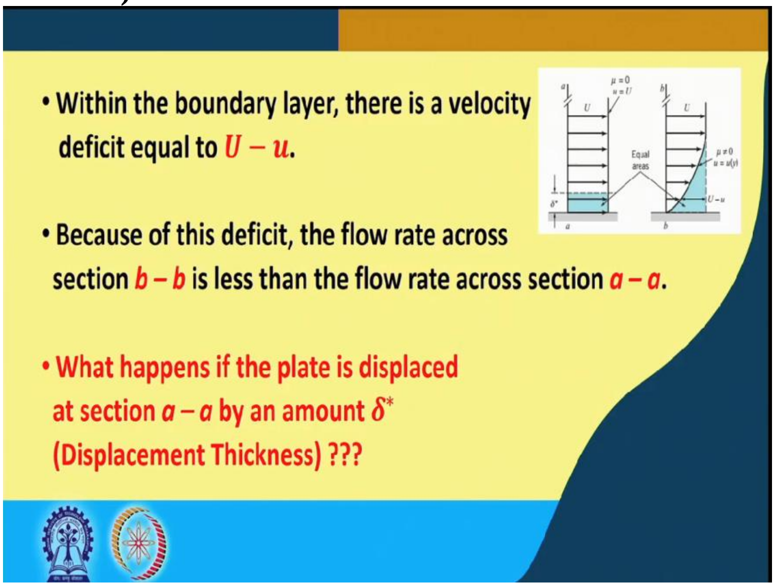
Hydraulic Engineering
Prof. Mohammad Saud Afzal
Department of Civil Engineering
Indian Institute of Technology Kharagpur

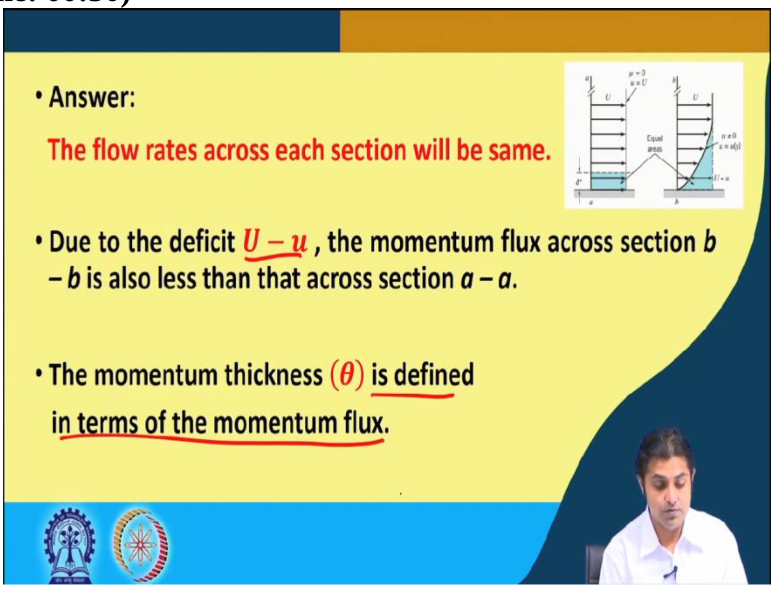
Lecture – 18
Boundary Layer Theory (Contd.,)

(Refer Slide Time: 00:29)



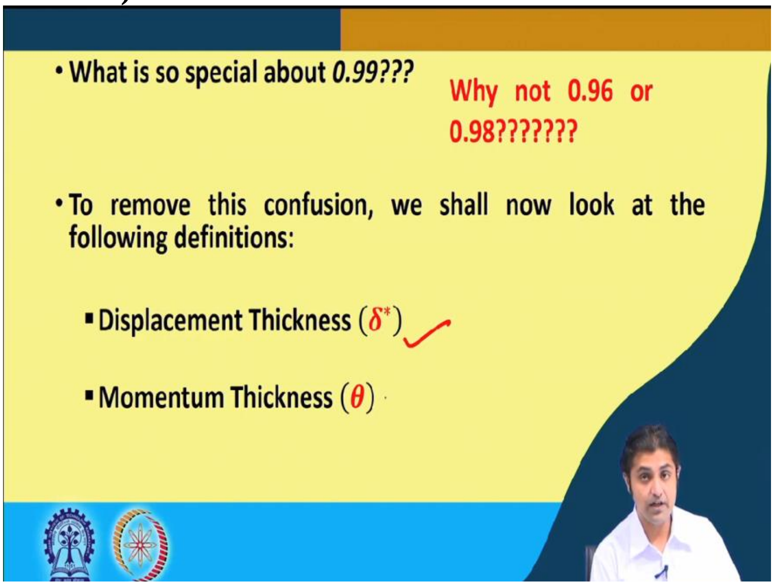
Welcome back to the second lecture of this module, boundary layer analysis. So, last class we finished the lecture by saying, what happens if the plate is displaced at a section a - a by an amount delta dash. So, we displace this plate to here, for example.

