**Soil Structure Interaction**
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**Lecture – 50**
**Use of Finite Difference Method for Soil Structure Interaction Problems (Contd.,)**

In this class I will discuss how to apply the finite difference method to solve the rectangular plate problem when the plate is resting on springs.

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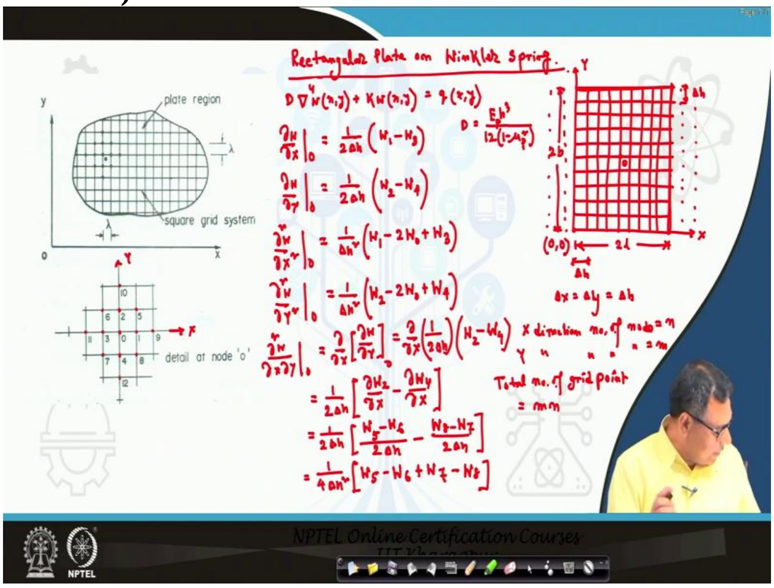


In the last class, the division of the rectangular plate into segments, numbering the nodes and the various expressions were discussed.

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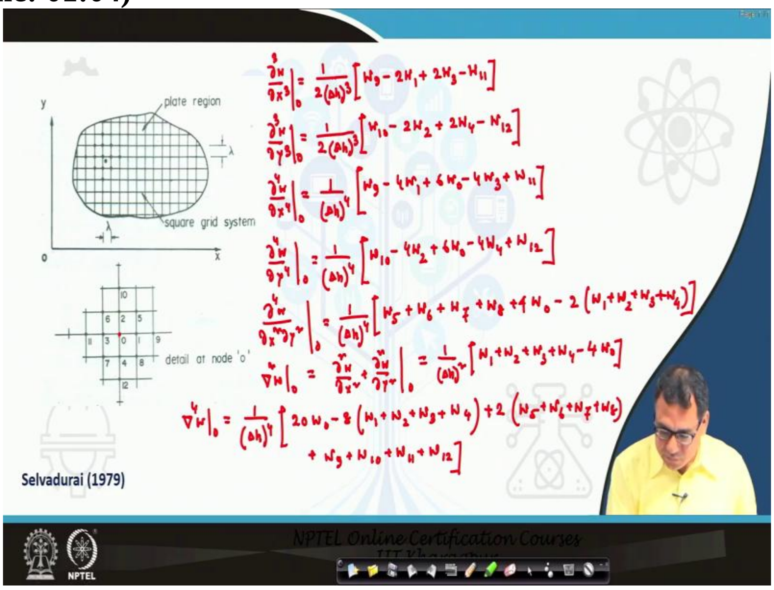


The expression for $\nabla^4 w$ was also given in the finite difference form for which 13 node points are required to solve. Now, let us apply these equations in the governing differential equation.

