**Soil Structure Interaction**
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**Lecture-43**
**Plates on Elastic Foundation (Contd.)**

In my previous lecture I have derived the basic differential or the governing differential equation for plate resting on Winkler spring and plate resting on 2 parameter soil medium. I have also derived the equation for the moment $M_x$, $M_y$ & $M_{xy}$ and shear forces $Q_x$ & $Q_y$. Today I will show the boundary conditions to solve these equations when the plate is subjected to UDL or the plate surface is subjected to UDL and the plate boundary end conditions are free.

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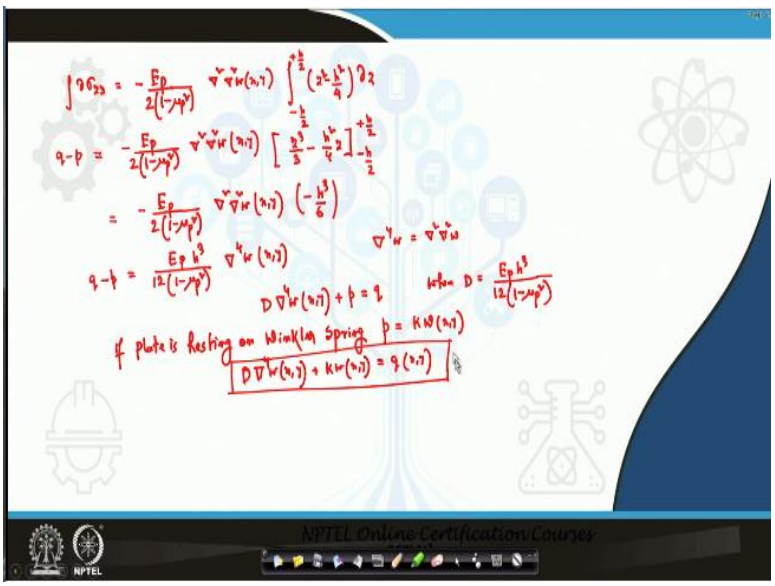


The basic differential equation for the plate resting on Winkler springs is shown in the above slide.

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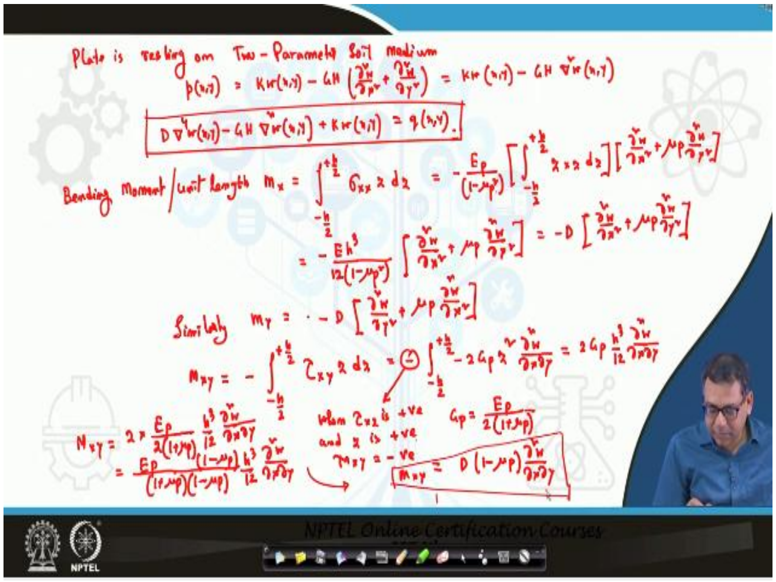


Similarly, the governing differential equation for the plate resting on 2 parameter soil medium is shown in the slide above. The expressions for moments $M_x$, $M_y$ & $M_{xy}$ are also shown in the slide above where the expressions for shear forces are given in the slide below.