(Refer Slide Time: 00:50)



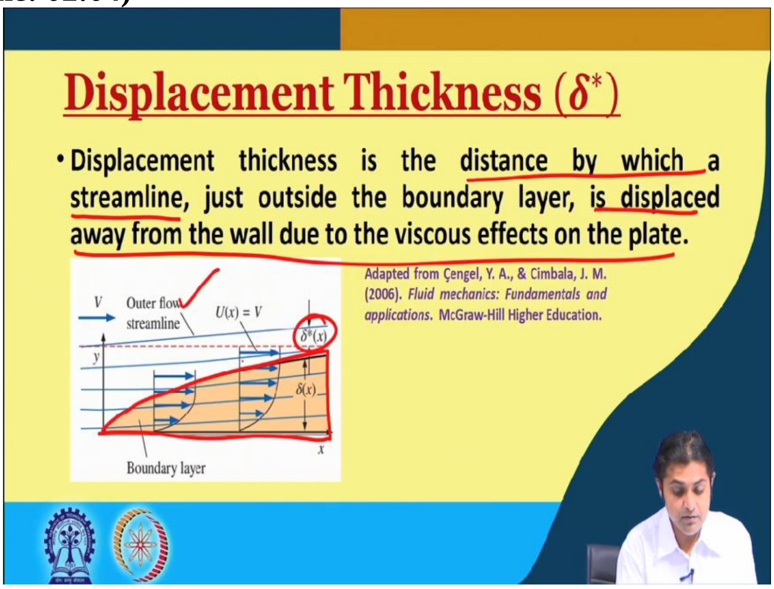
So, to answer that, we see that the flow rate across each section will be the same because the area is the same. But in this section b – b, due to the deficit U - u, the momentum flux across the section b - b is also less than that across this section a – a, true, because the velocity is different, it is a deficit of U – u. And therefore, the momentum thickness theta is defined in terms of the momentum flux. So, we saw the boundary layer thickness.

(Refer Slide Time: 01:50)



We saw the displacement thickness, the phenomenon. And we saw what a momentum thickness is.

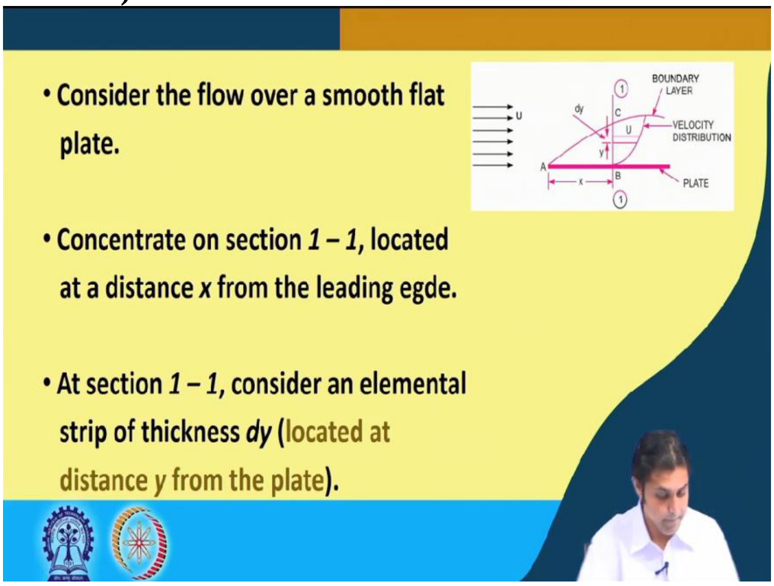
(Refer Slide Time: 02:04)



Now, what we are going to do is, we are going to derive the displacement thickness. First, the definition, displacement thickness is the distance by which a streamline, just outside the boundary layer, is displaced away from the wall due to viscous effects on the plate, as we have seen in the last slide and answer to the question, which we started the lecture with. So, displacement thickness is the distance by which a streamline, which is just outside the boundary layer, is displaced away from the wall due to the viscous effects on the plate.

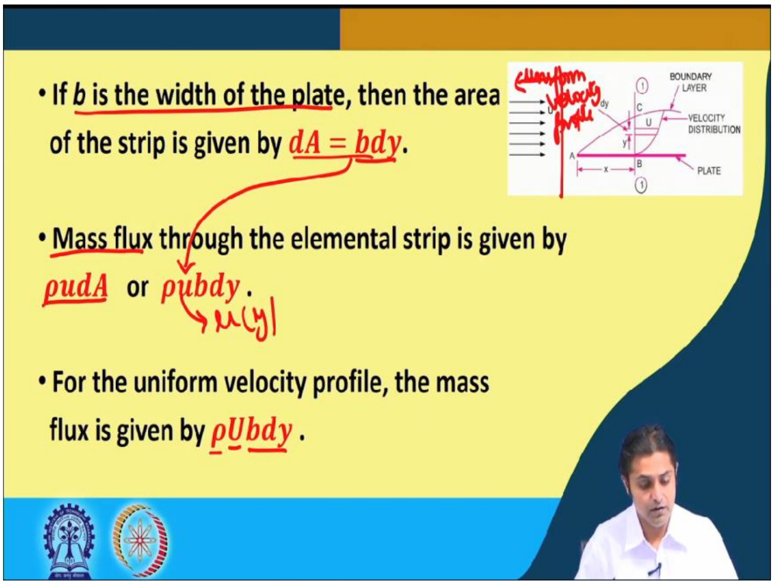
This is the boundary layer. Again, look at the definition, distance by which a streamline, just outside the boundary layer is, displaced away from the wall due to the viscous effects on the plate. What exactly is that thickness? Delta dash x. This is the outer flow stream line, and this is the boundary layer, and the velocity we was coming here, outside the boundary layer as you can see, the velocity is exactly U as a function of x is V.

(Refer Slide Time: 03:36)



Now, we consider the flow over a smooth flat plate, like this. So, there is a flow which is coming with a speed U, there is a flat plate and at any distance x, there is a section 1 - 1. I will remove this, but just to mark, this is the section 1 – 1 and, that is, located at a distance x, from the leading edge. So, this is the leading edge. The orientation is clear to you, I hope. Now, at section 1 - 1, we consider an elemental strip of thickness dy. Now, this dy so, this is the elemental thickness dy and it is located at a distance y from the plate. Again so, this is the elemental strip of thickness dy.

