Hydraulic Engineering
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Lecture- 15
Laminar and Turbulent Flow (Contd.)

Welcome back to this lecture of laminar and turbulent flow. We have left the last lecture before introducing the topic of shear stresses in turbulent flow.

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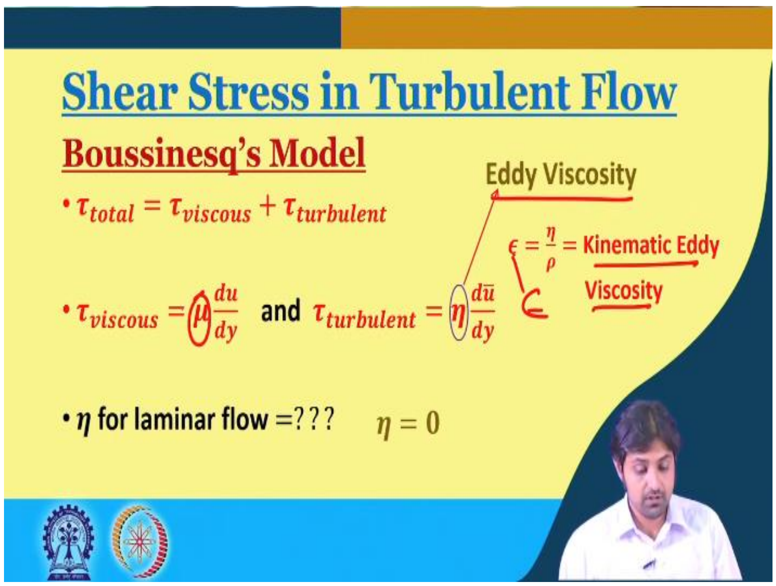

So, we are going to continue with this particular topic. So, shear stress in turbulent flow. We are going to talk about a model that is called Boussinesq's model, where the total shear stress, in case of laminar flow it was due to the viscosity viscous. Sorry. Yeah, that was only due to the viscous. But in a turbulent flow, there is an additional component of shear stress that happens because of the turbulence in the flow.

So, therefore, the shear stress in total is much, much larger than the viscous flow. At least it is definitely larger than the viscous flow because there is shear stress that is associated with turbulence too. So, Boussinesq's says as

$$\tau_{viscous} = \mu \frac{du}{dy}$$

for laminar flow. Therefore, the shear stress due to do the turbulence component is

$$\tau_{turbulent} = \eta \frac{d\bar{u}}{dy}$$

. Here, you see, this is similar. So, instead of μ there is something called η a new coefficient of viscosity, and this is called eddy viscosity.

So, we are not going to the derivation right now, at some point we can see these derivations when appropriate chapter comes. But now you have to take that the shear stress due to turbulence is eddy viscosity du / dy, very similar to the shear stress in the laminar flow. The coefficient is therefore different. And if we want to write a kinematic eddy viscosity then we write it by epsilon, for example. So, this can be written as eta / rho, similar type of definition as laminar flow.

What is eta for laminar flow, for example, or mu for laminar flow? That you already know, we have been doing that, it was 10 to the power - 3 Pascal second, No okay. So, eta for laminar flow will be 0.Yes. So, but mu for this laminar flow is 10 to the power - 3 Pascal second and if the flow is laminar eta is going to be 0 because it is related to the turbulent viscosity.

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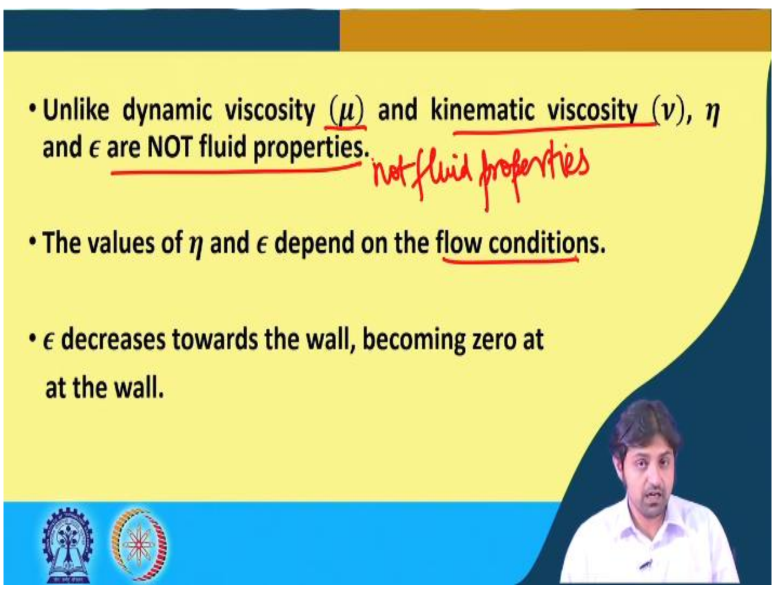

So, unlike the dynamic viscosity mu and kinematic viscosity nu, eta and epsilon are not fluid properties, they are not fluid properties. The values of eta and epsilon are dependent on the flow conditions. So, epsilon decreases towards the wall becoming 0 at the wall. So, the epsilon, that is, the eddy, kinematic eddy viscosity decreases as you move towards the wall and becomes 0 at the wall.

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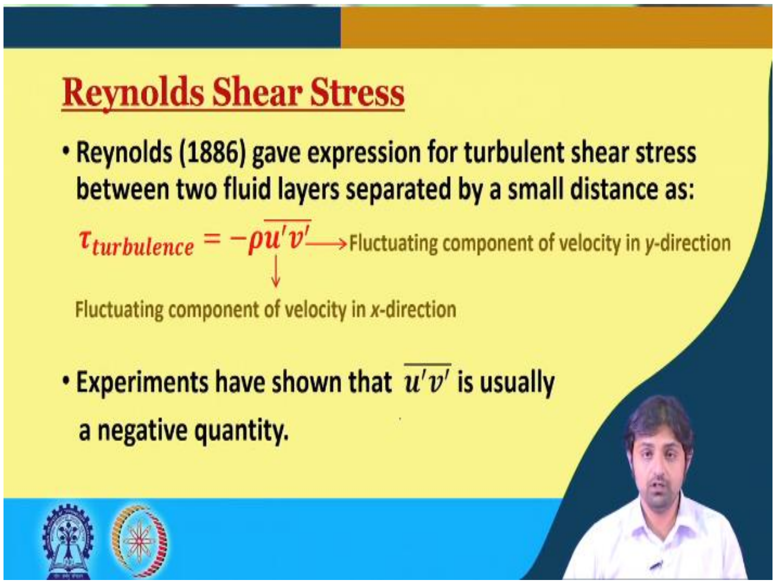

Now, coming to what is Reynolds shear stress. So, Reynolds in 1886 gave expressions for turbulent shear stress between two fluid layers separated by a small distance. And he said that the shear stress due to turbulence can be written as, minus rho u prime v prime whole bar. Actually, it is not an assumption, but this can actually be derived, which we will do at some point in this hydraulic engineering course, but not now.