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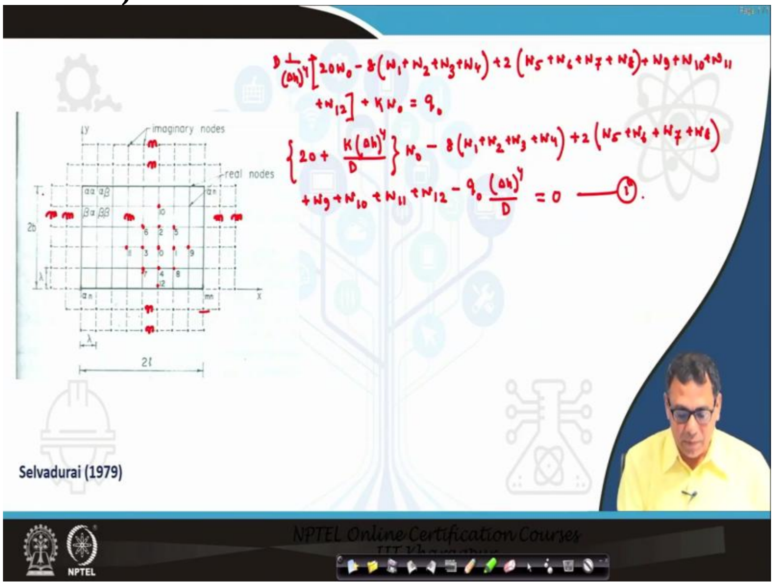


The governing differential equation was:

$$D\nabla^4 w(x,y) + kw(x,y) = q(x,y)$$

where, $D = \dfrac{E_p h^3}{12(1-\mu_p^2)}$; $E_p$ is the elastic modulus of the plate and $\mu_p$ is the Poisson's ratio of the plate.

Substituting the finite difference form:

$$D\frac{1}{(\Delta h)^4}\left[20w_o - 8(w_1 + w_2 + w_3 + w_4) + 2(w_5 + w_6 + w_7 + w_8) + w_9 + w_{10} + w_{11} + w_{12}\right] + kw_o = q_o$$

Rearranging the terms:

$$\left\{20 + \frac{k(\Delta h)^4}{D}\right\}w_o - 8(w_1 + w_2 + w_3 + w_4) + 2(w_5 + w_6 + w_7 + w_8) + w_9 + w_{10} + w_{11} + w_{12} - q_o\frac{(\Delta h)^4}{D} = 0 \rightarrow 1$$

This is the finite difference form of the governing differential equation.

Note that the $\nabla^4 w$ term should first be expressed in $\nabla^2 x$ and $\nabla^2 y$ form and then should be substituted in the finite difference form. Finally, the finite difference form of the governing differential equation for a plate resting on springs is obtained.

Now the condition and procedure mentioned is fine when being applied to any interior node. But if the equation is to be applied to a node on the boundary, all the 12 other nodes may not be available. So again here some imaginary nodes should be considered like that of in the beam problem. These imaginary can be solved by considering boundary conditions.

The m × n nodes are the real nodes but in addition to these, to solve these problems few imaginary nodes have to be considered. Now, the question is what would be the number of imaginary nodes that are to be considered.

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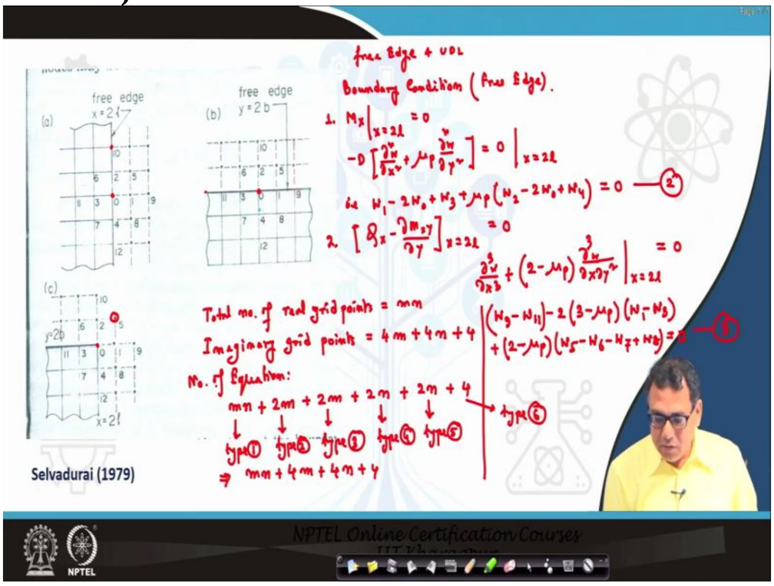


The 3 different cases where the imaginary nodes can be calculated are shown above. These three cases are like examples of wherever the imaginary nodes are required. The condition given in the question is that the plate is subjected to UDL and has free edge. The edge indicated by x = 2l is the vertical free edge towards the right and similarly, the top horizontal edge can be indicated by y = 2b.