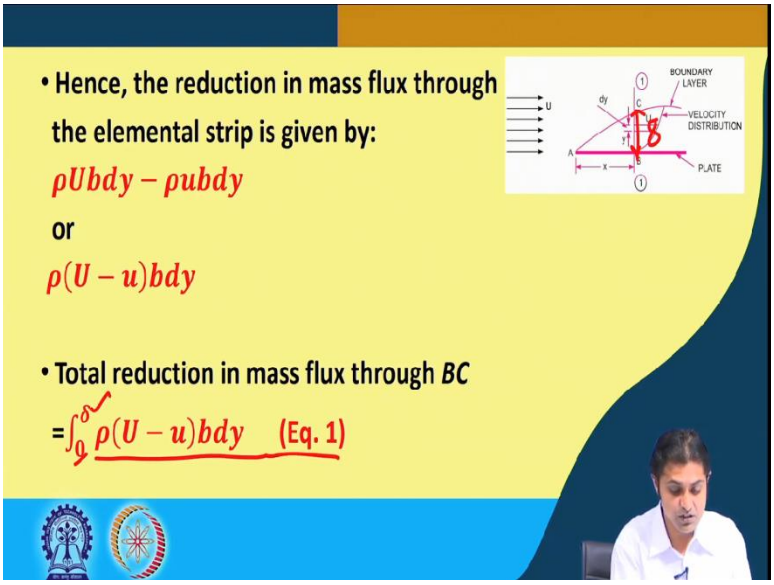
(Refer Slide Time: 04:52)



Suppose, if b is the width of the plate, that is, the width, then the area of the strip can be given by, b into dy. The strip will have the height dy and the width is b. Therefore, the area is simply bdy, here. Now, the mass flux through the elemental strip is given by, ρ u into dA. P u is the sort of a momentum, ρ into u, mass into velocity. Of course, we have not considered the volume, right now, but we multiply it with dA, to have the mass flux. So, dA if you put dy, this becomes ρ U b into dy.

Whereas, for the uniform velocity profile, so, the uniform velocity profile that exists in a region below this, the mass flux is given by, ρ into U, because this is the full velocity U. Here, it is u as a function of distance y, this one, so, here is ρ U b into dy, if there was the flow was uniform, and there was no plate, for example, and if you would have considered the same elemental area of thickness dy.

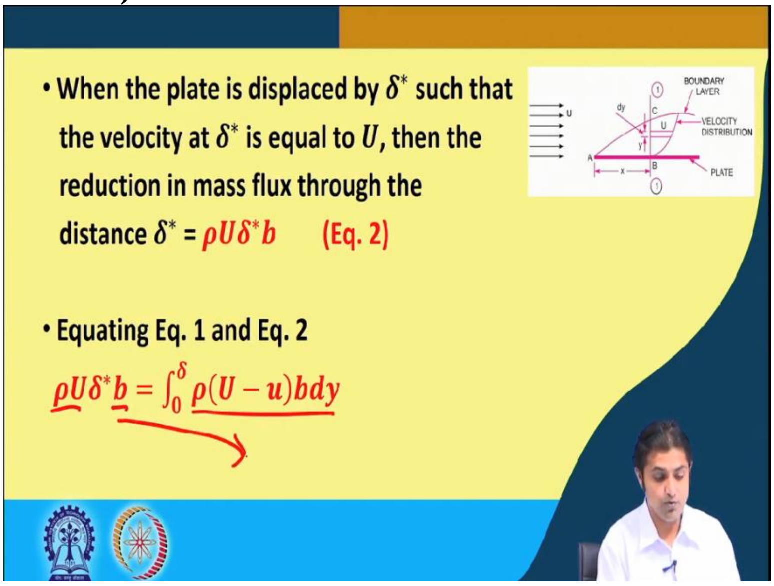
(Refer Slide Time: 06:46)



Therefore, the reduction in the mass flux through the elemental strip, compared to the, you know, uniform velocity profile will be the difference of the mass fluxes. This is due to uniform, that is, in the boundary layer, or if we take ρ outside, b and dy outside, it becomes ρ into U minus u into bdy.

Now, the total reduction in mass flux through BC, will be, we have considered and its thickness of dy. So, what we are going to do? We are going to simply integrate over this length. And let us say, this is so, let me, so, this distance is delta. So, the integration goes from 0 to delta and ρ U - u b dy, that is, the total reduction in mass flux through BC. I hope, this is clear, this is equation number 1.

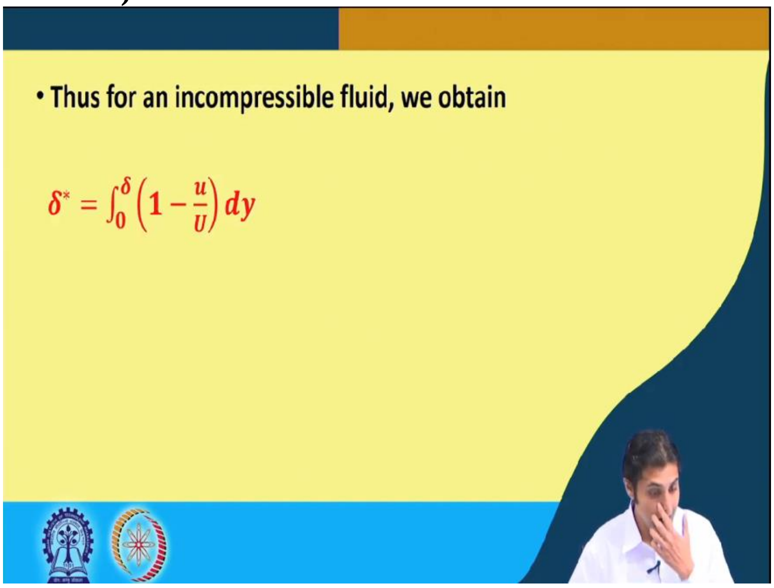
(Refer Slide Time: 08:19)



Now, when the plate is displaced by delta dash, such that, the velocity at delta dash is equal to U, then the reduction in mass flux through the distance delta dash is going to be, very simple. It is going to be ρ U because the velocity there is, U delta dash into b. So, we have assumed, that the place, I mean, the plate is displaced by delta dash, such that, the velocity at delta dash is equal to the uniform velocity U, that was, outside the boundary layer, then the reduction in the mass flux through that distance is going to be ρ U delta dash b, because at the plate it was 0.

