Soil Structure Interaction
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Lecture-44
Plates on Elastic Foundation (Contd.,)

In my last lecture I have shown you the governing differential equation and the expressions for bending moment and shear force for circular plate. Now, today, I will show you what happens if the loading condition and the end conditions of the plate are independent of θ or simply, the equation for axially symmetric case.

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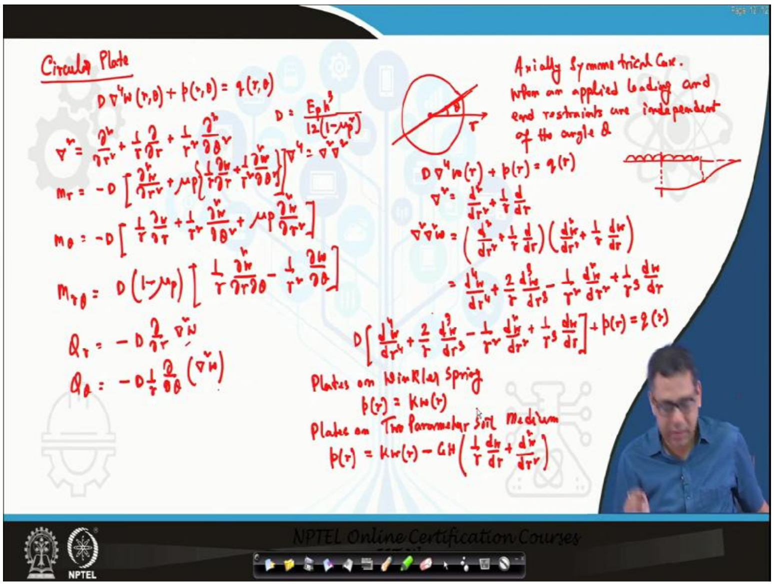

For example if a circular plate with free end condition is subjected to an UDL over its entire surface, then at any θ, the same boundary and loading conditions prevail. Also if concentrated loads are acting along the periphery of this free end plate, it can be called as an axially symmetrical case. In such case the del θ effect can be eliminated from the equations, simplifying the basic differential equation to:

$$D\nabla^4 w(r) + p(r) = q(r)$$

$$\text{where, } \nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$$

$$\nabla^4 w = \nabla^2\nabla^2 w = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)$$

$$\Rightarrow \nabla^4 w = \frac{d^4 w}{dr^4} + \frac{2}{r}\frac{d^3 w}{dr^3} - \frac{1}{r^2}\frac{d^2 w}{dr^2} + \frac{1}{r^3}\frac{dw}{dr}$$

Substituting this value of $\nabla^4 w$ in the differential equation, we get:

$$D\left[\frac{d^4w}{dr^4}+\frac{2}{r}\frac{d^3w}{dr^3}-\frac{1}{r^2}\frac{d^2w}{dr^2}+\frac{1}{r^3}\frac{dw}{dr}\right]+p(r)=q(r)$$

In the above expression, p(r) refers to the reaction from the springs or soil medium upon which the plate is resting. So, its value depends upon the type of medium the plate rests on and also the properties associated. If the plate rests on Winkler springs:

$$p(r)=kw(r)$$

If the plate rests on two parameter soil medium:

$$p(r)=kw(r)-GH\left(\frac{1}{r}\frac{dw}{dr}+\frac{d^2w}{dr^2}\right)$$

The expression for p(r) was derived in one of the previous lectures, may be lecture 16. The solution technique for the circular plate under an axially symmetrical condition is similar to that of the beam. Consider a section of the plate where the loading condition and edge condition would come into picture just like the way followed in beam. The procedure to determine the expressions for deflection below the loaded region and beyond that region are also same but, only the expressions are different.

When using the finite difference technique, the solution techniques for the beam and the circular plate under axially symmetrical condition are similar. That similarity will be shown. Here, the expressions for moment and the shear force under axially symmetrical condition are also to be given.