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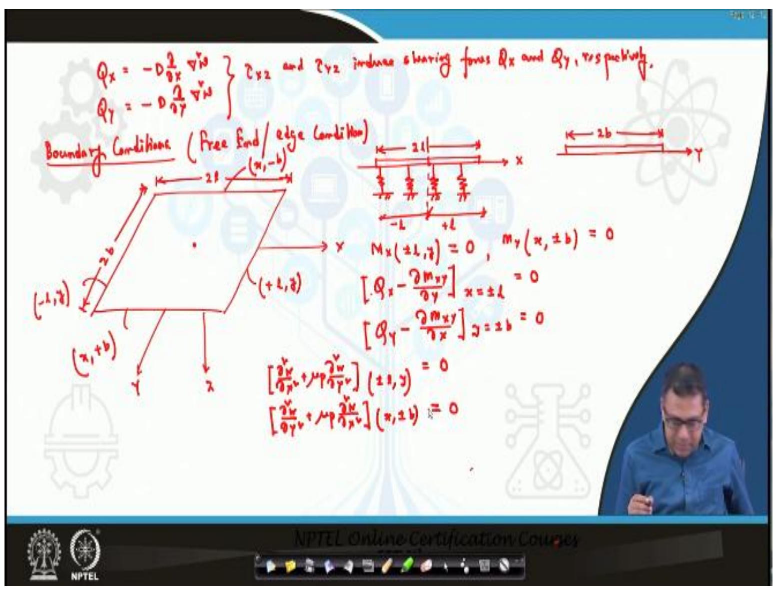


Now let us see how many boundary conditions are required to obtain the required solutions. Basically, 4 boundary conditions in two directions are required as the differential equation has term with power 4.

Consider that the dimension of beam in the X direction is 2l, and that of in Y direction is 2b. A UDL, q is acting over the entire surface of the plate and the direction of the plate thickness is the

Z axis. Two sections of the plate were considered and drawn in the slide above, one showing the 2l dimension and the other, 2b dimension. In the section showing 2l dimension, x = 0 is considered at the centre of the plate section and so the half towards the positive X axis is considered +l and that towards the negative X axis is considered –l.

So, co-ordinates of the edge towards positive X axis will be (+l, y) and the edge towards negative X axis will be (-l, y). As the plate is freely resting on the soil, the moment at the edges will be zero. So:

$$M_x(\pm l, y) = 0$$

Similarly, $M_y(x, \pm b) = 0$

For a free end, shear force will also be zero in addition to the moment.

$$\left[Q_x - \frac{\partial M_{xy}}{\partial y}\right]_{(x=\pm l)} = 0$$

$$\left[Q_y - \frac{\partial M_{xy}}{\partial x}\right]_{(y=\pm b)} = 0$$

These are the 8 boundary conditions that can be used to solve the differential equations. Consider the first equation showing the boundary conditions of $M_x$:

$$\left[\frac{\partial^2 w}{\partial x^2} + \mu_p \frac{\partial^2 w}{\partial y^2}\right]_{(\pm l, y)} = 0 \rightarrow 1,2$$

Consider the second equation showing the boundary conditions of $M_y$:

$$\left[\frac{\partial^2 w}{\partial y^2} + \mu_p \frac{\partial^2 w}{\partial x^2}\right]_{(x, \pm b)} = 0 \rightarrow 3,4$$

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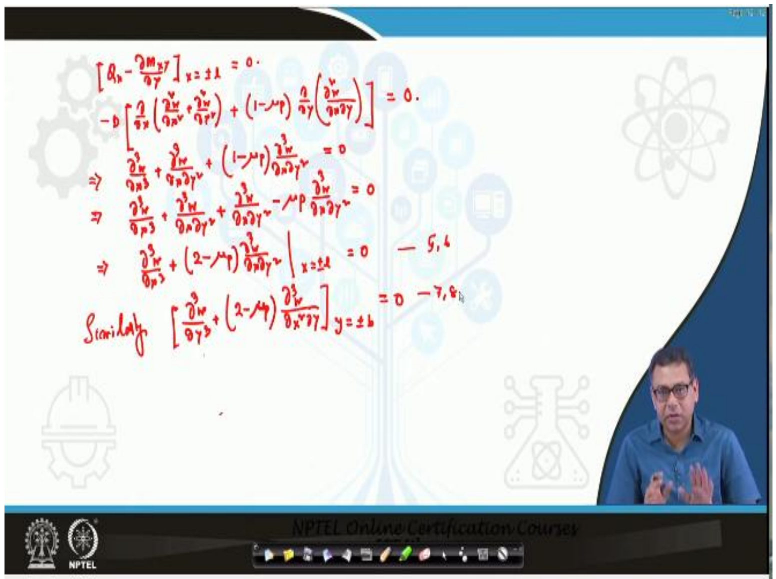


