Hydraulic Engineering
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Lecture-19
Boundary Layer Theory (Contd.,)

Welcome back to the continuation of the boundary layer analysis lecture.

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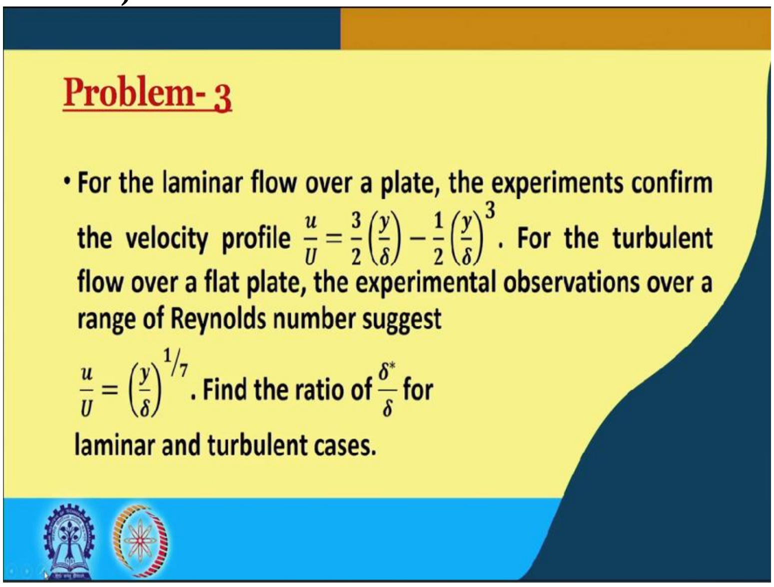
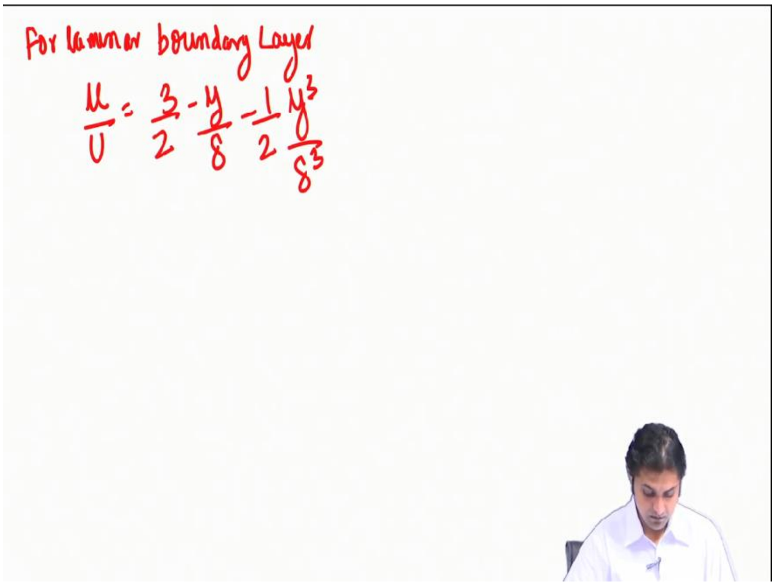


So, we are going to start this lecture by solving yet another problem, where the 2 velocity profiles is given, one for the laminar flow here, and one for the turbulent flow. This is for the turbulent flow. So, this is laminar and this is turbulent. We are going to the ratio of the displacement thickness, delta dash / ratio of delta for laminar and the turbulent cases, for 2 separate cases. So, to we will start. **(Refer Slide Time: 01:10)**



We will go to white screen and we will start doing the solution. For lamina boundary layer, u / U is given as, 3 / 2 - y / delta - half y cube by, so, let me just. Yeah, so, there is some mistake in my writing.

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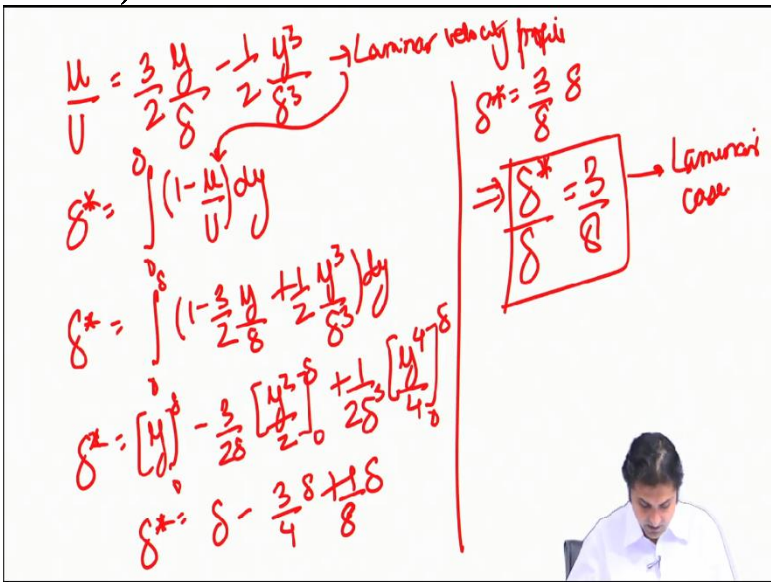

So, actually the laminar was u / U = 3 / 2 y / delta - half y cube / delta cube, this is laminar velocity profile. And therefore the, by definition, the displacement thickness is 0 to delta 1 - u / U dy and we substitute this velocity profile, here, and we are going to get, integral of 0 to delta 1 - 3 / 2 y / delta + half y cube / delta cube into dy and after integration, which we are going to get, y from 0 to delta - 3 / 2 delta and this is going to be, y square / 2 0 to delta + 1 / 2 delta cube y to the power 4 / 4 0 to delta.