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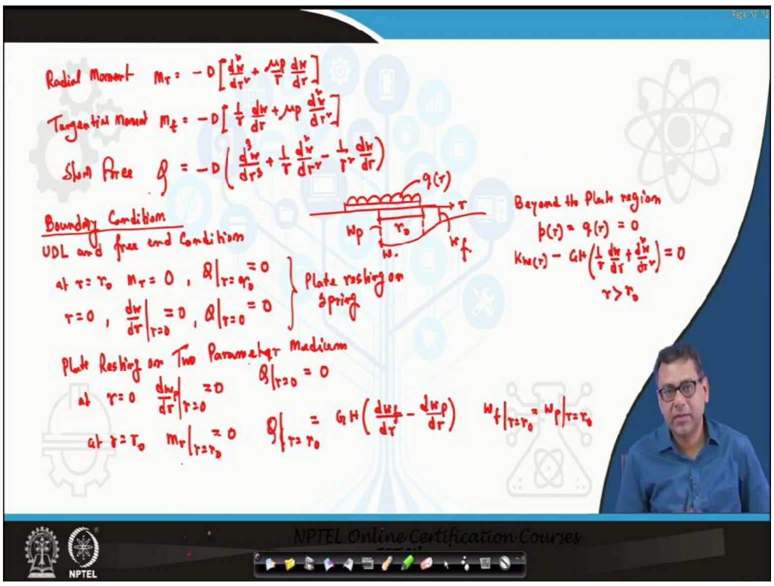

The expression for radial moment is:

$$M_r = -D\left[\frac{d^2w}{dr^2} + \frac{\mu_p}{r}\frac{dw}{dr}\right]$$

The expression for radial moment is:

$$M_t = -D\left[\frac{1}{r}\frac{dw}{dr} + \mu_p \frac{d^2w}{dr^2}\right]$$

The expression for shear force:

$$Q = -D\left(\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}\right)$$

To solve these, the boundary conditions should be considered. In the first case, consider that the UDL is uniformly distributed over the entire plate area and the end condition is free. The radius of the plate is considered as r_0, r represents the radial distance of any point considered and w indicates deflection. So, if $r = r_0$, the considered point is at the end or edge of the plate. Since this is a free end condition, the radial moment, M_r and the shear force, Q_r will be 0. Remember that the shear force at the edge of the plate will be 0 only if the plate is supported by Winkler springs. At r = 0, i.e., at the centre of the plate, the slope will be 0 because it is the point of maximum deflection for this loading condition.

The boundary conditions for a plate with free end condition resting on Winkler springs subjected to a UDL all over its area are:

At $r = r_0$: $\quad M_r\big|_{r=r_0} = 0,\ Q_r\big|_{r=r_0} = 0,$

At r = 0: $\quad \left.\frac{dw}{dr}\right|_{r=0} = 0,\ Q_r\big|_{r=0} = 0$

(shear force will be 0 only in case of Winkler springs)

If the plate rests on a two parameter soil medium:

At $r = r_0$: $\quad M_r\big|_{r=r_0} = 0,\ Q_r\big|_{r=r_0} = GH\left(\frac{dw_f}{dr} - \frac{dw_p}{dr}\right)$

$$\left\{ w_f\big|_{r=r_0} = w_p\big|_{r=r_0} \right\}$$

where, w_p is the deflection of the plate and w_f is the deflection beyond the loaded region

At r = 0: $\quad \left.\frac{dw}{dr}\right|_{r=0} = 0,\ Q_r\big|_{r=0} = 0$

At the centre (r = 0), shear force is 0 because w_f do not exist at the centre as it is within the loaded region and so, $dw_f/dr = 0$. The other term of the expression, dw_p/dr is also zero as the deflection is maximum at this point and the slope just changes its orientation i.e., slope is flat or zero.

So, I will show you how we can use these boundary conditions to solve these beam equation or determine the w slope, moment, shear force at any point within the beam by using finite difference technique, but these are the boundary conditions.

Now, the expression for the region beyond the loaded region is:

$$p(r) - q(r) = 0$$

$$kw(r) - GH\left(\frac{1}{r}\frac{dw}{dr} + \frac{d^2w}{dr^2}\right) = 0$$

This is the equation for the region beyond load or where $r > r_0$. Now, there are 2 equations just like the beam problem along with the boundary conditions.