And this should be equal to the reduction in the mass flux through the boundary layer thickness, for example, at that particular point at a distance x. So, ρ U delta dash b is equal to integral 0 to delta ρ U - u b dy.

(Refer Slide Time: 09:50)

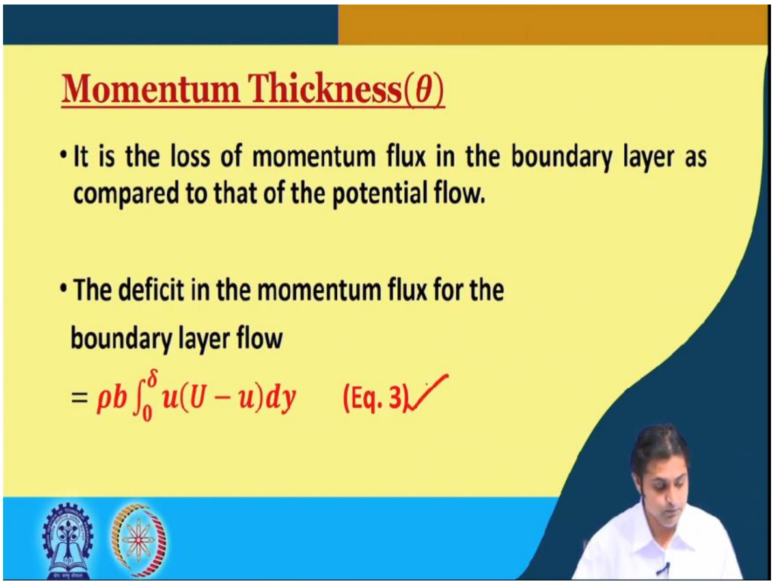


Thus, for an incompressible fluid, we obtain, delta dash is equal to, what we do is, we take these terms. So, we take ρ U and b down here, as a denominator and we simply write

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

and this is the equation for the displacement thickness. So, in the numerical, when you get, you are going to use this, I mean, in this equation for the, for finding the displacement thickness delta dash.

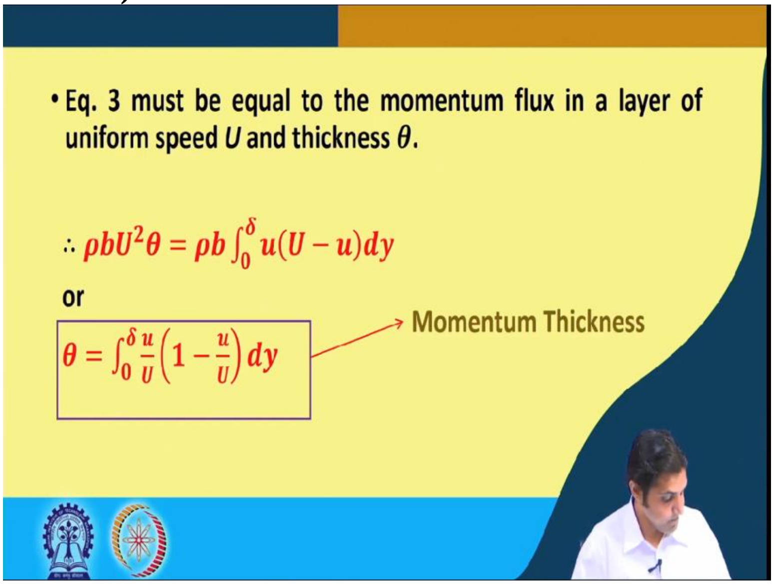
(Refer Slide Time: 10:42)



Now, proceeding, what is the momentum thickness theta? So, momentum thickness theta is the loss of momentum flux in the boundary layer, as compared to that of the potential flow. So, there is some loss in the momentum flux. And why that is lost? Because of the presence of the viscosity due to the presence of the plate in a viscous fluid flow. Therefore, this is the loss of the momentum flux.

Now, the deficit in the momentum flux for the boundary layer flow is written as, the same fundamental. Now, instead of, mass flux we have written momentum flux. Therefore, instead of, simply ρ b U minus u, there is a multiplication term of a u as well, here. So, this is the loss in the, the deficit in the momentum flux, given by, equation number 3.

(Refer Slide Time: 11:45)

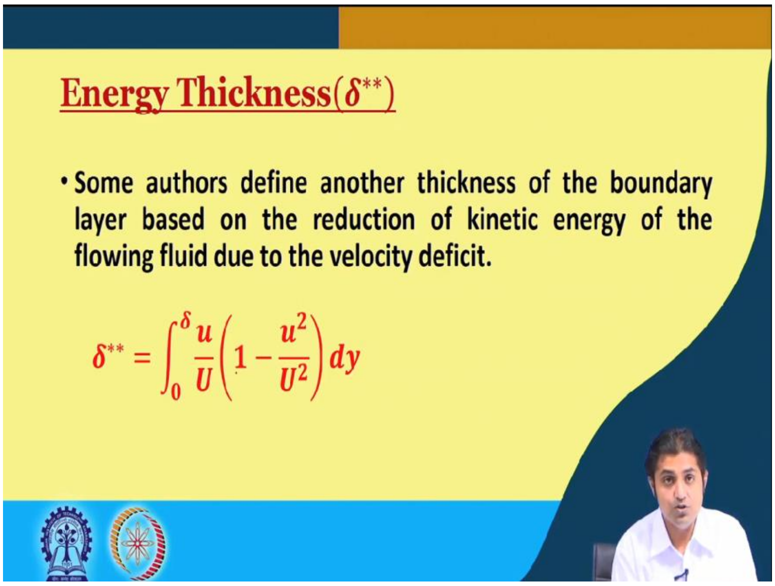


Now, the equation number 3 must be equal to the momentum flux in the layer of uniform speed U and thickness theta. That is how we derive the displacement thickness as well. So, we equate ρ b U square theta because this is the momentum flux in a layer of uniform speed U and thickness theta, or if we take this down here, as a denominator, ρ b, ρ b will get cancelled, this will become

$$\theta = \int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

. So, this is the momentum thickness.