And the displacement thickness is going to be, this first term will be, delta - 3 / 4 delta + 1 / 8, this is 8 delta. This will give us, delta dash is equal to 3 / 8 delta and this implies that delta dash / delta for laminar case is going to be, very simple. This we have been doing. Anyways. now, we will do for turbulent boundary layer.

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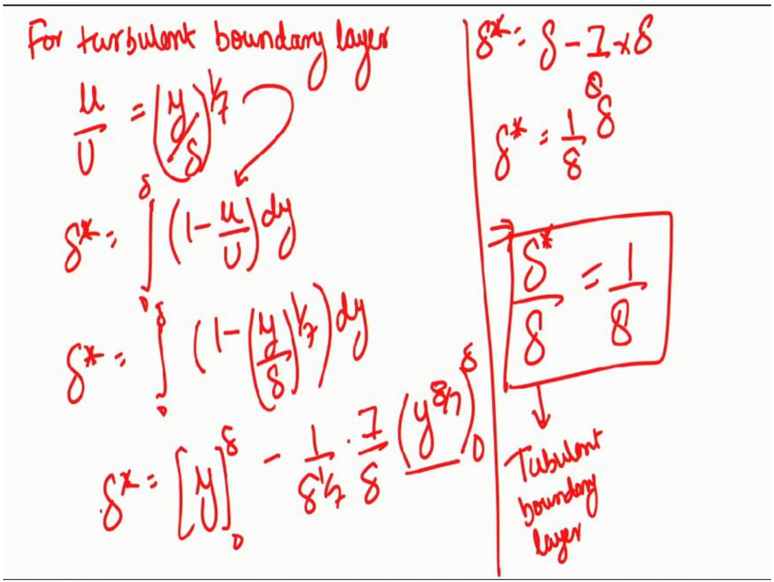

u / U is the velocity profile is y / delta to the power 1 / 7. So, the displacement thickness by definition is, 0 to delta 1 - u / U dy and we plug it in here. And therefore, what we get is, integral 0 to delta 1 - y / delta to the power 1 / 7 dy or we get, delta star as, after integrating and putting the limit, 0 to delta - 1 / delta to the power 1 / 7 into 7 / 8 into y to the power 8 / 7 divided, no, and this also goes from 0 to delta.

And therefore, this comes out to be, after limit delta, second term will be, 7 / 8 into delta or this is going to be, 1 / 8 delta. Therefore, displacement thickness and this is for turbulent. So, this is what the turbulent boundary layer. So, this question is complete.

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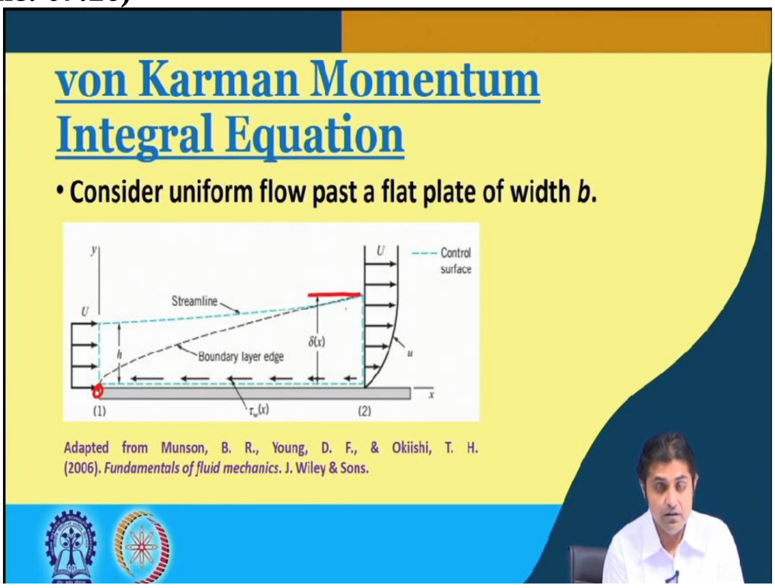

And now, we can proceed to another topic, that is, von Karman momentum integral equation. So, what we do is, we consider a uniform flow past a flat plate of width b. So, the width of the plate is B. So, this is the definitions are as usual, a uniform velocity U is coming and it strikes the leading edge here, and then the boundary layer would start to develop and at the end of the boundary layer where the boundary layer the thickness is delta x, there will be a shear stress developing tau w, as a function of x.

