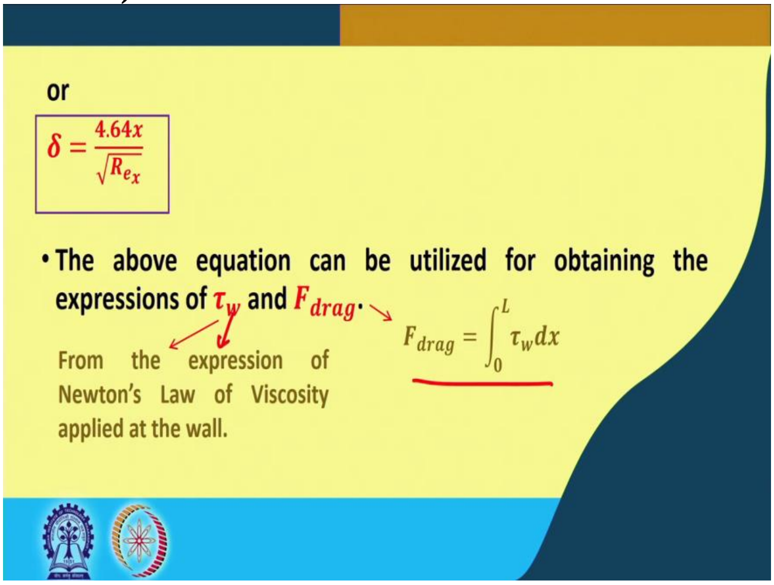

Or in other words, we get, delta = 4.64 x under root Reynolds number of x. Try to remember the question that we did, just finished in the last, I mean, at the beginning of this lecture. So, the above equation can be utilized for obtaining expressions for tau w and F drag because we get the boundary layer thickness. And from the expression of Newton's law of viscosity, we can obtain tau w. And F drag, we can simply use. How? That we integrate, tau w from 0 to over the entire plate. So, this is the way that it is done.

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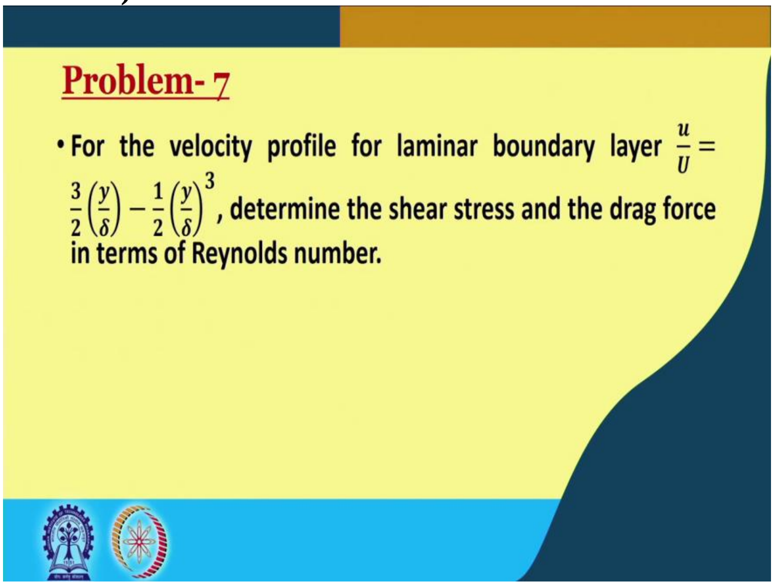

Now, we will solve 1 problem. The problem is, for the velocity profile for laminar boundary layer u / U = 3 / 2 y / delta - half y / delta to the whole cube, determine the shear stress and the drag force in terms of Reynolds number. We have done a similar problem but we will do another one now, to make it more clear.