Now consider the boundary condition where the shear force, $Q_x$ is 0.

$$\left[ Q_x - \frac{\partial M_{xy}}{\partial y} \right]_{(x=\pm l)} = 0$$

$$\Rightarrow -D\left[ \frac{\partial}{\partial x}\left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + (1-\mu_p)\frac{\partial}{\partial y}\left( \frac{\partial^2 w}{\partial x \partial y} \right) \right] = 0$$

$$\Rightarrow \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} + (1-\mu_p)\frac{\partial^3 w}{\partial x \partial y^2} = 0$$

$$\Rightarrow \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} + \frac{\partial^3 w}{\partial x \partial y^2} - \mu_p \frac{\partial^3 w}{\partial x \partial y^2} = 0$$

$$\Rightarrow \frac{\partial^3 w}{\partial x^3} + (2-\mu_p)\frac{\partial^3 w}{\partial x \partial y^2}\bigg|_{x=\pm l} = 0 \rightarrow 5,6$$

$$\text{Similarly, } \Rightarrow \frac{\partial^3 w}{\partial y^3} + (2-\mu_p)\frac{\partial^3 w}{\partial x^2 \partial y}\bigg|_{y=\pm b} = 0 \rightarrow 7,8$$

These are the 8 boundary conditions that can be used to solve this problem. Remember that these are the boundary conditions only when a UDL is acting on the entire plate and the edge condition is free. For a different loading condition or a different end condition of the plate, the boundary conditions also change.

Here, only the boundary conditions and the governing differential equations will be showed. The solution for this problem, how to solve these equations using the boundary conditions and to determine the deflection and other quantities will be discussed when discussing the numerical techniques.

That means the solutions will be determined numerically using finite difference technique and will be discussed when dealing with the numerical part. For now, the solution will be given directly.

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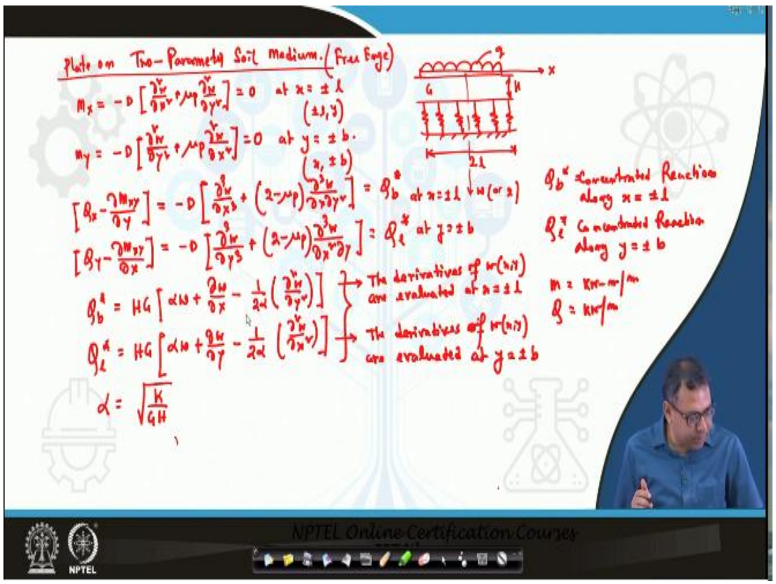


The conditions, for which the governing differential equations and boundary conditions are to be formulated, are shown in the figure in the slide above. A plate of dimension 2l in x direction rests on a shear layer of thickness H and shear modulus G. This shear layer is underlain by a soft or weak soil layer represented by springs. A UDL, q is acting all over the plate.

Similar to the previous model, the bending moments $M_x$ & $M_y$ will be zero at the edges of the plate in this case too as the end condition is free.