(Refer Slide Time: 13:04)



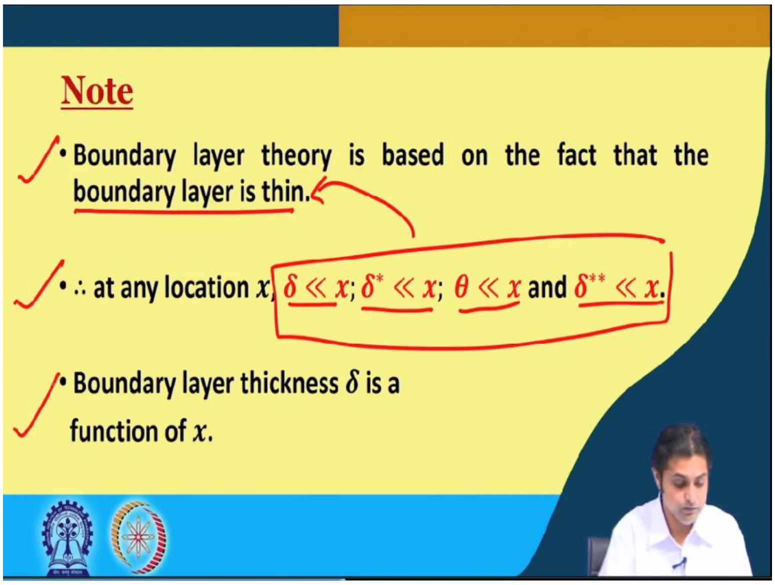
Now, there is something called energy thickness, delta double dash. We are not going to derive it. But some authors define, another thickness of the boundary layer, based on, the reduction of the kinetic energy of the fluid flow due to the velocity defect. So, the momentum thickness was due to the loss of the momentum flux, displacement was due to the loss of the mass flux and the energy thickness reduction of the kinetic energy of the fluid due to the velocity deficit. And we will just write down the equation here.

It is given as,

$$\delta^{**} = \int_0^{\delta} \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right) dy$$

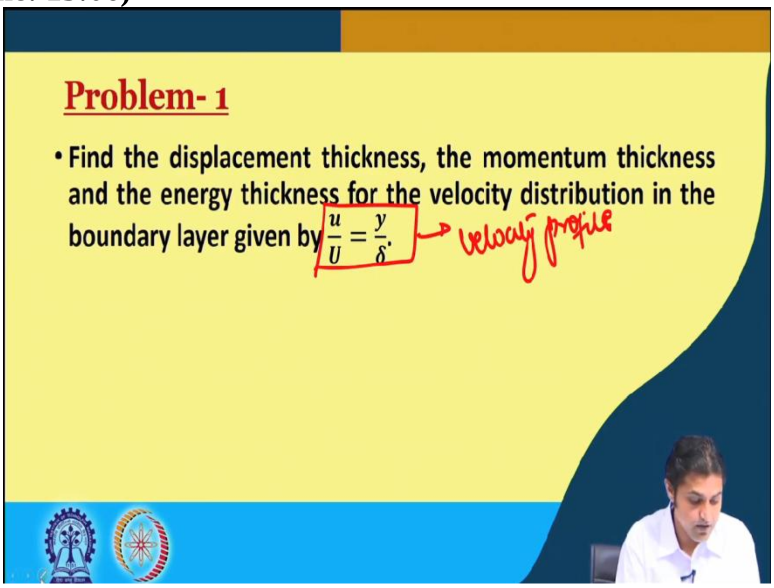
.

(Refer Slide Time: 14:03)



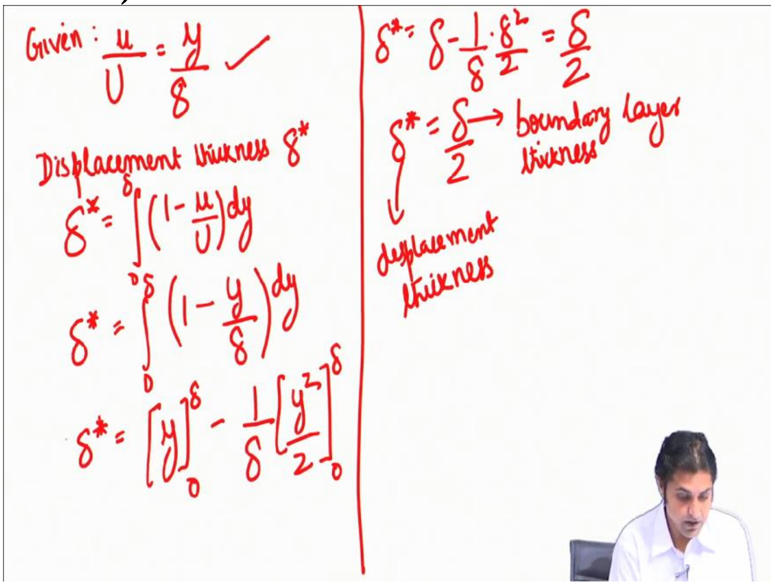
Now, we have to make note of some important point, that is, the boundary layer theory is based on the fact that the boundary layer is thin. So, at any location, at any location x, this x must be very much greater than delta, boundary layer thickness and also the displacement thickness. This x must also be greater than momentum thickness and also the energy thickness. This is these four things, means that, the boundary layer is thin. Boundary layer thickness, delta is a function of x. This has to be assumed, these are the 1, 2 and 3.

