Soil Structure Interaction
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Lecture - 49
Use of Finite Difference Method for Soil Structure Interaction Problems (Contd.,)

In this class I will discuss how to apply the finite difference method in plate problem under axi-symmetric loading condition and then for non-dimensional case. First I will discuss how to solve the plate problem under axi-symmetric circular loading condition.

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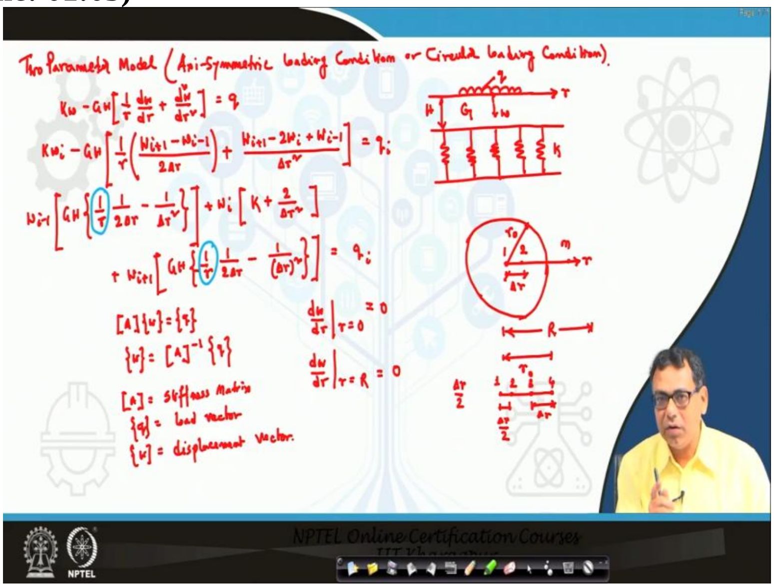

Consider a circular UDL, q acting on a two parameter soil medium. The subgrade modulus of the springs is k, thickness of the shear layer is H and shear modulus is G. The radius of the plate is r and the deflection direction is considered as w. The expression for the two parameter model under axi-symmetric loading condition was already derived (*In lecture-16*).

$$kw - GH\left[\frac{1}{r}\frac{dw}{dr} + \frac{d^2w}{dr^2}\right] = q$$

$$kw_i - GH\left[\frac{1}{r}\left(\frac{w_{i+1} - w_{i-1}}{2\Delta r}\right) + \frac{w_{i+1} - 2w_i + w_{i-1}}{\Delta r^2}\right] = q_i$$

$$w_{i-1}\left[GH\left(\frac{1}{r}\frac{1}{2\Delta r} - \frac{1}{\Delta r^2}\right)\right] + w_i\left[k + \frac{2}{\Delta r^2}\right] + w_{i+1}\left[GH\left(\frac{1}{r}\frac{1}{2\Delta r} - \frac{1}{\Delta r^2}\right)\right] = q_i$$

Using the above coefficients, the stiffness matrix should be developed by generating n equations. The displacement should be calculated from that matrix form:

$$[A]\ \{w\} = \{q\}$$

$$\{w\} = [A]^{-1}\ \{q\}$$

where, [A] is the stiffness matrix, {w} is the deflection matrix and {q} is the load vector.

The boundary conditions here will be:

$$\left.\frac{dw}{dr}\right|_{r=0} = 0$$

As the loading is symmetric, the slope at the centre will be 0. Similarly, slope will be zero at another point:

$$\left.\frac{dw}{dr}\right|_{r=R} = 0$$

where, R is the radius of the entire circular zone considered for modelling. In the beam problem this was in the terms of length denoted by L and was mentioned that L should be sufficiently large such that the deflection at x = L should be zero. The same rule applies here also and R should be considered large enough to keep the deflections at r = R as zero. Also, the radius of the loading region was considered to be r_o from the centre.

Remember that here any plate is not considered, but the UDL is directly applied on the two parameter model. Note that there is a 1/r term in the expression for this axi-symmetric loading condition. So, if r = 0 (i.e., at the centre) the value will be infinity and it cannot be solved. To avoid this problem, the first node should not be considered at x = 0, but very close to the centre. So, the first node may be considered at a distance of (Δr/2) from the centre and the remaining nodes can be considered as it is. This implies that the first segment will be shorter than all other segments.