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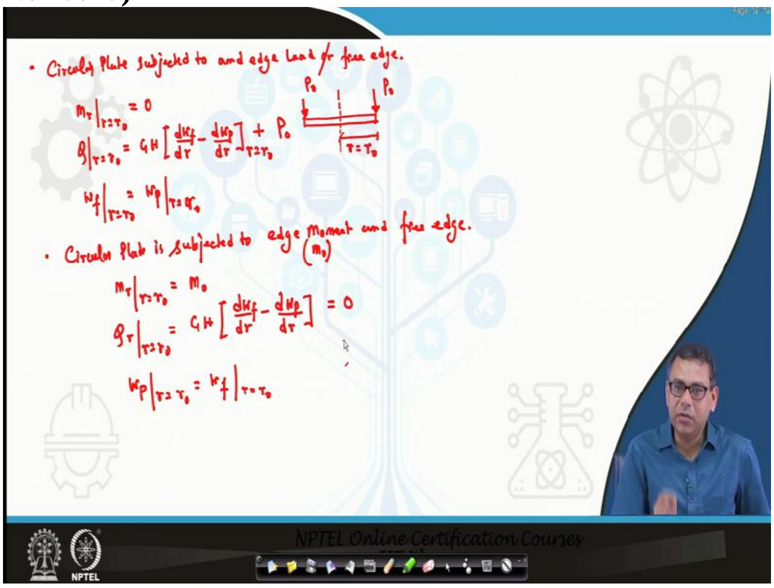

Now, if the loading condition is changed, let us see what happens. Consider that the circular plate with free end is now subjected to concentrated loads along its periphery. The boundary conditions for this case will be:

$$M_r\big|_{r=r_0} = 0$$

$$Q_r\big|_{r=r_0} = GH\left[\frac{dw_f}{dr} - \frac{dw_p}{dr}\right]_{r=r_0} + P_O$$

$$w_f\big|_{r=r_0} = w_p\big|_{r=r_0}$$

These are the edge boundary conditions if the loading conditions are changed.

Similarly if there is an edge moment acting at the free end of the plate, the boundary conditions will be:

$$M_r\big|_{r=r_0} = M_O$$

$$Q_r\big|_{r=r_0} = GH\left[\frac{dw_f}{dr} - \frac{dw_p}{dr}\right]_{r=r_0} = 0$$

$$w_f\big|_{r=r_0} = w_p\big|_{r=r_0}$$

When there is no UDL acting on a beam, or plate, the q term in the governing differential equation can be replaced with 0 to get the close form solution. Then, in the boundary condition, the effect of concentrated load will come into the picture.

If the end condition or edge condition changes, automatically these boundary conditions will also change. The boundary conditions for a fixed edge and also for a hinged edge in a plate problem will also be discussed further. So, those boundary conditions can be used when the edge condition changes.

In the beam problem, the closed form solution for infinite beam, semi infinite beam and beam with finite length were discussed. Closed form solution means a readymade solution that can be obtained from an expression. A solution or value can be obtained by mere substitution. So, it is always preferable to have a closed form solution.
Similarly, in the plate problem also closed form solutions can be obtained, but those are very complicated. So, in the plate problem no closed form solutions will be discussed. Instead, the numerical methods to solve these problems will be discussed. Basically the method to solve the plate problems using finite difference technique will be discussed. In other words discussion will be about how to solve these problems numerically.

Even in the beam problem, all the closed form solutions were discussed only for the linear springs. But if the spring is nonlinear, rather the soil response is nonlinear, it will be difficult to get the closed form solutions no matter the beam rests on Winkler springs or a two parameter soil medium. So, the finite difference techniques will be discussed for the beam problem also because it will be difficult in case of nonlinear response.

But the problem for numerical techniques is that a program should be developed on our own and only then, the solution can be obtained. So, in the next class, I will discuss how to solve these problems numerically.

Few problems I have already discussed the closed form solution, but few problems like the plate problem, I have not discussed the closed form solution. But I will discuss about the numerical solutions to all the problems (both beam and plate problems) because it is difficult to get closed form solutions even for the beam problems if the response is nonlinear.

But remember that I will not discuss any programming part, but I will just show the steps of how to solve, how to apply the boundary conditions, how to apply the finite difference techniques in the governing differential equation to get the solution. I will show all these cases in detail in the next class. Thank you.