When a point is considered as the 0 node, we need another 12 points surrounding to this point. So, 2 nodes right side to that 0 node, 2 nodes left side to that 0 node, 2 nodes above that 0 node and 2 nodes below that 0 node along with 4 additional nodes, diagonally adjacent to the 0 node are required. These nodes are required to apply the governing differential equation along the edge also.

If the 0 node is on the x = 2l edge, there are two imaginary nodes required in that line. Along with it, one node each in the line above and below it is also required (diagonally adjacent nodes). But, these 2 diagonal nodes will be added to the consideration when the 0 node is considered in that particular line. So, 2 imaginary nodes are required in each horizontal division.

(*Each column has m nodes and each row has n nodes*)

In the Y direction, there are m such horizontal divisions and it means that every column has m nodes in it. Out of those m nodes in the x = 2l edge, we found that for one such node, two imaginary nodes are needed. So, two more columns of m nodes each are needed for the x = 2l edge and also for the x = 0 edge. The x = 0 edge is also similar to the x = 2l edge and so four columns, each of m nodes are required. So, number of imaginary grid points or nodes required will be 4m. This is the number of imaginary nodes considering only the vertical edges. Similar way when considering the y = 2b edge and y = 0 edge, four rows of imaginary nodes are required. But the row is divided by n grid points (number of nodes in X direction) and so here, the number of imaginary nodes required will be 4n.

Considering the first two cases (cases- a & b) shown in the slide, the imaginary nodes required are 4m + 4n. Now, consider the third case (case-c) showing a corner node. Till now, only edge nodes are dealt with and for them the number of imaginary nodes needed is found out. When considering a corner node, we can see from the figure that the nodes 1, 8, 9, 2, 6 and 10 are already counted in the 4m +4n nodes. The only node that is needed in addition to those for the corner node is the fifth node. So, one additional node for each corner node and as there are four corner nodes, we need four more nodes. So, the total number of imaginary grid points needed to solve the rectangular plate problem is 4m +4n +4.

By applying the governing differential equation to all the real nodes, 'mn' number of equations will be obtained. But the unknowns are more than that and so, to solve them, the boundary conditions should be applied.

The first boundary condition is that the moment in the X direction at x = 2l is zero as this is a free edge condition.

$$1.)\ M_x\big|_{x=2l} = 0$$

$$-D\left[\frac{\partial^2 w}{\partial x^2} + \mu_p \frac{\partial^2 w}{\partial y^2}\right] = 0\Bigg|_{x=2l}$$

$$\Rightarrow w_1 - 2w_o + w_3 + \mu_p\left(w_2 - 2w_o + w_4\right) = 0 \rightarrow (2)$$

Similarly the shear force also will be zero at the edge:

$$2.)\left[Q_x - \frac{\partial M_{xy}}{\partial y}\right]_{x=2l} = 0$$

$$\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu_p)\frac{\partial^3 w}{\partial x \partial y^2}\right] = 0\Bigg|_{x=2l}$$

$$(w_3 - w_{11}) - 2(3-\mu_p)(w_1 - w_3) + (2-\mu_p)(w_5 - w_6 - w_7 + w_8) = 0 \rightarrow (3)$$

For the free edge, x = 2l, the boundary conditions that bending moment and shear force are 0 are applied. Similarly, these boundary conditions can be applied to the y = 2b edge.