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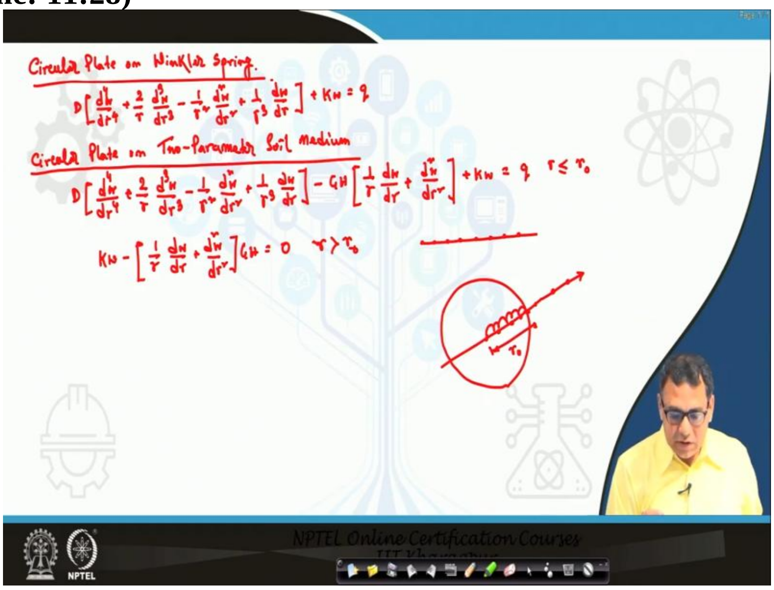

The case of a circular plate resting on Winkler springs can also be solved in the similar way discussed. The numerical solution process of circular plate is similar to that of the beam, but the only difference is in the equation (but the solution technique is similar). For the circular plate condition, the θ term is being eliminated because the solution is independent of the angle under an axi-symmetric loading condition. In that case the governing differential equation for the circular plate is:

$$D\left[\frac{d^4w}{dr^4}+\frac{2}{r}\frac{d^3w}{dr^3}-\frac{1}{r^2}\frac{d^2w}{dr^2}+\frac{1}{r^3}\frac{dw}{dr}\right]+kw=q$$

If the circular plate is resting on two parameter model soil medium, the equation will be:

$$D\left[\frac{d^4w}{dr^4}+\frac{2}{r}\frac{d^3w}{dr^3}-\frac{1}{r^2}\frac{d^2w}{dr^2}+\frac{1}{r^3}\frac{dw}{dr}\right]-GH\left[\frac{1}{r}\frac{dw}{dr}+\frac{d^2w}{dr^2}\right]+kw=q \quad \text{for } r \leq r_o$$

$$kw-GH\left[\frac{1}{r}\frac{dw}{dr}+\frac{d^2w}{dr^2}\right]=0 \quad \text{for } r > r_o$$

Now that the basic differential equations are formulated, the boundary conditions should be applied to the above equations and can be solved. This procedure is similar to that of the beam problem which is already discussed in the previous lectures. In the beam, the zone was similar to a line and that was divided into number of linear segment. Even here, any line can be considered and divided into segments because it is independent of θ.

The second equation is valid only for the region beyond the loaded area ($r > r_o$). This expression is applicable for the plate resting on two parameter soil medium only, because in case of Winkler springs, the deflections are constrained to the loaded region only. But the solution techniques are same as that of the beam problem.