So, you have to understand, Reynolds shear stress is given by

$$\tau_{turbulence} = -\rho \overline{u'v'}$$

and it does not have only one component, it has minus u dash w dash, it will have minus v dash w dash. So, there are different, there are some normal shear stress, but this is one of the shear stress component. Whereas, what is u prime? That is the fluctuating velocity component in x direction, v prime is fluctuating component of velocity in y direction. Experiments show that u prime v prime is usually a negative quantity. Therefore, the tau turbulence or minus rho u prime v prime whole bar is total positive quantity, it has negative correlation that we will see.

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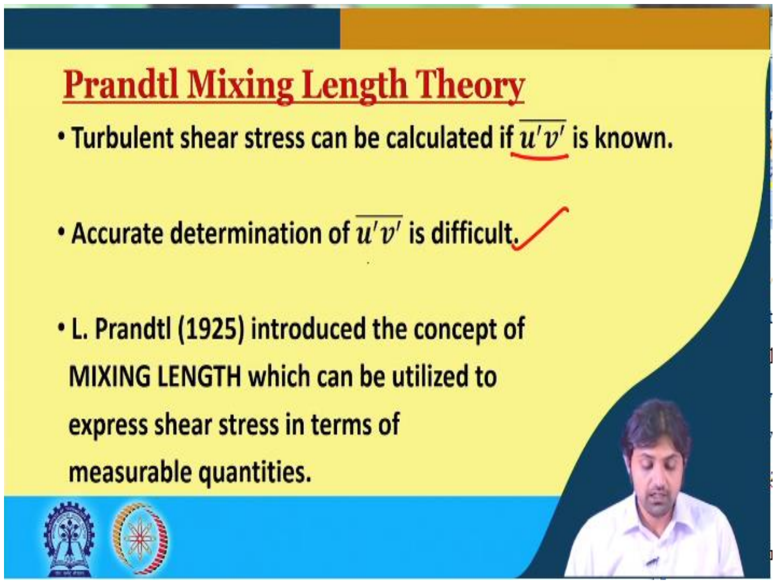

Now, there is a concept of Prandtls mixing length theory. So, turbulence shear stress can be calculated if this thing is known, u prime v prime whole bar is known. Because as we see, the tau turbulence by Reynolds was given by minus rho u prime v prime whole bar. So, what a nice thing it would be if we can calculate u prime v prime bar because that is unknown until now. So, accurate determination of u prime v prime whole bar is very difficult. Therefore, in 1925 Prandtls introduced the concept of mixing length, which can be utilized to express the shear stress here, in terms of some measurable quantity.

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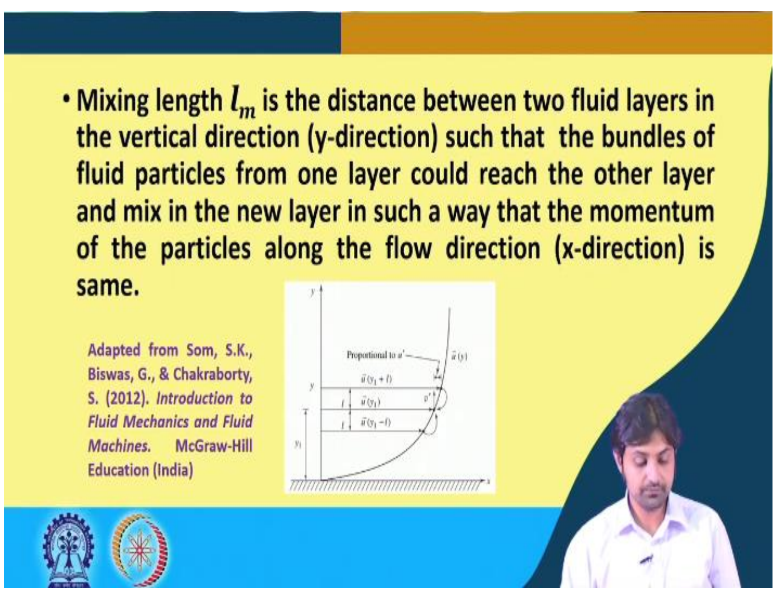

So, mixing length lm, he said it can be described in terms of mixing length l m. He said mixing length l m is the distance between 2 fluid layers in the vertical direction, in the y direction, such that, the bundles of fluid particles from one layer could reach the other layer and mix in the new layer in such a way that the momentum of the particle along the flow direction is the same. So, he related it to mixing.

And he said that the mixing length is the distance between 2 fluid layers in the vertical direction, such that, the bundles of fluid particles from one layer could reach the other layer and the mixing can happen, something like this. So, this is the velocity profile and he says he divided the fluids in 2 layers and this he says is the mixing length. We are going to explore this in more detail in the next slide.