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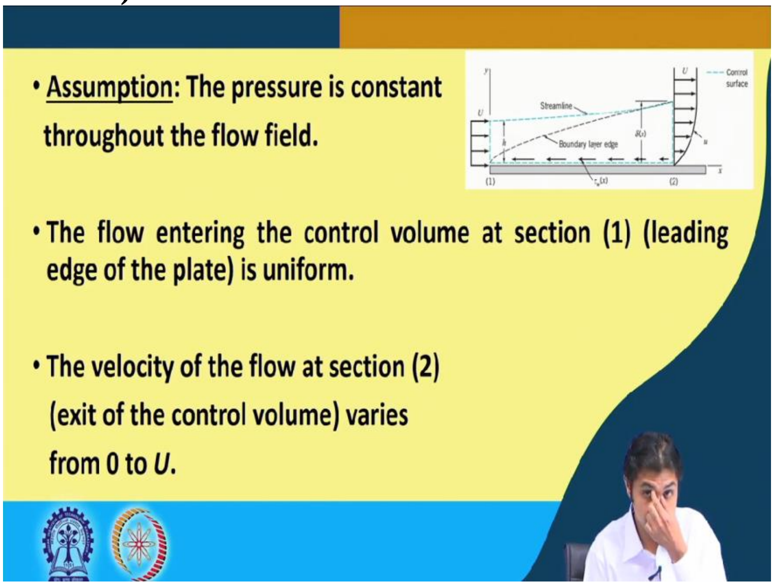

So, we have written the, I mean, we have shown the figure again on the right hand side. So, what is the assumption? The assumption is that the pressure is constant throughout the flow field. The flow entering the control volume at section 1, here, this is uniform, leading edge of the plate. So, the flow entering the control volume at section 1, section 1 is the leading edge of the plate is uniform.

And the velocity of the flow at section 2, here, varies from 0 to U, and this happens because of the boundary layer. So, this is the assumption that we start with.

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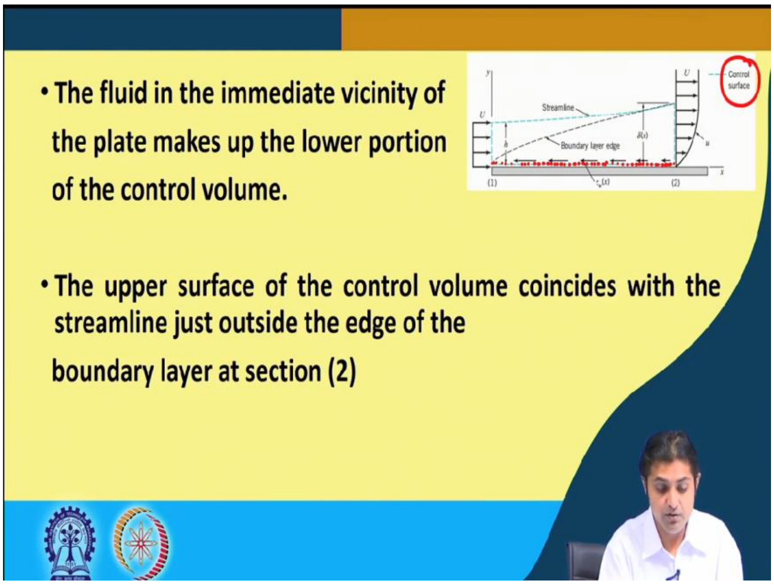

So, the fluid in the immediate vicinity of the plate makes up the lower portion of the control volume. So, you see the control surface here. So, we have chosen a control surface here. This control surface, it says, that the fluid in the immediate vicinity. So, this one you see, the dash, dash line, now, I am marking it red, red, red dots, this is the lower portion of the control volume. And the upper surface of the control volume coincides with the streamline just outside the edge of the boundary layer. So, you see, this is the top most surface of the control volume.

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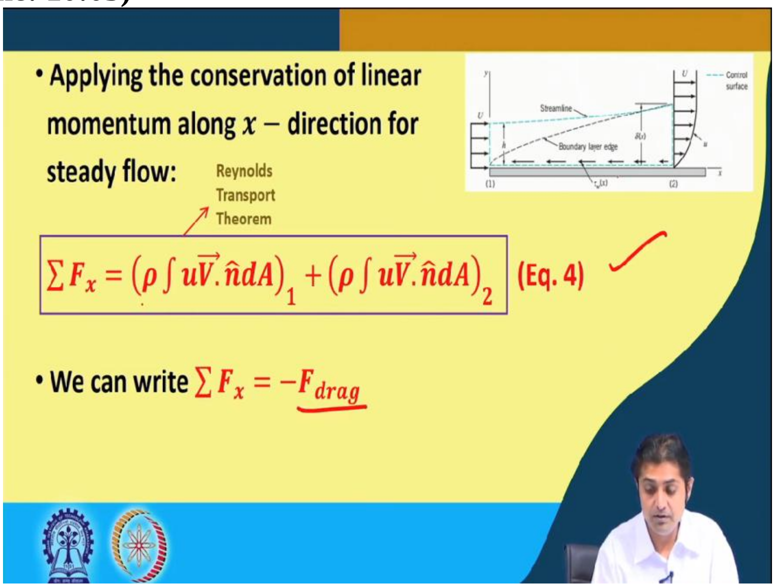

Now, if you apply the conservation of linear momentum along x direction for steady flow, what do we get? So, you do you remember this equation from somewhere, just a quick thought? Yes, it is Reynolds transport theorem that we did in week number 2. So, the conservation of linear momentum along steady along the x direction for the steady flow gives us

$$\sum F_x = \left(\rho \int u\vec{V}.\hat{n}dA\right)_1 + \left(\rho \int u\vec{V}.\hat{n}dA\right)_2$$

For steady flow therefore, there is no time variation.