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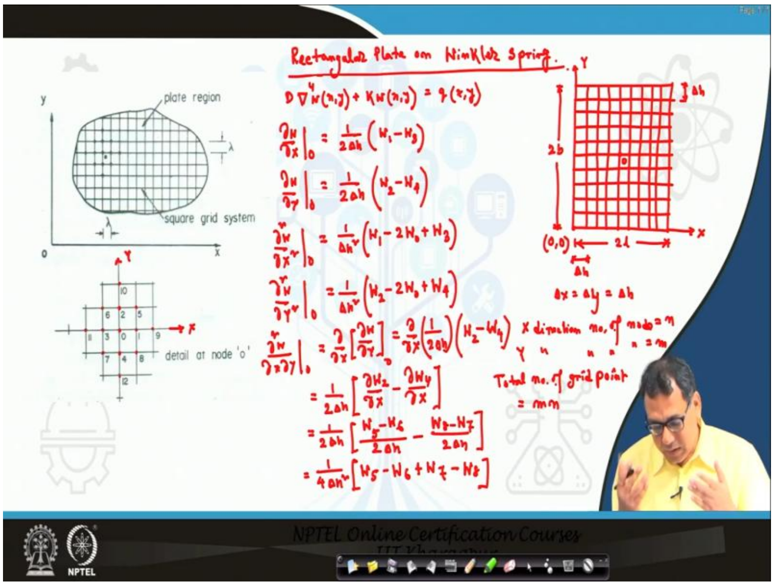

But the solution technique for the rectangular plate is slightly different. Now let us discuss about the rectangular plate resting on Winkler springs. Consider a rectangular plate with its centre at (0,0). The dimension of the plate in the X direction is considered as 2l and in the Y direction as 2b.The plate is divided into number of segments, as this is a 2D problem.

As this is not a straight line, it is divided into small area segments. For that, the divisions should be made in both the directions (X & Y). The dimension of each segment in the X direction can be considered as Δx and that in the Y direction as Δy. But in both the directions, the distance between two successive nodes is same and so we can write: $\Delta x = \Delta y = \Delta h$.

The origin (0,0) is named as point O. Let the number of nodes is m in the X direction and in the Y direction be n. The origin or the point O can be considered as the zero point at first and once a node is considered as zero point, the nodes around it will be numbered as shown in the left side figure. If the differential equation has to be applied at one point, that point or node may be considered as the zero point. To apply this to one node, 12 such nodes are needed.

The governing differential equation for the plate resting on Winkler springs is:

$$D\nabla^4 w(x,y) + kw(x,y) = q(x,y)$$

The above equation should be applied to all the nodes. As the number of nodes in X direction and Y direction is m and n respectively and so the total number of nodes is m × n. Now, applying the above equation to a node, which is considered the zero node the first equation:

$$\left.\frac{\partial w}{\partial x}\right|_0 = \frac{1}{2\Delta h}(w_1 - w_3)$$

As the differentiation is considered in the X direction, the nodes in the X direction are considered in the above equation. Similarly, in the Y direction:

$$\left.\frac{\partial w}{\partial y}\right|_0 = \frac{1}{2\Delta h}(w_2 - w_4)$$

Higher order expressions can be formulated as:

$$\left.\frac{\partial^2 w}{\partial x^2}\right|_0 = \frac{1}{\Delta h^2}(w_1 - 2w_o + w_3)$$

$$\left.\frac{\partial^2 w}{\partial y^2}\right|_0 = \frac{1}{\Delta h^2}(w_2 - 2w_o + w_4)$$

The procedure to derive the following equation will be discussed in-detail that can be followed in a similar way for other equations too.

$$\left.\frac{\partial^2 w}{\partial x \partial y}\right|_0 = \frac{\partial}{\partial x}\left[\frac{\partial w}{\partial y}\right]_0 = \frac{\partial}{\partial x}\left[\frac{1}{2\Delta h}(w_2 - w_4)\right]$$

$$\Rightarrow \left.\frac{\partial^2 w}{\partial x \partial y}\right|_0 = \frac{1}{2\Delta h}\left[\frac{\partial w_2}{\partial x} - \frac{\partial w_4}{\partial x}\right]$$

$$\Rightarrow \left.\frac{\partial^2 w}{\partial x \partial y}\right|_0 = \frac{1}{2\Delta h}\left[\frac{w_5 - w_6}{2\Delta h} - \frac{w_8 - w_7}{2\Delta h}\right]$$

$$\Rightarrow \left.\frac{\partial^2 w}{\partial x \partial y}\right|_0 = \frac{1}{4\Delta h^2}\left[w_5 - w_6 + w_7 - w_8\right]$$

The positive direction horizontally is considered to be to the right side and in the vertical direction it is upwards. In other words, the right ward and upward directions are considered as the forward directions and the other two as the backward directions. Note that in the above derivation, at every step, the deflection at a node in the backward direction is subtracted from the deflection at a forward direction node. For example, when the derivative of y is considered (in the first step), it is (w_2 – w_4) as node-2 is towards the upward direction and node-4 is towards the backward direction. Similar procedure should be followed for all the equations.