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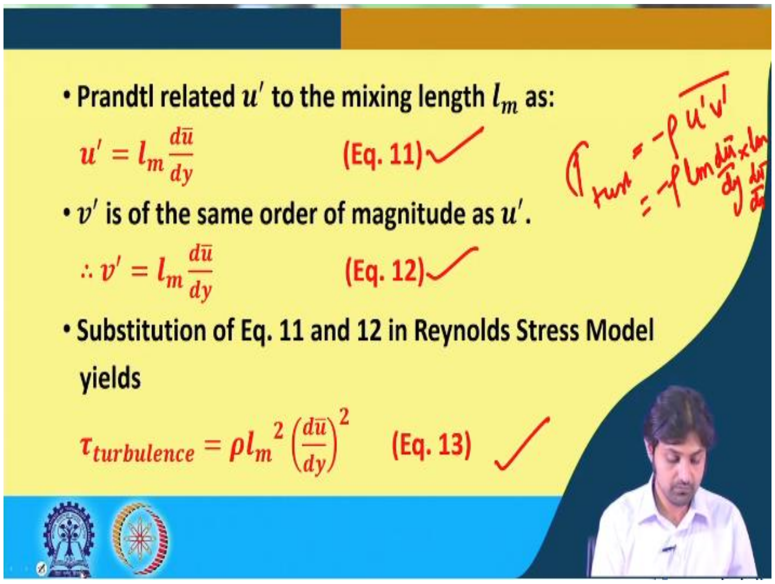

So, Prandtl related u prime to mixing length lm. He said that this u prime, as you can see in the figure here, he said proportional to u prime but I am going to write it in the next slide. He related u prime to the mixing length lm as, he said this u prime can be written as, mixing length lm multiplied by the gradient of the average velocity. So, he said u prime, this is very important to know, lm as the du bar / dy.

This is what his assumption was, where v prime is also of the same order of magnitude as u bar and similarly, this can also be written as v prime is equal to lm du bar / dy, similar type of equation. And if you substitute equation 11 and equation 12 in Reynolds stress model, which was it was tau turbulent is equal to minus u prime v prime whole bar you get, so, minus and minus will become positive it will become tau turbulence is

$$\tau_{turbulence} = \rho {l_m}^2 \left(\frac{d\bar{u}}{dy}\right)^2$$

You can just substitute and see, l m du bar / dy multiplied by l m du / dy bar, it is the same thing. So, it becomes rho l m square du bar by dy, this is equation 13. So, this is one of the, this is the mixing length theory of the Prandtl.

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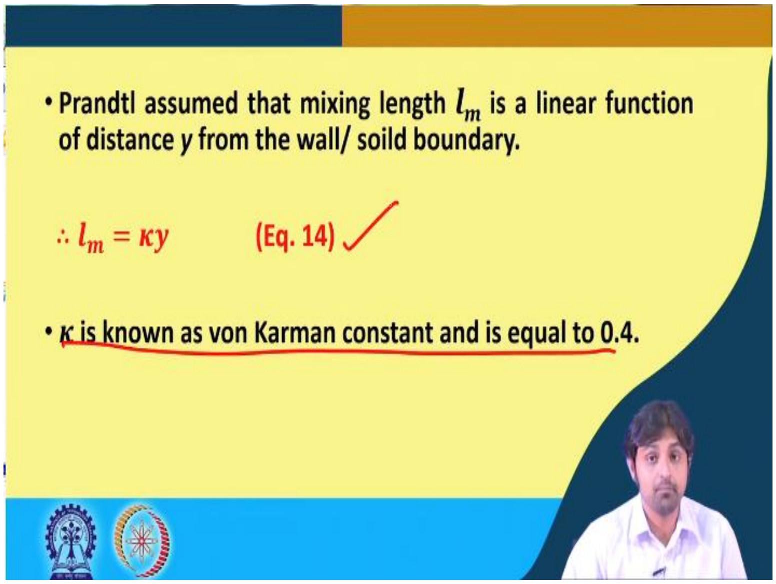

So, Prandtl also assumed that the mixing length lm is a linear function of distance y from the wall or any solid boundary. Therefore, he said l m can be written as ky. So, if you look go back and see here now rho is known, du / dy can be calculated because we are dealing in terms of average velocity, which can be measured. Now the only unknown is l m. So, how do we find this l m now? So, the problem is becoming less and less complex we are going from one variable to the other. Now, the only unknown thing is l m.

So, Prandtl needed to relate this to something. So, he assumed that lm is a linear function of distance y and he said l m is equal to kappa into y, where kappa is known as von Karman constant and it has been found to be equal to 0.4. So, he said mixing length is 0.4 times y, this is quite an important result. So, now, we know everything, for example. Shear stress in turbulent flow was related in terms of u prime v prime which was made by Prandtl as, u prime and v prime or of the same order of magnitude and is equal to, which was equal to, which was proportional to du dy l m and l m is equal to kappa into y.

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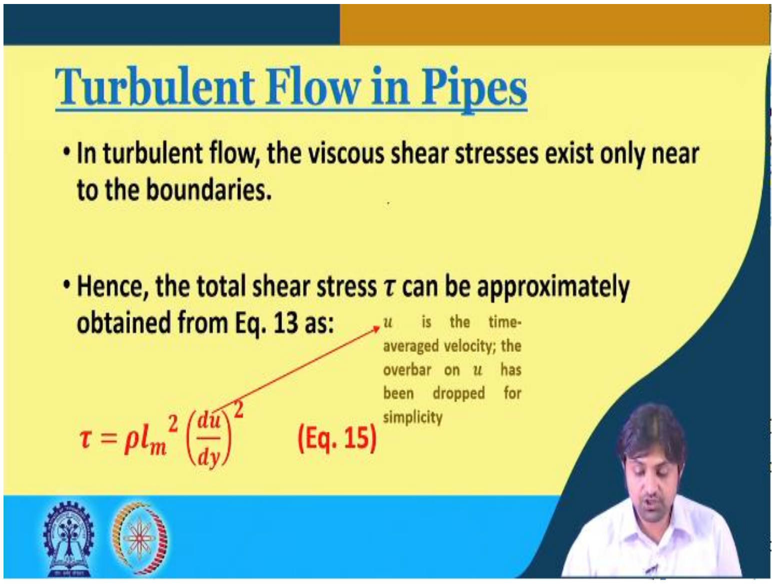

So, now we will see the turbulent flowing pipes now. So, in turbulent flow the viscous shear stresses exists only near the boundary and most of the region is dominated by the turbulence. So, near the boundary the viscous shear stress will act and that are the only places where its existence is. Hence, the total shear stress can be approximately obtained from equation 13 as the total, you know, this was so, this was equation 13.

So, we call this now equation 15 but because most of the shear stress in turbulent flow is due to the turbulent shear stress. So, we can neglect the viscous shear stress. We say tau is equal to rho lm square du / dy whole square. Where, u is the time-averaged velocity, the over bar on u has been dropped for just for simplicity.