(Refer Slide Time: 15:06)



So, we are going to solve some problems that will demonstrate this momentum thickness, displacement thickness and energy thickness. So, the question is, find the displacement thickness, the momentum thickness and the energy thickness for the velocity distribution in the boundary layer which is given by, u / U = y / delta. This is the velocity profile. So, this is one of the most simplest problems that we are going to see in this part. So, how to solve that? What I am going to do is, I will go to my usual white screen.

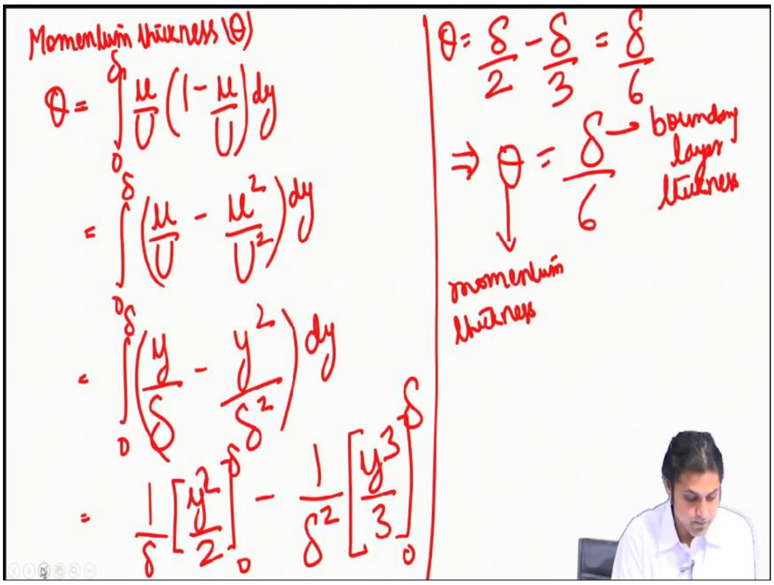
(Refer Slide Time: 16:03)



So, we will start writing the things that are given. So, the things that are given is, u / U is equal to y / delta. By definition, displacement thickness, delta dash is given as, 0 to delta 1 minus u / U and into dy. So, using this term here, we can use, we can write, delta dash again as, integral 0 to delta 1 minus, so, this is bracket, 1 - y / delta dy. Delta dash can be written as, after integration, 0 to delta - 1 / delta because this an integration with respect to y, this will become y square / 2 0 to delta. Draw a line here.

So, delta dash is going to be, delta - 1 / delta into delta square / 2 is equal to delta / 2 and this delta here is boundary layer thickness. This is as we told, this is displacement thickness. So, now going to momentum thickness, this is very simple. So, we have written displacement thickness, in terms of boundary layer thickness. So, what I am going to do here, I will erase all ink on this slide.

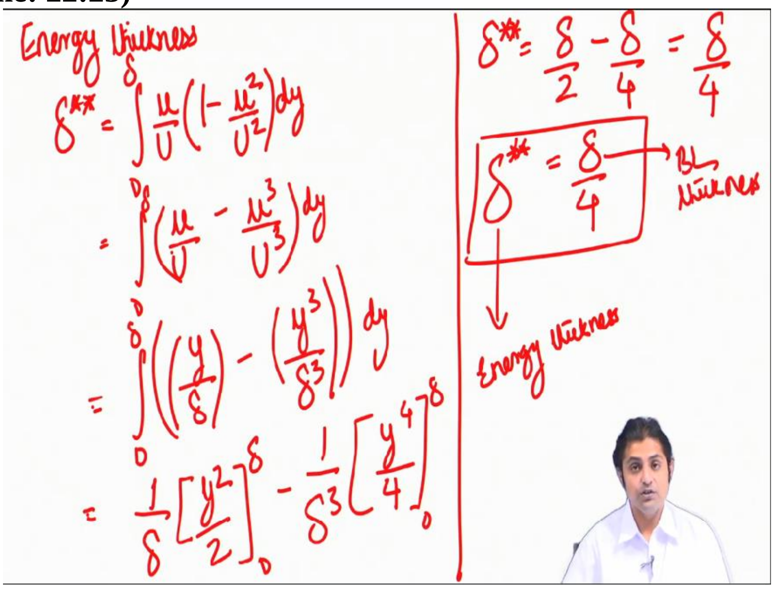
(Refer Slide Time: 19:27)



Momentum thickness theta, so, theta definition is equal to 0 to delta u / U into 1- u / U dy and this is going to be 1 to delta. This is going to be u / U – u square / U square into dy and using the velocity profile, we can write, integral 0 to delta u / U is going to be, y / delta minus y square / delta square into dy or removing the integration sign, 1 / delta would come out and this is going to be, y square / 2.

So, this is y square and the limits is going to be 0 to delta minus, this delta square is constant it is coming out and this becomes, y cube / 3 and the limit is 0 to delta. So, momentum thickness theta, can be written as, delta / 2 - delta / 3 here, is equal to delta / 6, which implies, delta / 6. So, this is momentum thickness.