We can also write, so, drag sigma Fx is equal to - of F drag. So, drag will have happened because of this shear stress here. And this is as I told, is due to taw W.

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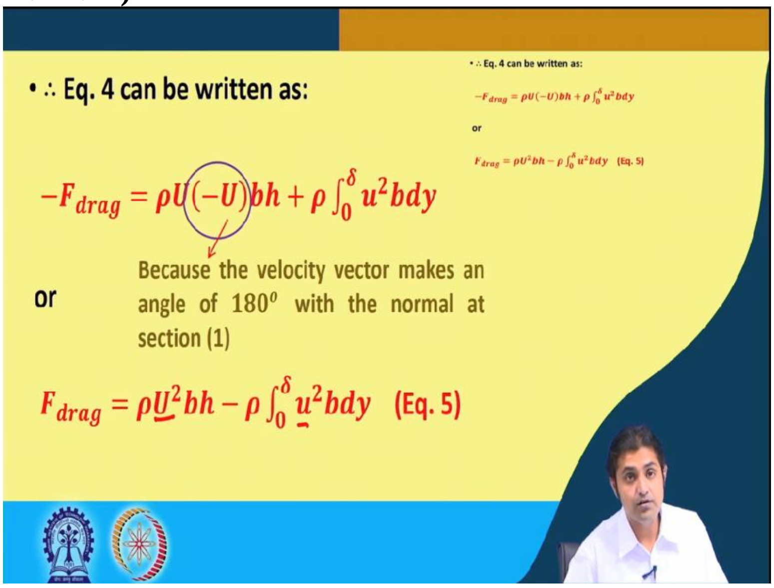

Now, this equation number 4, can be written as, - F drag. So, I will, so, this is the equation, we write,

$$-F_{drag} = \rho U(-U)bh + \rho \int_0^\delta u^2 bdy$$

And this gives us equation 5. What we did it? We just use substituted the equation 4 and input in the velocities. And this is section 1, where uniform flow was there and here the boundary layer. And now, you can understand why 2 different forms of momentums we have written.

And this is minus U, because the velocity vector makes an angle of 180 degree with the normal. So, this was the control surface, you remember, something like this. So, the velocity was in this direction and the normal to this surface was here. So, that is why 180 degrees, or simply, we multiplied - 1 on both side. This becomes F drag is equal to ρ U square bh - ρ integral 0 to delta u square bdy. This U is capital, this u is small.

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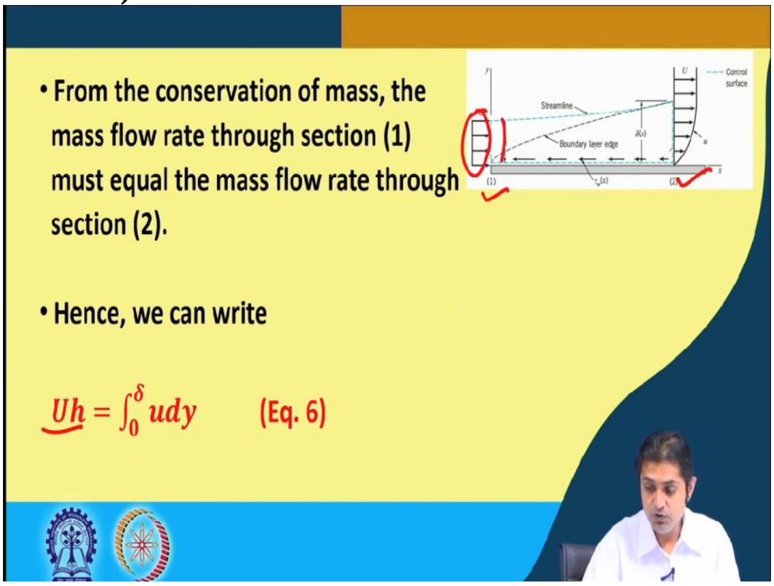

From the conservation of mass, the flow rate through section 1, here, must equal the mass flow rate through section 2, that is, correct. Therefore, we can write, U into h. So, U into h is equal to integral u dy. If you assume, you know, thickness element of thickness and at distance y, the velocity is u, so, it becomes u dy. Very simple.