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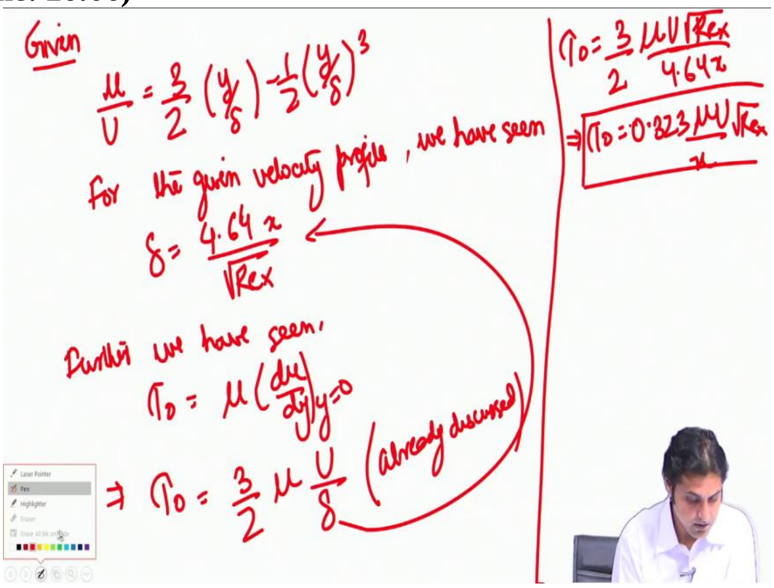

So, first write what is given to us. So, u / U = 3 / 2 y / delta – 1 / 2 y / delta to the whole cube. For the given velocity profile, we have seen that delta came out to be 4.64 x under root

Reynolds number at x. This was the last derivation that we saw. Further, we have seen, tau 0 = mu du / dy at y = 0 implies tau 0 = 3 / 2 mu U / delta. This has already been discussed.

So, delta, we know, from here. So, tau 0 = 3 / 2 mu U under root R e x 4.64 x. Or finally, tau 0 can be written as 0.323 mu U / x under root R e x. Now, the second part is, how do we find the drag force? Very simple, we have laid down the procedure.

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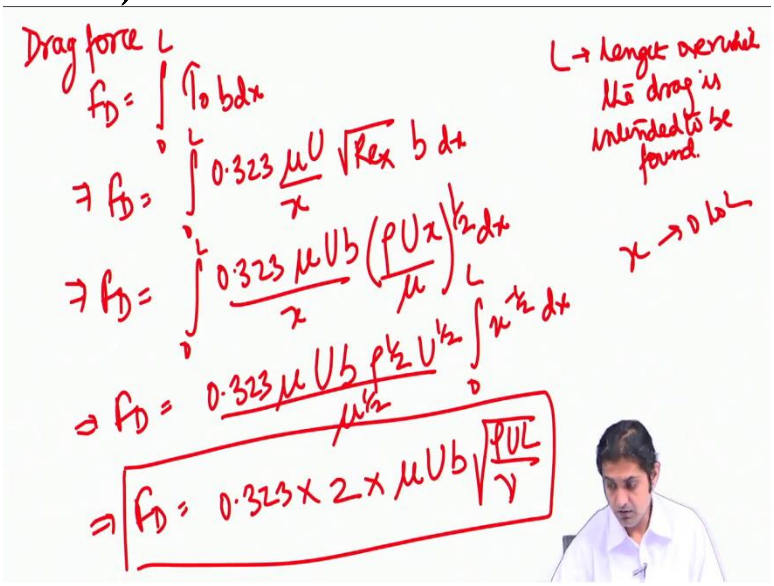

So, drag force F D, can be found out using integral 0 to L tau 0 b dx implies drag force F D = integral 0 to L 0.323 mu U / x under root R e x b dx. Here, L is the length over which the drag is intended to be found. So, implies, drag force integral 0 to L 0.323 mu U b x rho U x / mu to the power 1 / 2 dx. So, implies, F D actually can be written as, after integration 0.323 mu U b rho to the power 1 / 2 it will come out, U to the power 1 / 2 it will come out, mu to the power 1 / 2 it will come out integral 0 to L x to the power – 1 / 2 dx.

So, after integrating and substituting the values of x from 0 to 1, 0 to L x goes from 0 to L. F D will be 0.323 into 2 into mu U b into under root mu rho U L / mu. So, this is the drag force. More importantly, it is this way that it has to be found, I mean, in your questions you will be mostly given an objective question. If you know the way how to find it out, I mean, it will not be so, tedious calculation.

So, I would like to end the lecture at this point in time and we will go through the some more concepts and some more questions in our next class. Thank you so much.