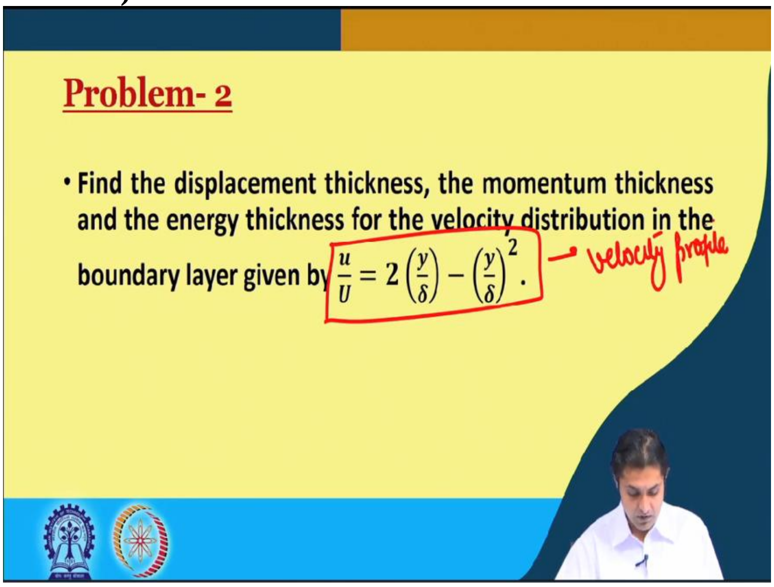
(Refer Slide Time: 22:23)



Now, the energy thickness, delta double dash, the definition is, integral 0 to delta u / U into 1 - u square / U square into dy and this is going to be, 0 to delta u / U - u cube / U cube into dy and after substituting the velocity profile, this is going to be, 0 to delta, the limits, y / delta - y cube / delta cube and all into dy. So, we have not yet integrated it. Now, the next step is to integrate and after integrating this, what we get is, 1 / delta is common, so, it comes, y square / 2, 0 to delta minus, this comes out to be this is constant.

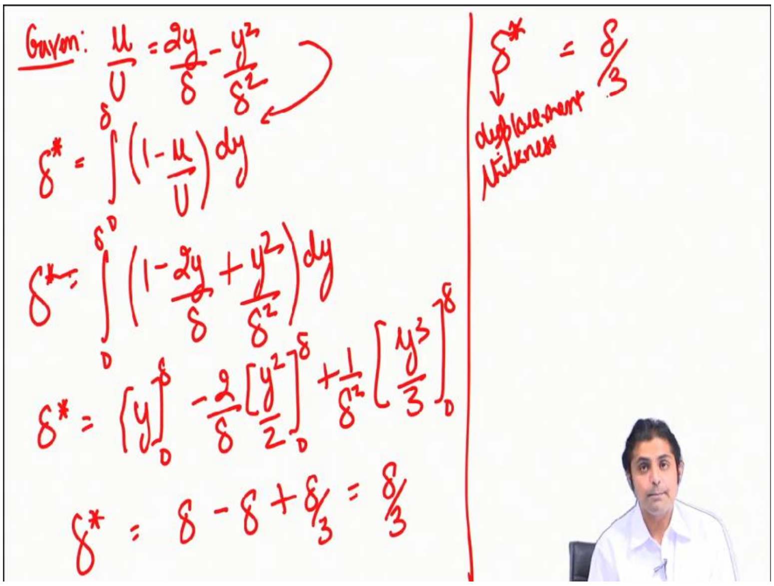
So, this comes out and y cube becomes y to the power 4 / 4 and the limit is 0 to delta. And therefore, the energy thickness here is, this will come out to be, delta / 2 and this comes out to be, delta / 4. So, the energy thickness is coming out to be, delta / 4 energy thickness and this is boundary layer thickness. So, this is one of the simplest problems in the boundary layer analysis. So, you have to attack all the problems like this.

(Refer Slide Time: 25:07)



So, we proceed to the next problem. So, our next problem is, find the displacement thickness, the momentum thickness and the energy thickness for the velocity distribution in the boundary layer, that is, given by u / U equal to. So, we have another velocity profile, similar to problem number 1. So, we will solve this problem, as well. So, for doing that we are going to have white screen again.

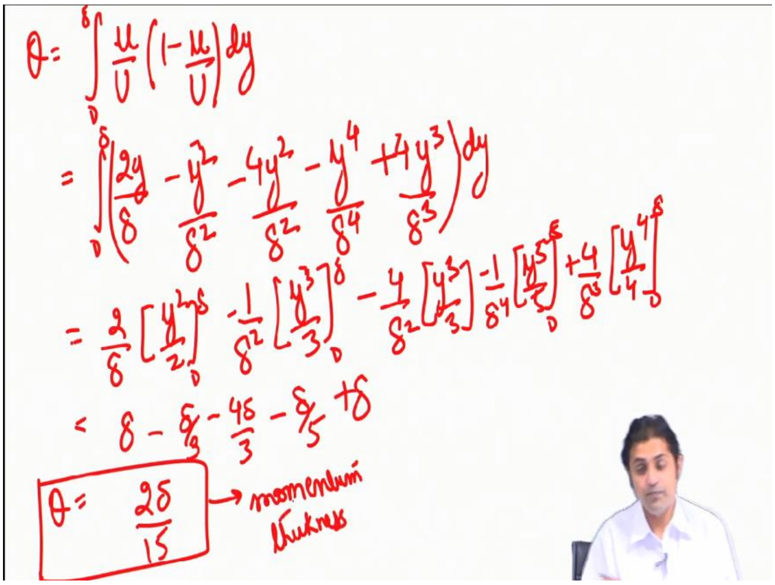
(Refer Slide Time: 25:56)



And as always, we start with the thing, that is, given. What is given to us? u / U equal to 2y / delta - y square / delta square. And according to definition of the displacement thickness, that is, delta star, definition is 0 to delta 1- u / U into dy. So, it is quite simple, we use this velocity profile here, we plug it in, and we first write, integral 0 to delta is equal to 1 – 2y / delta plus, this minus becomes plus, y square / delta square into dy. So, now there are 3 terms to integrate.