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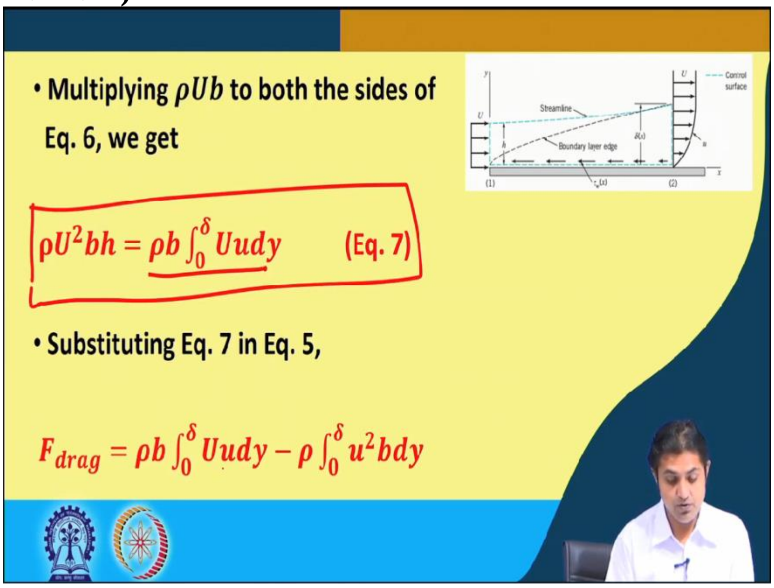

So, if we multiply ρ Ub to both side of the equation 6, this is equation 6. So, we get, ρ U, capital U square, so, we multiply this term, ρ U b. So, for the left hand side becomes ρ U square bh, is equal to ρ b U u dy and this is integrated to 0 to delta. Now, if we substitute this equation, equation 7 into equation number 5. I will take you to equation number 5 again.

So, this was equation number 5, with the drag force equation. What we get? F drag is going to be, ρ b integral. So, the idea was to substitute the first term in equation number 5 by this.

So, it will be written as, ρ b integral 0 to delta U u dy and this was already there, in the equation number 5.

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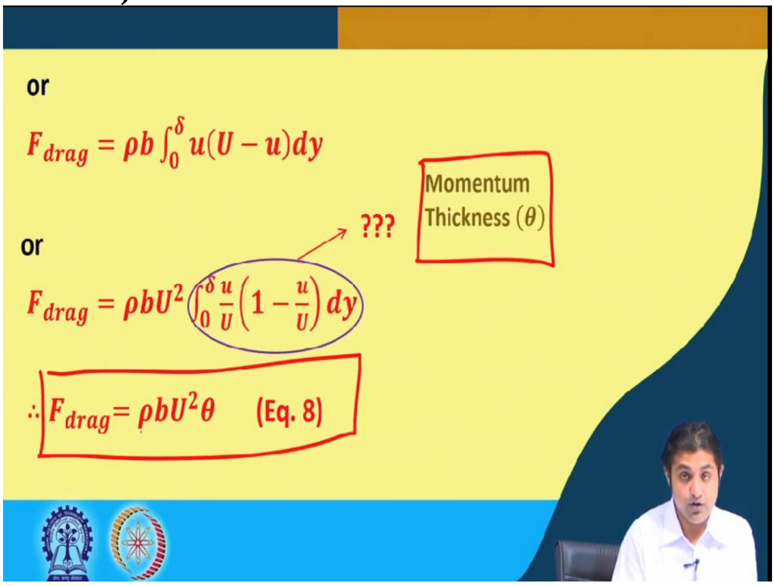

Or simply, if we take some terms outside and take the common, it becomes forced drag, F drag will become ρ b integral of 0 delta u multiplied by the velocity deficit into dy. Simply, in other words, if you take, if you or, I mean, you multiply this side by U square and divide this, so, it will not make a difference U then you get, ρ b U square integral of u / U into 1 - u / U dy, very simple.

Simply, I will write, multiply and divide RHS with U square then you can get this. So now, what is this? You remember? This is the momentum thickness, theta. Remember from our last lecture? So, F drag can simply be written as $\rho b U^2 \theta$,where ρ is the density, b is the width of the plate, U is the free stream velocity and theta is the momentum thickness.

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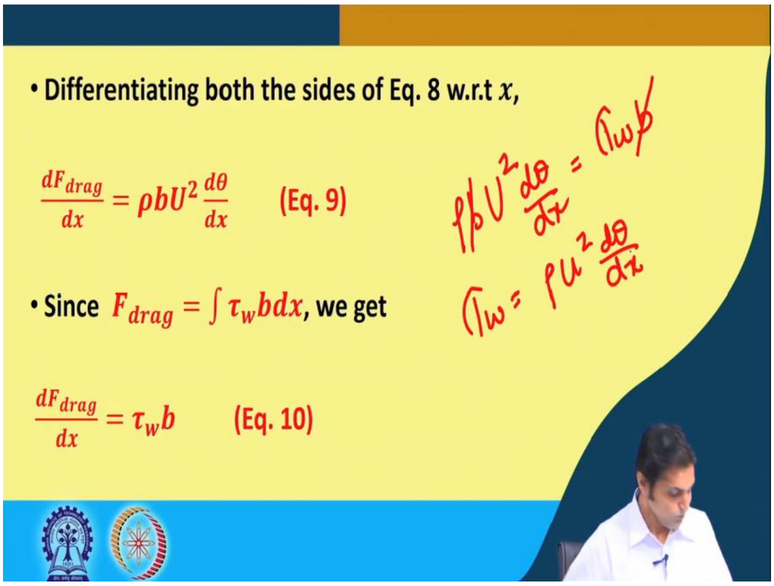