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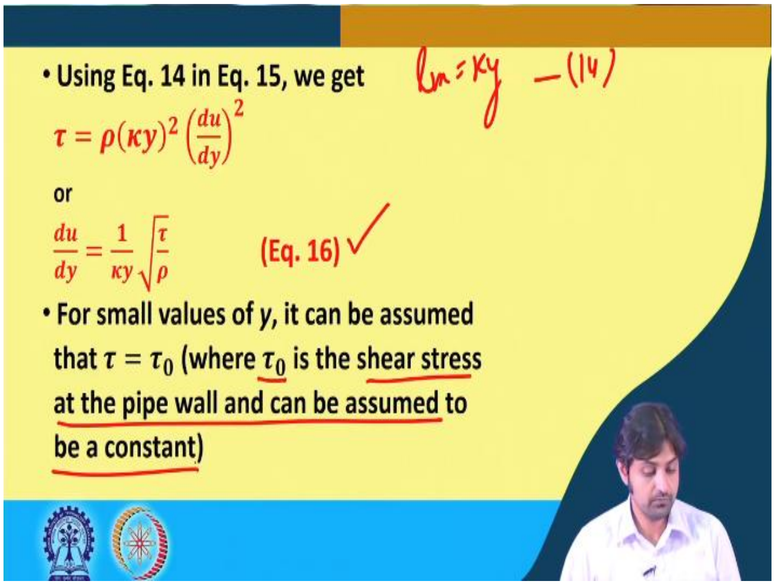

So, if we use equation 14 in equation 15, what was equation 14? l m was kappa into y, this was what we said in equation number 14. So, we can simply write

$$\tau = \rho(\kappa y)^2 \left(\frac{du}{dy}\right)^2$$

or we can simply write

$$\frac{du}{dy} = \frac{1}{\kappa y}\sqrt{\frac{\tau}{\rho}}$$

and this is equation number 16, very simple. So, for small values of y it can be assumed. So, if the y is very small we can assume that tau is equal to tau not, where tau not is the shear stress at the pipe wall and can be assumed to be a constant. So, at the wall the shear stress is assumed to be constant and equal to tau not.

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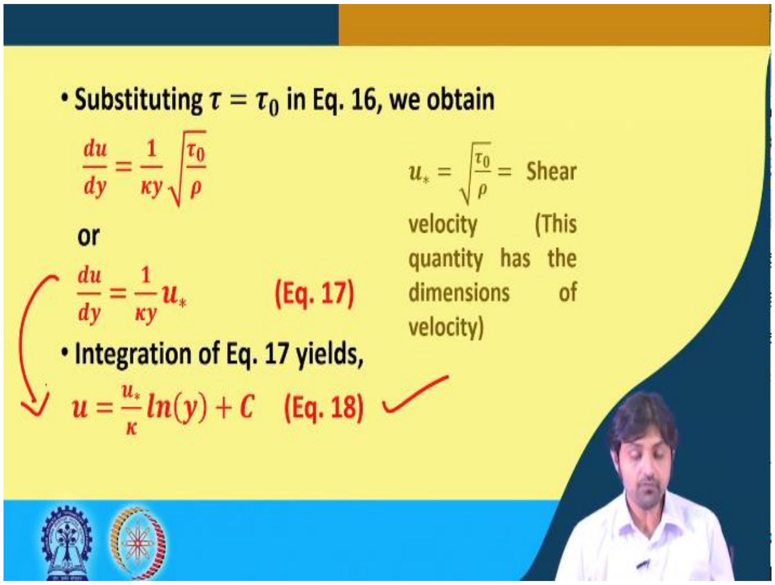

And therefore, what we can say, if we substitute tau is equal to tau not in equation 16, we can obtain

$$\frac{du}{dy} = \frac{1}{\kappa y}\sqrt{\frac{\tau_0}{\rho}}$$

or du / dy, this quantity actually tau not can be written as rho. So, but the catch here is, what is the catch? We have considered small value of y. So, du / dy can be written as, 1 / kappa y and under root tau / rho is rho u *. So, it becomes

$$\frac{du}{dy} = \frac{1}{\kappa y}u_*$$

and this u * under root tau not / rho is the sheer velocity and this has the dimension of velocity.

And if you integrate the equation number 17, so, what we can get is, simple integration, it will get

$$u = \frac{u_*}{\kappa} ln(y) + C$$

. This is very simple integration, from here to here, you can attempt it.

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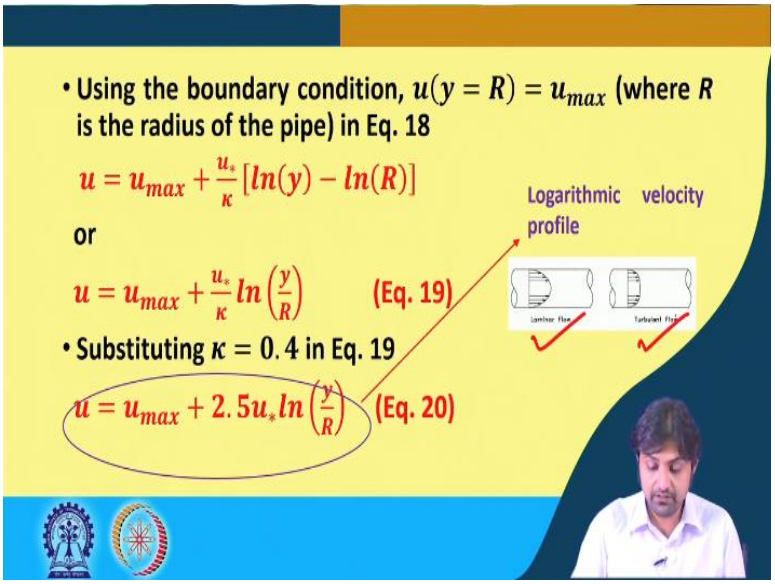

Then using the boundary conditions, what are the boundary conditions? So, u at y is equal to R, where R is the radius of the pipe. We will get, u is equal to u max. That is what we have seen at the center line of the pipe the velocity is going to be the maximum. So, if we use this boundary condition u at y is equal to R is u max, we can get u is equal to, you know, we put u max here, y will be R and therefore, we can obtain C.