So, we are going to do, delta dash is equal to this 1 into dy, will be, y 0 to delta - 2 / delta y square / 2 0 to delta plus 1 / delta square is constant. So, this becomes, y cube / 3 0 to delta. So, displacement thickness is going to be, delta – delta + delta / 3 and this gives us delta / 3. So, the displacement thickness is coming out to be, delta / 3. Now, similarly, we have to find the momentum thickness.

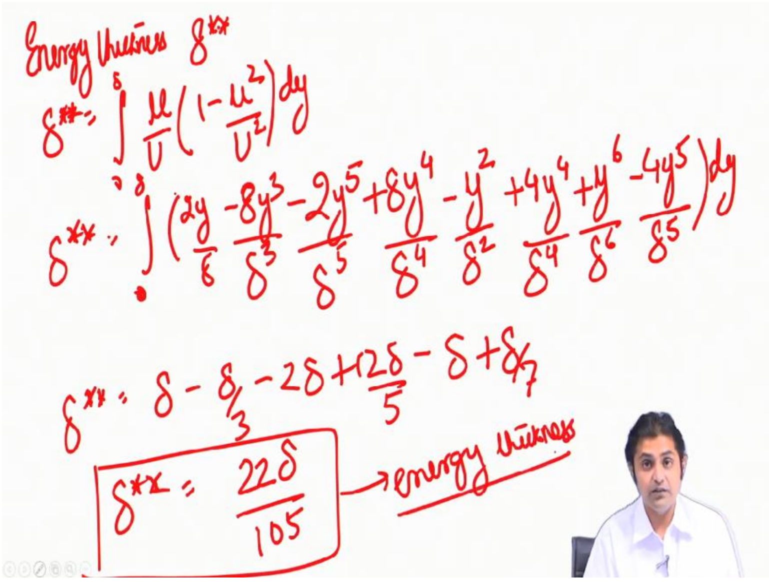
(Refer Slide Time: 28:26)



So, theta is the momentum thickness. Going by the definition, 0 to delta u / U into 1 minus u / U dy. And we substitute the velocity profile here, so, it will be u / U - u square / U square. So, it will become, 2y / delta - y square / delta square - 4y square / delta square - y to the power 4 divided by delta to the power 4 + 4y cube / delta cube the whole dy. And now, we can integrate, this will be, 2 / delta y square / 2 from 0 to delta - 1 / delta square and this y square will turn out to be, y cube / 3 0 to delta - 4 / delta square y cube / 3 and this will be, minus 1 to the power 4 delta 4 and this will be come y to the power 5 / 5 from 0 to delta and the last term is going to be, 4 / delta cube outside, and y cube becomes y to the power 4 / 4 0 to delta.

And this will come out to be, the first term is going to be, delta. Second term is going to be, minus delta / 3 - 4 delta / 3 - delta / 5 + delta. So, momentum thickness is going to be, 2 delta / 15. So, this is the momentum thickness. Now, the last part before we conclude todays

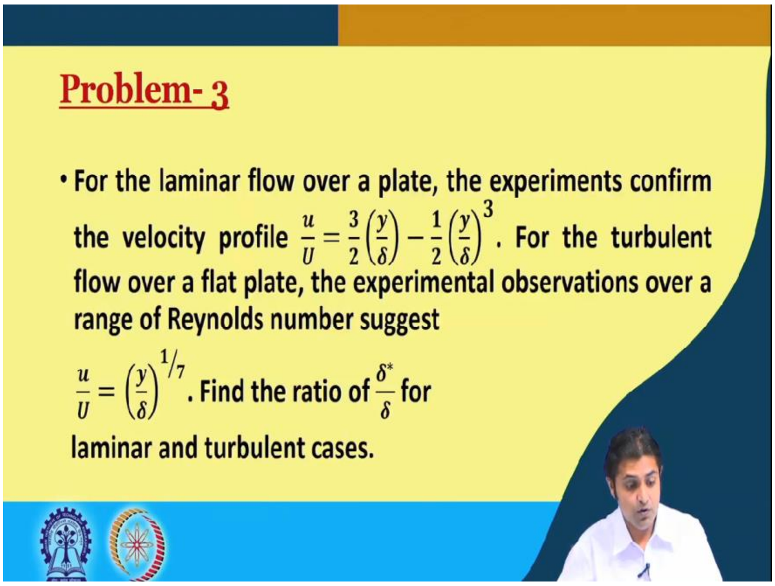
(Refer Slide Time: 31:07)



Is the energy thickness, delta double dash. So, delta double dash, first, what we do is, we write the equations, 0 to delta u / U 1 - u square / U square dy. And after we substitute the velocity profile, what we get? Integral 0 to delta, 0 to delta, it is going to be, so, 2 y / delta - 8y cube / delta cube - 2y to the power 5 / delta to the power 5 + 8, bear with me, because so many terms, but this is the beginning, so I am not skipping anything, not trying to take anything for granted, plus 4y to the power 4 / delta to the power 4 + y to the power 6 / delta to the power 6 - 4y to the power 5 / delta to the power 5 dy.

And now on integrating actually, and putting in the limits, what you are going to get is, delta - delta / 3 - 2 delta + 12 delta / 5 – delta + delta / 7. So, the energy thickness is going to be 22 delta / 105. This is the energy thickness for this particular problem. So, this problem is also solved.

(Refer Slide Time: 33:39)



So, now, we will close this lecture and start our new lecture, lecture number 3 by solving this problem number 3 and then we proceed to the next topic. So, that is enough for today and I will see you in the next lecture. Thank you so much.