If we differentiate both sides, with respect to x, this equation, equation number 8, if we differentiate, we can get,

$$\frac{dF_{drag}}{dx} = \rho b U^2 \frac{d\theta}{dx}$$

, other things are constant become d theta dx and we can write this as equation number 9. Also, we know that, this drag was due to integration of shear stress or the entire area, integral tau w b dx. We get. if we differentiate this, with respect to x, we get, simply, this integration sign will go and it will become dF drag / dx is equal to taw w b.

And then, you know, what needs to be done. We are going to substitute, I mean, equate equation number 9 and equation number 10 and we get, taw w is equal to ρ U square d theta / dx. I will just, I mean, very elementary, but let me show you, ρ b U square . So, ρ b U square d theta dx is equal to tau w b. So, b will get cancelled here. So, tau w will become

$$\tau_w = \rho U^2 \frac{d\theta}{dx}.$$

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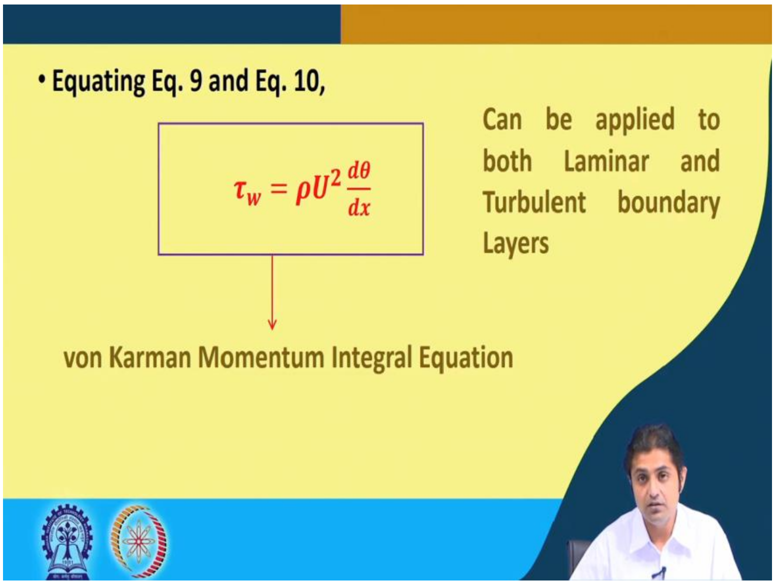

So, this is what it is written here and this famous equation is called von Karman Momentum Integral Equation, a core component of this module and the hydraulics, in general. And this, the beauty of this equation is, it can be applied to both laminar and turbulent boundary layers. So, this is the general equation, because we have not at any point in time assumed whether the flow is laminar or turbulent.

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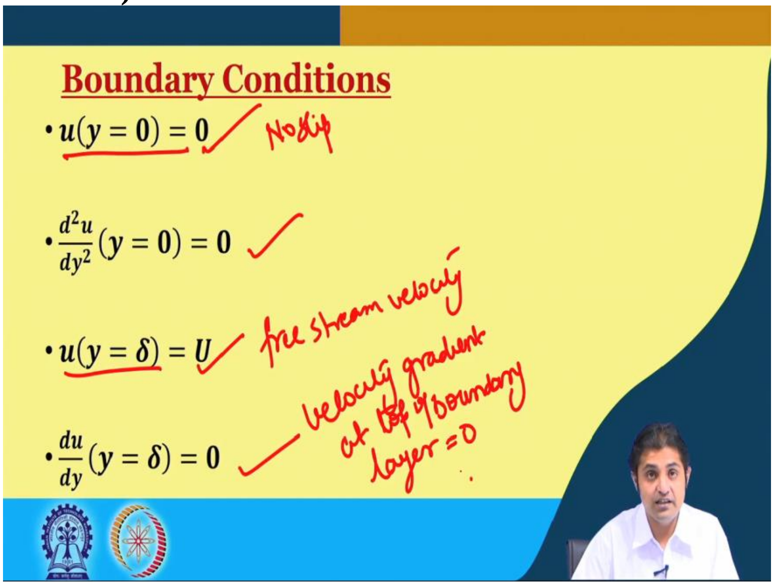

Now, what are the boundary conditions at the plate? It is very simple, u, the velocity at the plate is going to be 0. The second condition is, d square u / dy square at y is equal to 0. This is the second boundary condition. Now, the third boundary condition is at the top of the boundary layer. This means that the velocity at the top of the boundary, that is, the free stream velocity and this is no slip and the velocity gradient at the top of the boundary layer is going to be 0. Now, we will use all these boundary conditions in our further analysis.