C will be u max minus u * / kappa ln R and if we substitute this as C, then we can get equation

$$u = u_{max} + \frac{u_*}{\kappa}[ln(y) - ln(R)]$$

or

$$u = u_{max} + \frac{u_*}{\kappa} ln\left(\frac{y}{R}\right)$$

or if when we substitute kappa as 0.4, we can get

$$u = u_{max} + 2.5u_* ln\left(\frac{y}{R}\right)$$

and this is equation number 20. This is just simple manipulation and as you can see we have derived a logarithmic velocity profile starting with the Prandtl mixing length theory for turbulent fluid flow.

So, laminar flow was something like this, a parabolic profile. Here, a profile is little different u is u maximum plus a logarithmic profile. So, it looks like something like this. Now, the equation 20 this equation 20 can be expressed as,

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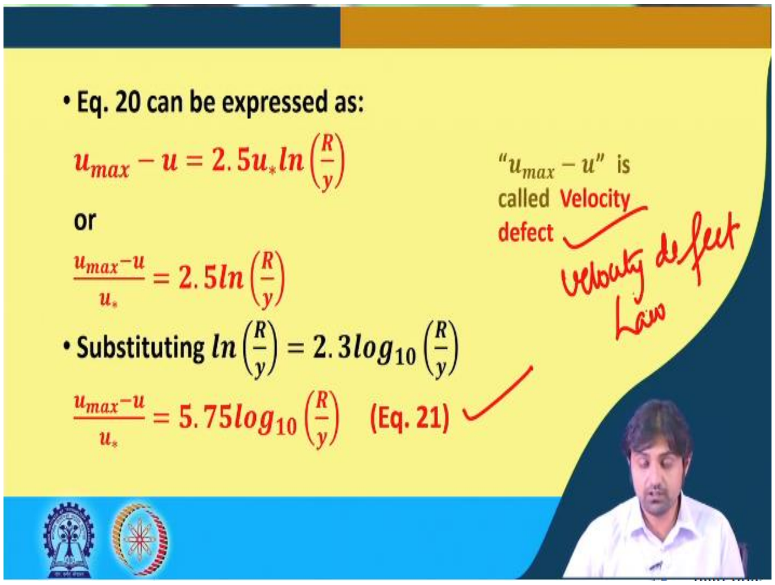

u max, so, what we do is, we bring u on the other side. So, we bring u on this side and we take this whole side component this side, then what the result is, u max minus u because u max will always be larger than u is equal to 2.5 u * ln R / y. y will always be less than R or we bring u frictional velocity down, then we get u max, you bring it down here by dividing then you get u max minus u / u * equals to 2.5 ln R / y. And, so, this is ln.

So, we can put it in form of log. This is simple manipulation, we can get u max minus u / u * is equal to 5.75 log to the base 10 R / y. u max minus u is called the velocity defect or velocity defect law, this is velocity defect law. This is just simple, you know, manipulation of these terms here.

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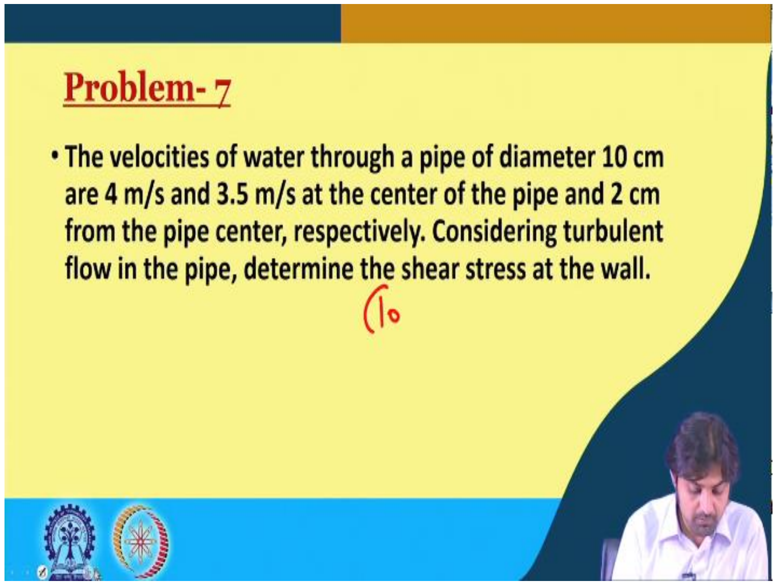

So, now we are going to solve one of the problems, problem number 7. And what it says is, the velocity of water. So, what we have learned in this particular lecture is about the turbulent flow and this problem 7 will help you in solving any problem that is based on this particular concept. So, it says the velocities of water through a pipe of diameter 10 centimeter are 4 meters per second and 3.5 meters per second at the center of the pipe and 2 centimeters from the pipe center, respectively. Considering turbulent flow in pipe, determine the sheer stress at the wall. So, we need to determine tau not. So, let us see how are we going to solve this problem. We are going to have a white screen first.

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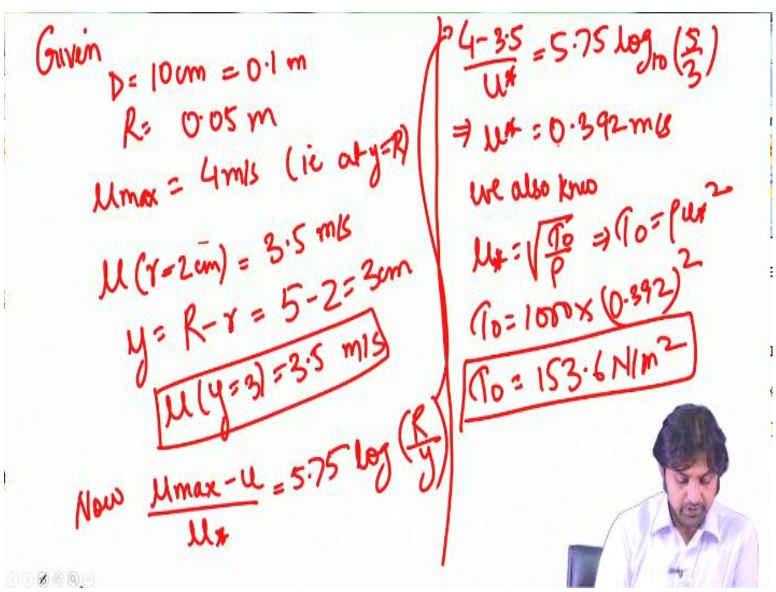

As always what we do we solve, we write given, diameter is given as 10 centimeter, try to always write down in SI units. So, we write 0.1 meter. So, diameter is 10. So, radius is going to be 0.05 meter. u max is given, is given as 4 meters per second, that is, at y is equal to R. And this is also given, u at r is equal to 2 centimeter is given 3.5 meters per second, that is, y is equal to R - r. So, y is going to be 5 - 2 is equal to 3 centimeter.

