Soil Structure Interaction

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Lecture – 45
Use of Finite Difference Method for Soil Structure Interaction Problems

In this lecture I will start a new topic, which is use of finite difference method for soil structure interaction problems. In my previous lectures I derived governing differential equations for different soil structure interaction problems. And then I showed closed form solutions and the boundary conditions to get that closed form solutions for different conditions. Now, in this class I will show how to solve those problems and get the deflection and other quantities by using finite difference method.

It is always preferable to have a closed form solution, because by giving some inputs, the required quantities can be obtained, but sometimes it is very difficult to get the closed form solution. In such cases, this approximate type of solutions can be adopted and one of them is the finite difference technique. Here, any software or programming part will not be discussed. Rather, the procedure using which we can form the basic governing differential equation to the matrix where the equations are solved to get the required quantities will be discussed.

To form a basic differential equation, a set of equations need to be prepared and then that set of equations should be solved. The different solution techniques will not be covered in this course. But, the methodology involved in obtaining these equations, in solving these equations will be discussed along with the ways to use or implement the boundary conditions and loading conditions.

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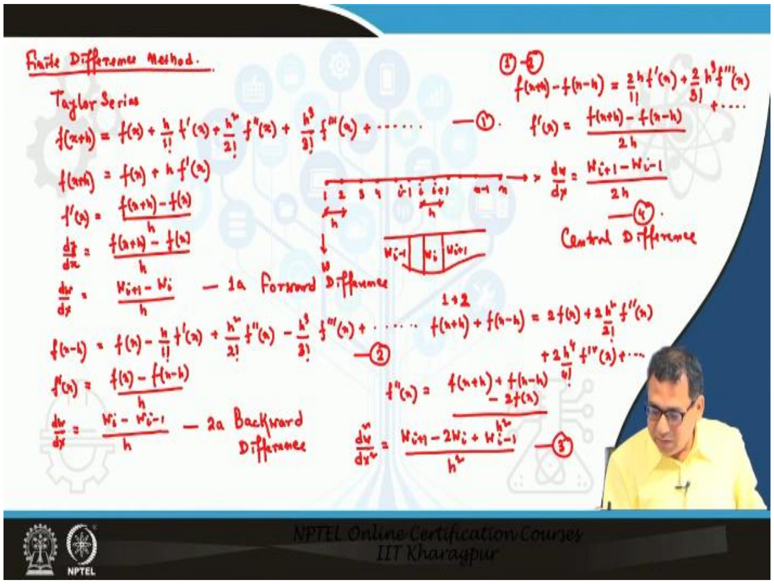

Before starting the above mentioned part, let us have a brief introduction about the finite difference method. Before applying those methods to the governing differential equations, the forms of difference derivatives should be obtained. Consider the general expression of Taylor series:

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + ... \rightarrow (1)$$

As the value of h is so small, the terms with power more than one can be neglected:

$$f(x+h) = f(x) + hf'(x)$$

$$\Rightarrow f'(x) = \frac{f(x+h) - f(x)}{h}$$

The above expression can also be written as:

$$\Rightarrow \frac{dy}{dx} = \frac{f(x+h) - f(x)}{h}$$

Consider a structure which is to be studied, may be a beam of length, l. It is divided into a number of imaginary segments of very small lengths, h. Each segment has two nodes on either of its ends and these nodes are name 1,2,3,…..(i-1), i, (i+1),…….(n-2), (n-1), n. The vertical direction is now considered as w because in most of the formulations till now, w has been used for the deflection (in the vertical direction). So, the above equation can be re written as:

$$\frac{dw}{dx} = \frac{w_{i+1} - w_i}{h} \rightarrow (1a) \quad \{\text{forward difference}\}$$

w_{i+1} and w_i are the deflection values at the $(i+1)^{th}$ and i^{th} nodes respectively.