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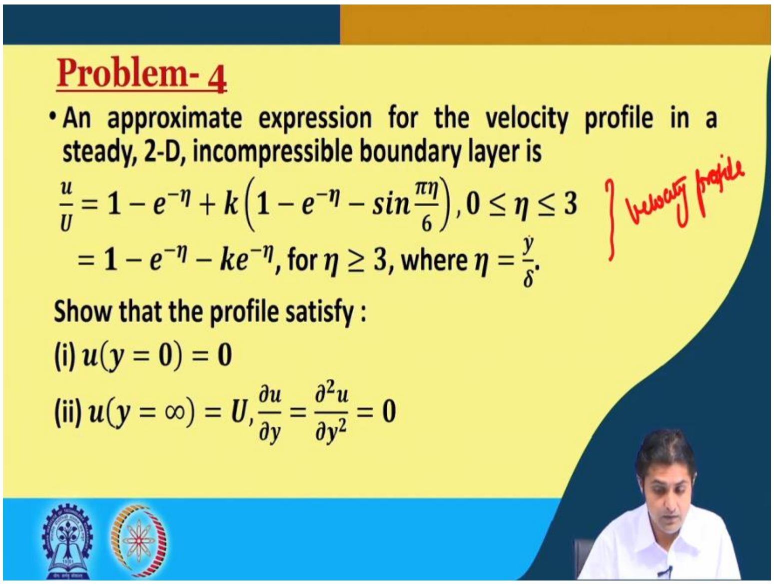

With this, we are going to start solving one problem. The question says, an approximate expression for the velocity profile in a steady 2-D incompressible boundary layer is given by the following function. Now, we have to show that the profile satisfy u at, I mean, we have to see that the boundary conditions are actually satisfied or not. These are the velocity profile. And here, eta is y / delta. So, to solve the question we are going to go to the white screen.

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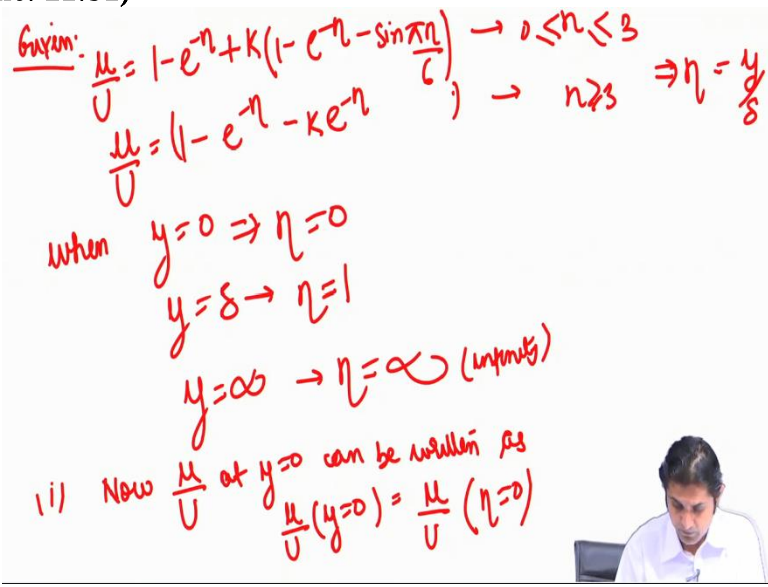

So, we have been given, u / U is equal to 1 - e to the power – eta + K into 1 - e to the power - eta - sin pi eta / 6. And this is for eta, greater than or equal to 0 or greater than or equal to 3. Also, u / U is equal to 1- e to the power - eta - K e to the power - eta and this is for, eta greater than or equal to 3.

Here, eta is given as, y / delta. So, actually, when y is equal to 0, implies, eta is equal to 0. When y is equal to delta, eta is equal to 1, and when y is equal to infinity, implies, eta is equal to infinity. So, first part where, now, u / U at y is equal to 0, can be written as, u / U y is equal to 0 is equal to u / U. Simply, y is equal to 0, means, eta is equal to 0.

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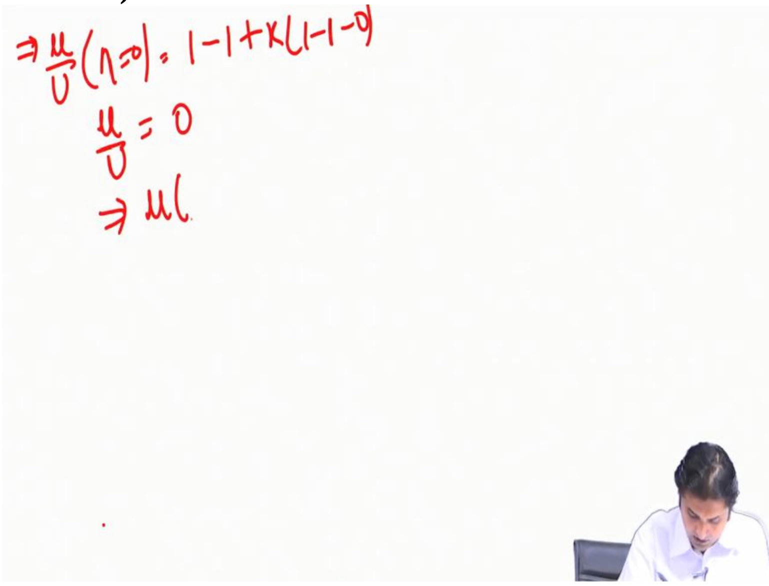

Therefore, u / U at eta is equal to 0, can be written as, 1 – 1 + K into 1 - 1 - 0. So u / v is equal to 0, implies, sorry, not v, u / U, implies, u at y is, I will go to the white screen again.