So, u at y is equal to 3 is equal to 3.5 meters per second. So, now u max we are using the minus u / u * was 5.75 log R / y. So, substituting the values here, this from here, this equation, 4 - 3.5 divided by u * is equal to 5.75 log base 10 5 / 3. This will give us, u * as 0.392 meters per second. We also know, u* is under root tau not / rho or tau not is rho u* whole square. Therefore, tau not rho is 1000 and u * we already got, 0.392 whole square.

So, tau not is coming out to be 153.6 Newton per meter square. This is the solution to the question that we have at hand. So, going back again to the slide, so, what we got was approximately 153 Newton per meter square the sheer stress at the wall.

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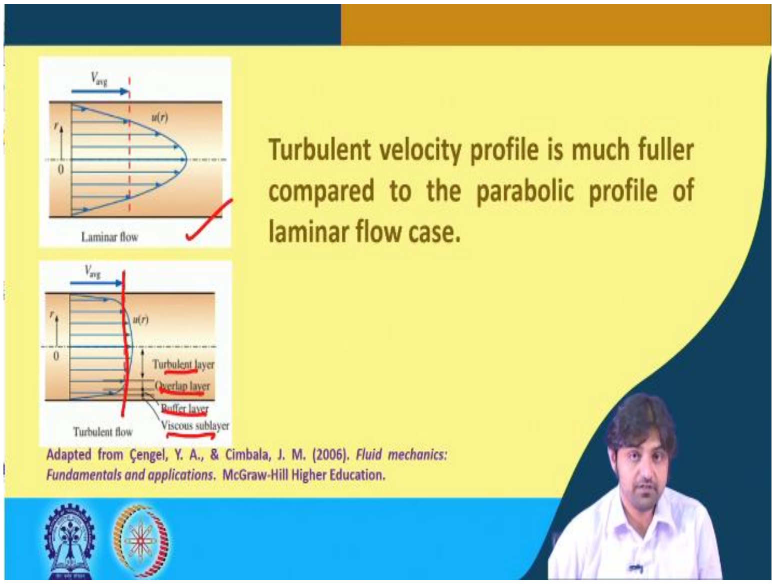

So, now, the turbulent velocity profile is much fuller compared to the parabolic profile of laminar flow case. Actually this is the flow, this is the true picture, this is a laminar flow that we have seen before. But below is, this is the V average and the velocity fluctuates or deviates from these depending upon the flow condition. So, this is the V average line. There are several other layers, viscous sublayer, buffer layer, overlap layer and turbulent layer. So, as I told you in the last slide, there are different layers, different layers in turbulent flow.

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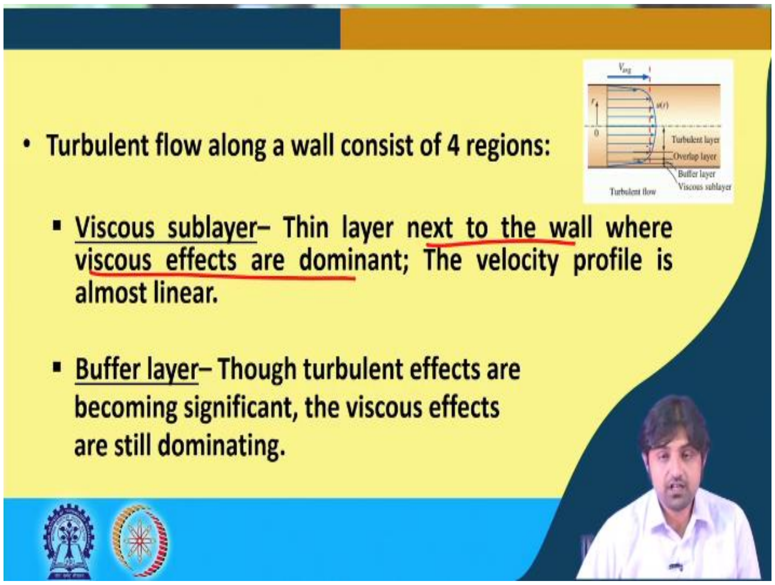

And we are going to talk about that. Turbulent flow along a wall consists of 4 regions. Viscous sublayer, this layer is thin layer next to the wall. So, this is the closest to the wall where the viscous effects are dominant and the velocity profile is all most linear. So, in viscous sub layer the viscous effects are dominant and the velocity profile is linear. In the buffer layer, though turbulent effects are becoming significant, the viscous effects are still dominating.

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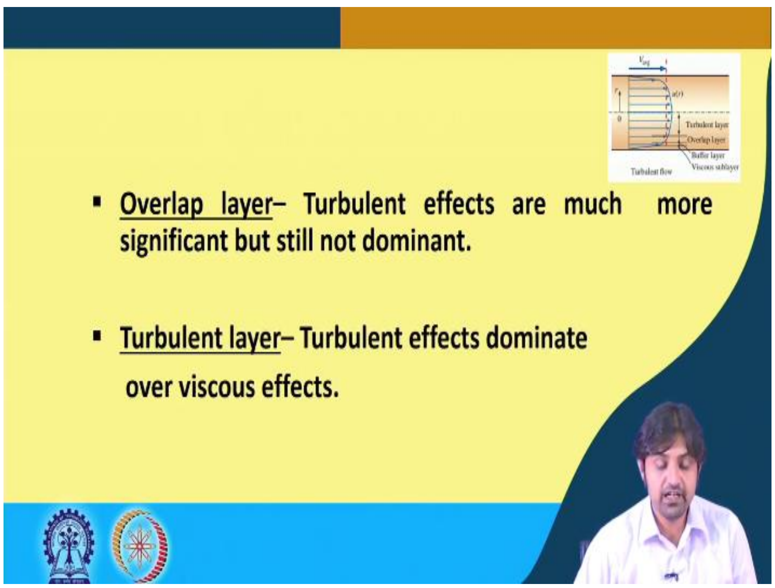

In the overlap layer, the turbulent effects are much more significant but still not dominant, in the overlap layer. In the turbulent layer, the turbulent effects dominate over these viscous effects.