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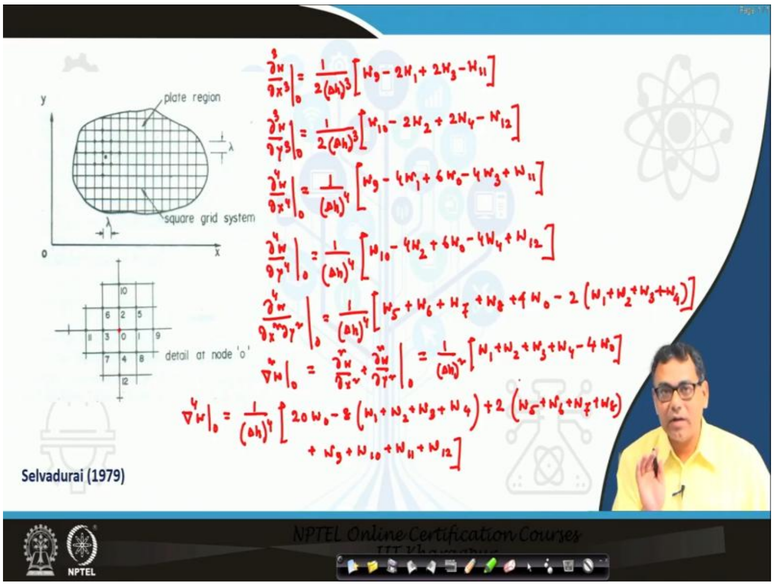

Similar way, few other expressions can be obtained:

$$\left.\frac{\partial^3 w}{\partial x^3}\right|_0 = \frac{1}{2(\Delta h)^3}(w_9 - 2w_1 + 2w_3 - w_{11})$$

$$\left.\frac{\partial^3 w}{\partial y^3}\right|_0 = \frac{1}{2(\Delta h)^3}(w_{10} - 2w_2 + 2w_4 - w_{12})$$

$$\left.\frac{\partial^4 w}{\partial x^4}\right|_0 = \frac{1}{(\Delta h)^4}(w_9 - 4w_1 + 6w_o - 4w_3 + w_{11})$$

$$\left.\frac{\partial^4 w}{\partial y^4}\right|_0 = \frac{1}{(\Delta h)^4}(w_{10} - 4w_2 + 6w_o - 4w_4 + w_{12})$$

$$\left.\frac{\partial^4 w}{\partial x^2 \partial y^2}\right|_0 = \frac{1}{(\Delta h)^4}\left[w_5 + w_6 + w_7 + w_8 + 4w_o - 2(w_1 + w_2 + w_3 + w_4)\right]$$

$$\left.\nabla^2 w\right|_0 = \left.\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right|_0 = \frac{1}{(\Delta h)^2}\left[w_1 + w_2 + w_3 + w_4 - 4w_o\right]$$

Finally, the fourth order derivative, that is in the differential expression:

$$\left.\nabla^4 w\right|_0 = \frac{1}{(\Delta h)^4}\left[20w_o - 8(w_1 + w_2 + w_3 + w_4) + 2(w_5 + w_6 + w_7 + w_8) + w_9 + w_{10} + w_{11} + w_{12}\right]$$

The above equation clearly shows that to apply the governing differential equation to a particular node or grid point, a total of 13 nodes are required.

In the next class, I will apply the finite difference form in the governing differential equation. Then I will show you how to get the governing differential equation in finite difference form and what are the boundary conditions and then how to solve them. Thank you.