Now consider the Taylor series expansion for f(x-h):

$$f(x-h) = f(x) - \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \ldots\ldots\ldots \rightarrow (2)$$

Again neglecting the terms with h^2 and higher power:

$$f(x-h) = f(x) - hf'(x)$$

$$\Rightarrow f'(x) = \frac{f(x) - f(x-h)}{h}$$

$$\frac{dw}{dx} = \frac{w_i - w_{i-1}}{h} \rightarrow (2a) \text{ \{backward difference\}}$$

Adding the equations (1) and (2), we get, (1) + (2):

$$f(x+h) + f(x-h) = 2f(x) + 2\frac{h^2}{2!} f''(x) + 2\frac{h^4}{4!} f''''(x) + \ldots.$$

Neglecting the terms with h^4 and higher, we get:

$$f(x+h) + f(x-h) = 2f(x) + 2\frac{h^2}{2!} f''(x)$$

$$\Rightarrow f''(x) = \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$

$$\frac{d^2w}{dx^2} = \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} \rightarrow (3)$$

Subtracting the equation (2) from (1), we get, (1) - (2):

$$f(x+h) - f(x-h) = 2\frac{h}{1!} f'(x) + 2\frac{h^3}{3!} f'''(x) + \ldots\ldots$$

Neglecting the terms with h^3 and higher, we get:

$$\Rightarrow f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

$$\frac{dw}{dx} = \frac{w_{i+1} - w_{i-1}}{2h} \rightarrow (4) \text{ \{central difference\}}$$

We have seen 3 methods: the forward difference method or forward difference scheme, the backward difference scheme and the central difference scheme. In all the three methods, we have neglected some part of the expression. It is h^2 and higher for the forward difference and backward difference, but it is h^3 and higher for the central difference scheme. So, in the central difference scheme, we are neglecting the least value out of all three and hence it is the most accurate of all three methods.

For example, say, in a problem the h value is 0.1. So, h^2 will be 0.01 and h^2 will be 0.001. So, as the power increases, the value gets smaller and so, h^3 term is smaller than h^2. That means smaller value is being neglected in the central difference scheme as compared to the forward and backward difference schemes. So, the central difference scheme is always preferable as it will have less error compared to the other methods.

So, in most of the equations we are about to see, the central difference scheme will be used. But sometimes, when it is inevitable to use the forward difference or backward difference scheme, the central difference scheme will not be adopted. Otherwise we try to use central difference scheme always and always in a particular equation try to use the same technique for all the nodes.

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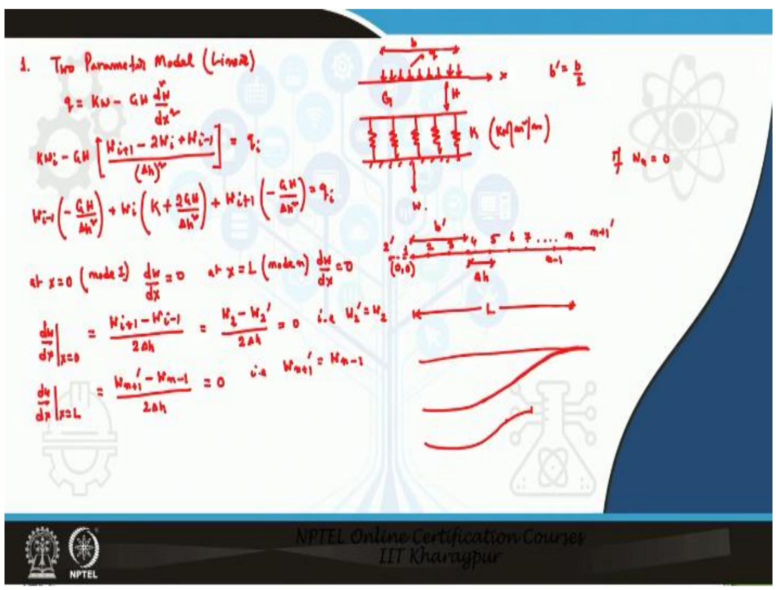

Now let us use these schemes in the actual soil structure interaction model. Consider a two parameter model with linear response. The loading condition, dimensions and the constants involved are all shown in the figure above. The equation for this case under plane strain condition is:

$$q = kw - GH\frac{d^2w}{dx^2}$$

where, H is the thickness of the shear layer, G is the shear modulus of the shear layer, k is the modulus of subgrade reaction of the springs.

As this particular case is a symmetric problem, it is sufficient to solve for half the portion of this. The other half would be the mirror image of this half. That means if the loaded region extends over a length of b, the solution will only be for b′ (where, b′ = b/2). The total length of the half model is L (from centre of the load to the end of the model is L) within which the loaded region is up to b′ and beyond the b′ up to L is the unloaded region. The length L is divided into number of segments with the help of n nodes.