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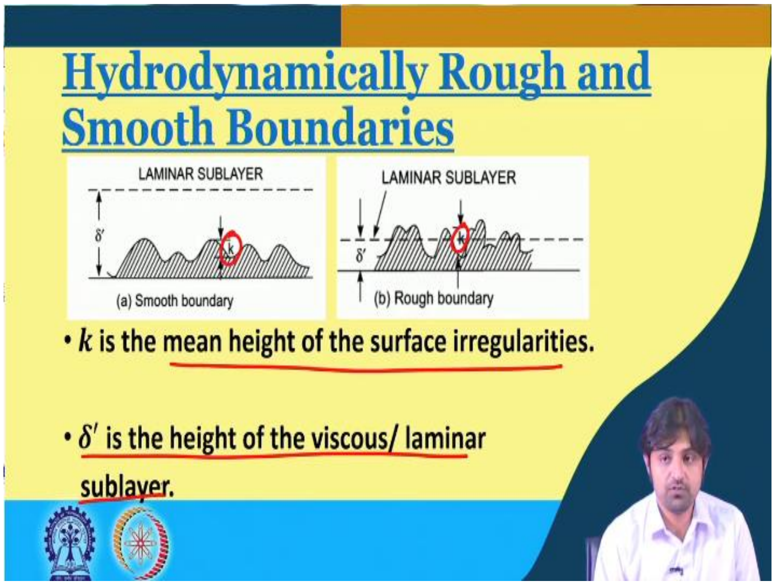

Now, when it comes to these beds and these regimes, some of the important terms that are there is hydro dynamically rough and smooth boundaries. So, this is the, if you see, there is a term called k. Here, if in here, so, k here is the mean height of the surface irregularities. We talked in the beginning that the turbulence could occur due to the presence of irregularities on the surface. So, let us say, the mean height of the surface irregularities is k. And delta dash, for example, is the height of viscous or laminar sublayer, the first layer that we talked, the viscous sub layer that was where the velocity profile was almost linear.

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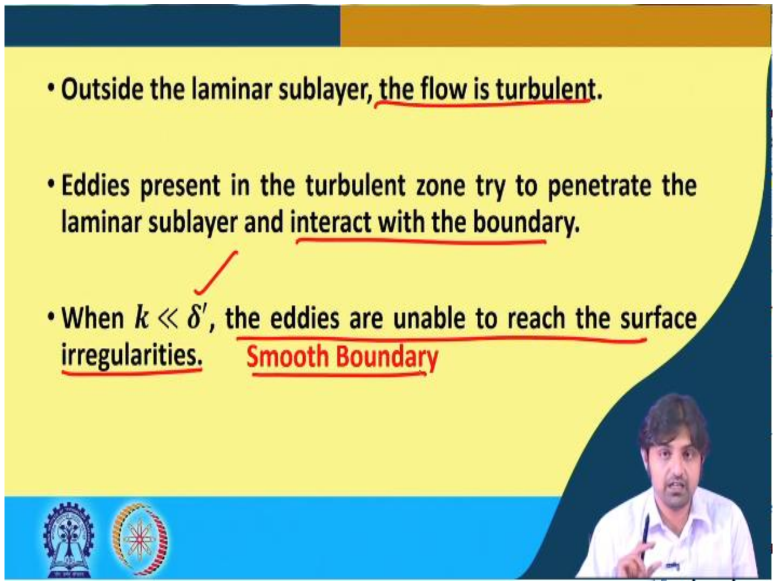

So, outside the laminar sublayer the flow is turbulent, that is, what we have talked about. Eddies present in the turbulent zone try to penetrate the laminar sublayer and interact with the boundary. But when the surface irregularities are much smaller than delta dash, the height of the viscous sublayer, the eddies are unable to reach the surface irregularities when the roughness height is much less. Therefore, we define that boundary as smooth boundary.

So, smooth boundary are the one, where the thickness of the viscous sublayer is much larger than the surface irregularities. We will see, what those surface regularities here, represented by k.

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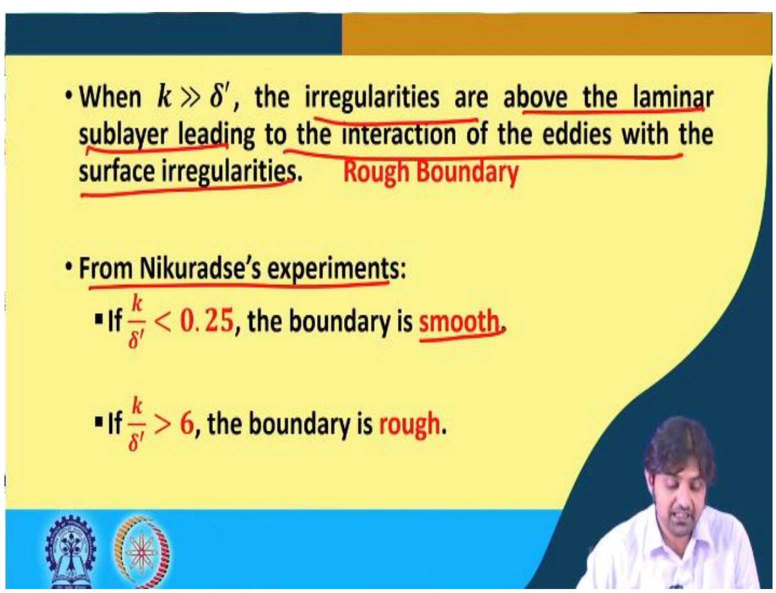

When k is much larger than the delta dash, that is, the thickness of viscous sub layer, the irregularities are above the laminar sublayer leading to the interaction of eddies with the surface irregularities and therefore, these are called rough boundaries. From Nikuradse's roughness, k / delta dash if it is less than 0.25. So, these values which we are going to talk about, has been derived from experiments by Nikuradse. Nikuradse said if k which is the height of the irregularity is divided by the thickness of viscous sublayer is less than 0.25, the boundary is smooth, if k / delta dash is greater than 6, the boundary is for sure rough.