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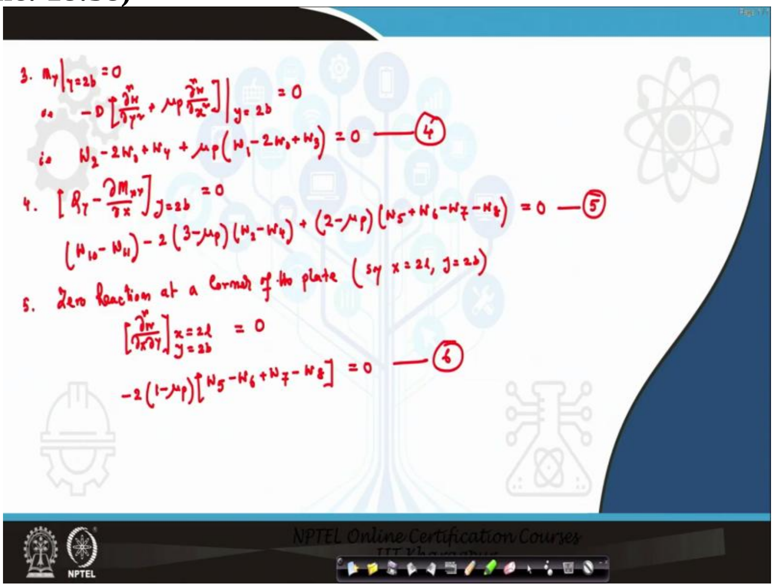


The moment in y direction at y = 2b will be:

$$3.)\ M_y\Big|_{y=2b} = 0$$

$$-D\left[\frac{\partial^2 w}{\partial y^2} + \mu_p \frac{\partial^2 w}{\partial x^2}\right] = 0\Bigg|_{y=2b}$$

$$\Rightarrow w_2 - 2w_o + w_4 + \mu_p(w_1 - 2w_o + w_3) = 0 \rightarrow (4)$$

Shear force at y = 2b will be:

$$4.)\left[Q_y - \frac{\partial M_{xy}}{\partial x}\right]_{y=2b} = 0$$

$$(w_{10} - w_{11}) - 2(3-\mu_p)(w_2 - w_4) + (2-\mu_p)(w_5 + w_6 - w_7 - w_8) = 0 \rightarrow (5)$$

Now that the boundary conditions are applied, let us see what will be the number of equations that we will get. It is important to calculate the number of equations because we need equations as many as the number of unknowns. Firstly, there are mn real nodes where the equation-(1) can be applied and mn equations can be obtained from that. Let us all these equations as type-1 equations.

Next, the first boundary condition will be applied to the edges x = 2l and x = 0 where the moment is 0 because the end condition here is free. As there are m nodes in each column, there are 2m nodes considering both the edges x = 2l and x = 0. So, there will be 2m equations of type-(2). Similarly, by applying the shear force boundary condition at these two edges, there will be 2m more equations, but of type-(3). The same substitution will be adapted to the horizontal edges too (y = 2b & y = 0). From that there will be 2n equations of type-(4) and 2n equations of type-(5). Till now, there are mn + 4m + 4n equations but there are mn + 4m + 4n + 4 unknowns including that of the corner nodes. So, four more equations are required to solve the problem.

Now another boundary condition should be applied which is that the reaction at the corner of a plate will be 0. Applying the boundary condition of zero reaction at the plate corner to the x = 2l & y = 2b corner.

$$\left[\frac{\partial^2 w}{\partial x \partial y}\right]_{\substack{x=2l \\ y=2b}} = 0$$

$$\Rightarrow -2(1-\mu_p)[w_5 - w_6 + w_7 - w_8] = 0 \rightarrow (6)$$

So, the above boundary condition can be applied to the four corners and four more equations can be obtained. The total number of equations and the unknowns are equal.

$$\underbrace{mn}_{type-1} + \overbrace{2m}^{type-2} + \underbrace{2m}_{type-3} + \overbrace{2n}^{type-4} + \underbrace{2n}_{type-5} + \overbrace{4}^{type-6}$$

By applying to all the relevant nodes, equations can be obtained and with those equations, the matrix can be formed. By solving the matrix the deflection at all the real nodes and the required quantities using the deflection values can be determined.

This problem is solved for the free edge condition. But if the edge condition changes, only the boundary condition will change. For example, here we have considered moment and shear force at the edges are zero, but if it is a fixed end, the slope and deflection will be considered zero. But the procedure will be same if it is a rectangular plate.

In the next class I will discuss how to express these equations in the non-dimensional form. After that I will show some applications of the models that I have discussed. There are number of applications, I will show few of them. Thank you.