Now, this basic governing differential equation will be applied to all the nodes because as mentioned earlier, these equations should be converted to a set of equations which can be solved to get the deflection and other quantities. For this purpose, first express this equation in central difference scheme.

$$kw_i - GH\left[\frac{w_{i+1} - 2w_i + w_{i-1}}{(\Delta h)^2}\right] = q_i$$

In the equation, the length of each segment or the distance between any two consecutive nodes is written as Δh to indicate that its value is very small. The above equation represents that it is used for the i^{th} node. If this equation is used for all the n nodes, n equations can be obtained and the number of unknowns will also be n which can be solved. But the problem is when using the equation for i^{th} node, there is a $(i-1)^{th}$ term. So, when this equation is used for the first node, a term pertaining to the node before the first is required to solve it. Similar problem appears when the equation is applied to the last or the n^{th} node.

Here the boundary conditions will play an important role. Remember that to obtain the closed form solution also, the boundary conditions were used to get the solution. So here, 2 boundary conditions are needed because the differential equation has the d^2w/dx^2 term and here the problem is also with 2 nodes. So, the 2 boundary conditions should be used on these 2 nodes.

First rewrite the equation in terms of coefficients of the deflection terms:

$$w_{i-1}\left[-\frac{GH}{(\Delta h)^2}\right]+w_i\left[k+\frac{2GH}{(\Delta h)^2}\right]+w_{i+1}\left[-\frac{GH}{(\Delta h)^2}\right]=q_i$$

Now let us write down the boundary conditions. As the loading condition is UDL, at the centre of the node or x = 0 or 1st node, slope will be 0:

$$\text{At x = 0: } \frac{dw}{dx}=0$$

The length of the model, L was considered sufficiently long such that the loading effects are not felt after some nodes or, L is too large to show deflections at the n^{th} node due to the loading on the beam.

$$\text{At x = l: } \frac{dw}{dx}=0$$

Now, using the central difference scheme for the first node:

$$\left.\frac{dw}{dx}\right|_{x=0}=\frac{w_{i+1}-w_{i-1}}{2(\Delta h)}=\frac{w_2-w_2'}{2(\Delta h)}=0 \text{ i.e., } w_2 = w_2'$$

To obtain the solution two imaginary nodes that do not exist are considered in the equations. Because of the two extra nodes, two extra unknowns will also add up making the number of unknowns to n+2. So, with the n equations for n nodes, (n+2) unknowns are to be solved. So for the extra two nodes, boundary conditions can be applied to find a solution.

So, now for (n + 2) number of unknowns, there are (n + 2) number of equations. A relationship will be established between these 2 imaginary unknowns to the real unknowns so that ultimately the number of unknowns will also be n and there are n number of equations.

The $(i\text{-}1)^{th}$ node when solving for the first node does not exist. So, an imaginary 2′ node is considered to complete the solution. As the slope at x = 0 is already known to be zero, it implies that w_2 and w_2' are equal. Similarly, while solving for the nth node also, a imaginary (n+1)′ node is assumed which is similar to the $(n\text{-}1)^{th}$ node in all ways.

$$\left.\frac{dw}{dx}\right|_{x=L}=\frac{w_{n+1}-w_{n-1}}{2(\Delta h)}=\frac{w_{n+1}'-w_{n-1}}{2(\Delta h)}=0 \text{ i.e., } w_{(n+1)}' = w_{(n-1)}$$

By using the two boundary conditions, relations were established between the imaginary and real unknowns. If the boundary condition of w = 0 at the nth node is also used, the (n+1)′ imaginary node itself need not be considered. The only purpose of the (n+1)′ imaginary node is to complete the solution for the n^{th} node. But, from the boundary condition if that value is already known, then the need for imaginary node here will not be there.

For example, consider that we have applied the boundary condition and formulated that $w_n = 0$. Then, while solving for the $(n-1)^{th}$ node, the w_n and w_{n-2} will only come into picture which are real nodes and can be solved for.

In the next class, I will form the n number of equations and then using those equations, I will discuss how to solve and get the w value. Thank you.