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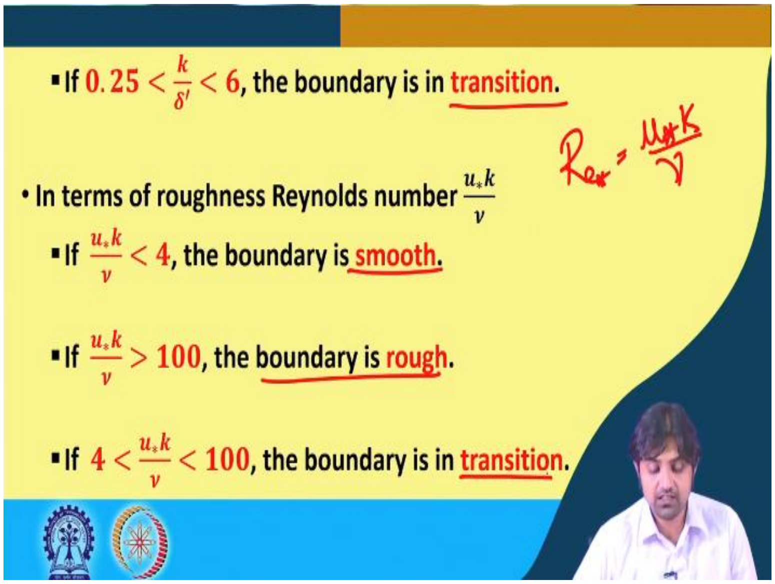

But if it lies in between 0.25 and 6, the boundary is transitional. In terms of roughness Reynolds number, so actually, there is something called roughness Reynolds number that is dependent upon k the height of the irregularities. So, in terms of roughness Reynolds number, if this Reynolds number is less than the 4, the boundaries is smooth, if it is more than the 100 then the boundary is rough and if it is lies between 4 and 100 the boundaries is transitional.

So, either we can calculate it in terms of k / delta, where k is this height of the irregularities and lambda dash is the viscous sublayer or more it is more easy to calculate, u * k / nu. If this is less than 4, it is smooth, if it is more than 100 then rough otherwise in between it is a transitional boundary.

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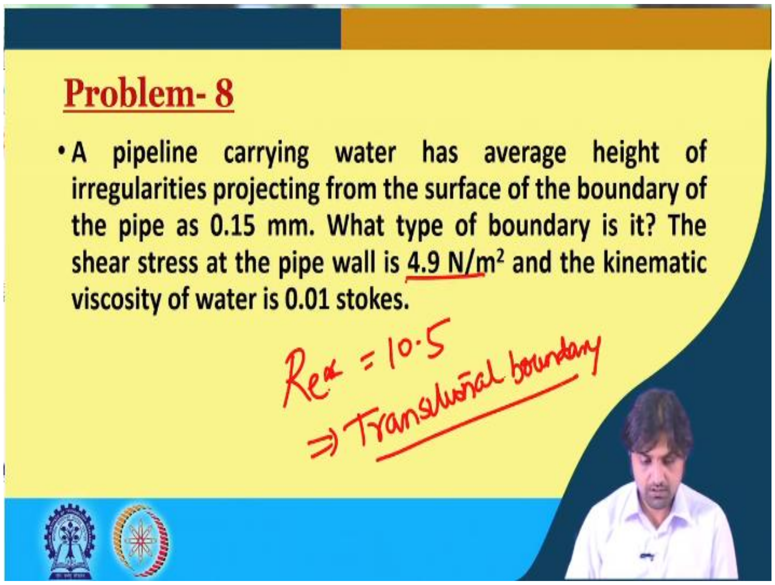

Now, we will solve one problem about this particular concept. So, the question is, a pipeline carrying water has average height of irregularities projecting from the surface of the boundary of the pipe as 0.15 millimeter. What type of boundary it is? We have to estimate the rough or smooth or transitional boundary. The shear stress at the pipe wall is 4.9 Newton per meter square and the kinematic viscosity is 0.01 Stokes. So, shear stress at the wall is given. So, we will be able to calculate u* from here. But better that we go and start doing the problems as we have been doing.

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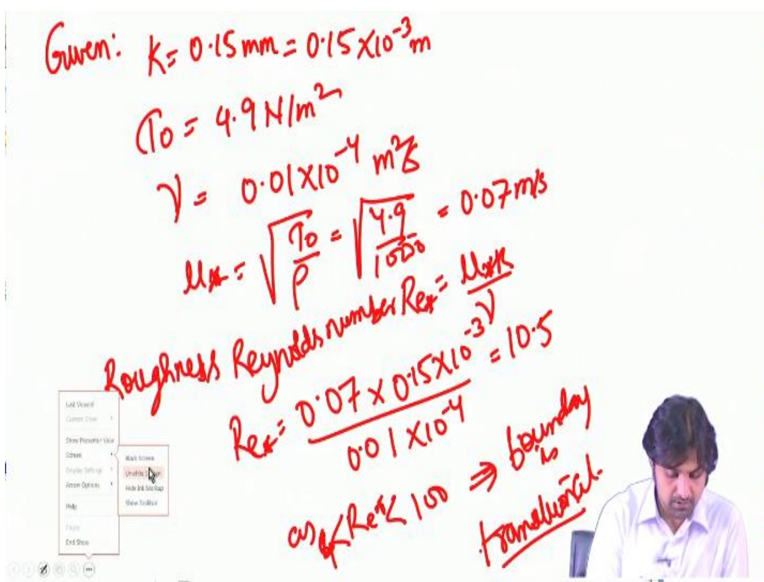

So, we have to write the things that we it has been given to us. k is given as, 0.15 millimeter, it is always a good habit to write it into SI unit into 10 to the power - 3 meter, tau not is actually given here, 4.9 Newton per meter square and nu is also given 0.01 into 10 to the power minus 4 meter square per second. Therefore, we can simply calculate u* under root tau not / rho, as I told you and this will come out to be under root 4.9 / 1000, so, it will come out to be 0.07 meters per second, very simple.

So, best is to calculate the roughness Reynolds number Re* and that is given as, u* k / nu. So, Re* is, u* is 0.07, k is 0.15 into 10 to the power - 3 and nu is 0.01 into 10 to the power - 4 and that comes to be 10.5. So, as Re* lies between 4 and 400, this implies that the boundary is transitional. So, just going back to that screen, so, what we have got is Re* is 10.5 implying transitional boundary.

So, this is the place where we will end this lecture of ours today and resume in the next lecture and will talk about turbulent flow in smooth pipes. So, I will see you in the next lecture. Thank you.