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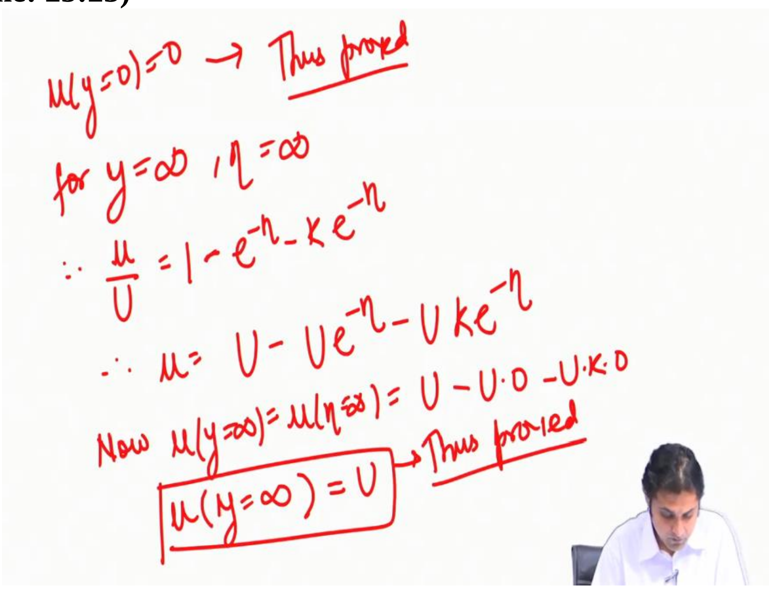

Simply, u at y is equal to 0 is 0, thus proved. For y is equal to infinity, eta is equal to infinity. Therefore, u / U is equal to 1 - e to the power - K eta to the power. Therefore, u is equal to U - U K e to the power 1 - eta. Now, u at y is equal to infinity is equal to u at eta is equal to infinity is equal to U - U into 0 - U into K into 0. So, therefore, u at y is equal to infinity is equal to U and thus second part is also proved.

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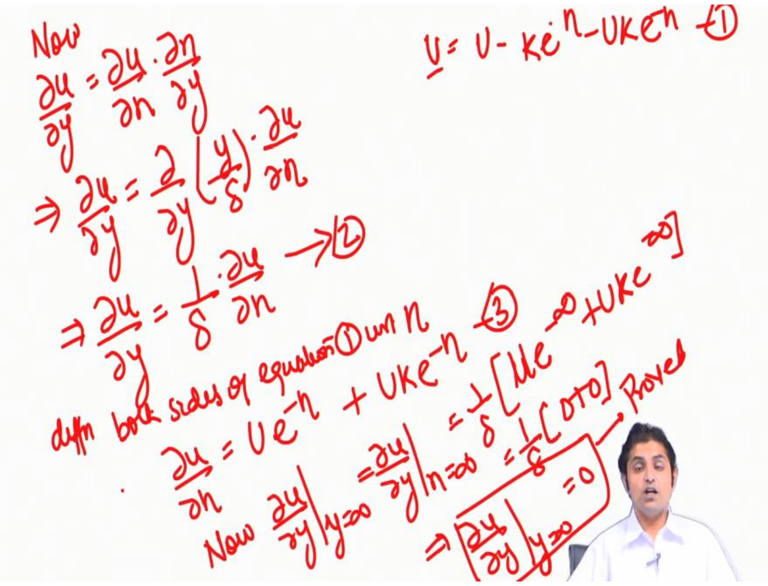

So, now, del u / del y is equal to del u / del eta into del eta / del y, implies, del u / del y is equal to del del y of y / delta into del u / del eta is equal to 1 / delta into del u / del eta. This is another equation. And we have seen the equation number 1 before, that U was, which we just wrote it, U is equal to U – K e to the power - eta - e to the power – eta, this was 1 actually. So, differentiating both sides of equation 1,with respect to eta, we get, del u / del eta is equal to, this is equation 1, equal to U e to the power - eta + U K e to the power - eta, that is, equation number 3. Now, del u / del y at y is equal to infinity is equal to del u / del y at eta is equal to, n is equal to infinity using both and this comes out to be, 1 / delta u e to the, capital U e to the power – infinity + U K to the power - infinity and this comes out to be, 1 / delta 0 + 0. Thus, del u / del y at y is equal to infinity is 0.

So, there is one more part left. For del square / del square u / del square, del square u / del y square. But I will leave it up to you to prove it. Try yourself at home using this maths that we have seen. So, this concludes the problem for today. And we will start with another problem, problem number 5 when we resume next. Thank you so much.