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \mu_p \frac{\partial^2 w}{\partial y^2}\right] = 0 \qquad \text{at } x = \pm l;\ (\pm l, y)$$

$$M_y = -D\left[\frac{\partial^2 w}{\partial y^2} + \mu_p \frac{\partial^2 w}{\partial x^2}\right] = 0 \qquad \text{at } y = \pm b;\ (x, \pm b)$$

Unlike the previous case, here the shear forces $Q_x$ & $Q_y$ will not be zero even at the edges because there will be some shear force induced due to the shear layer. $Q_b^*$ is nothing but the concentrated reaction along x = ±l whereas $Q_l^*$ is the concentrated reaction along y = ±b from the shear layer. $Q_b^*$ and $Q_l^*$ can be calculated by the following expressions:

$$\left[Q_x - \frac{\partial M_{xy}}{\partial y}\right] = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu_p)\frac{\partial^3 w}{\partial x \partial y^2}\right] = Q_b^* \qquad \text{at } x = \pm l$$

$$\left[Q_y - \frac{\partial M_{xy}}{\partial x}\right] = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\mu_p)\frac{\partial^3 w}{\partial x^2 \partial y}\right] = Q_l^* \qquad \text{at } y = \pm b$$

In case of the plate resting on Winkler spring, this expression or the $Q_b^*$ and $Q_l^*$ were 0, but here due to the shear layer, they have a specific value which can be determined by the above expressions. Remember that the zone for deflection is considered to be within the plate region only. Till now the region beyond the plate has not been considered. This can also be considered in a similar way to that of the beam problem.

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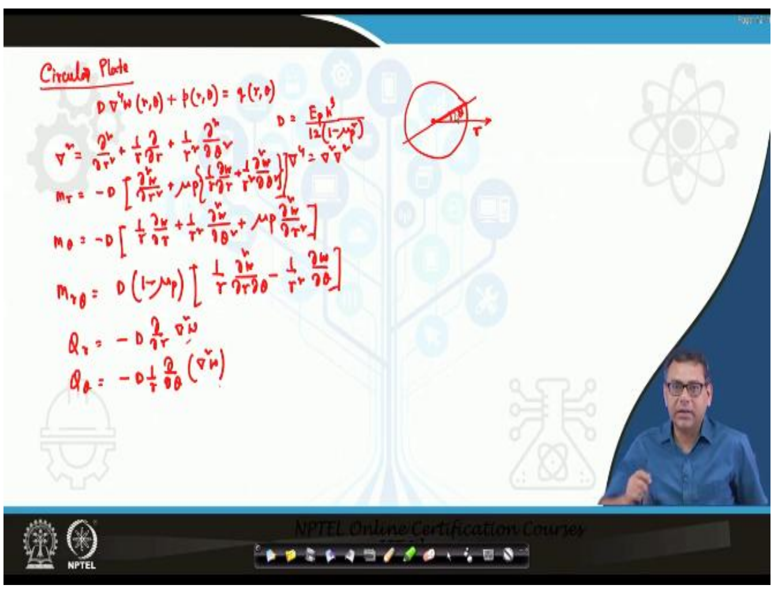


Now let us get started with the circular plate by looking into the governing differential equation and the boundary conditions required to solve it. The governing differential equation for the circular plate is:

$$D\nabla^4 w(x,\theta) + p(x,\theta) = q(r,\theta)$$

$$\text{where, } \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial r^2};\ D = \frac{E_p h^3}{12(1-\mu_p^2)} \text{ and } \nabla^4 = \nabla^2\nabla^2$$

($E_p$ is the elastic modulus of the plate; $\mu_p$ is the Poisson's ratio of the plate; h is the thickness of the plate)

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \mu_p\left\{\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\right\}\right]$$

$$M_\theta = -D\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \mu_p\frac{\partial^2 w}{\partial r^2}\right]$$

$$M_{r\theta} = D(1-\mu_p)\left[\frac{1}{r}\frac{\partial^2 w}{\partial r\partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right]$$

$$Q_r = -D\frac{\partial}{\partial r}\nabla^2 w$$

$$Q_\theta = -D\frac{1}{r}\frac{\partial}{\partial \theta}\left(\nabla^2 w\right)$$

Note one thing here that if the loading condition and the end condition are such that they are independent of the angle, then the θ terms can be eliminated further simplifying the equations. It means that irrespective of any θ value considered, the same loading condition and boundary condition will be obtained.

In the next class I will first derive that equation from these equations. If it is a symmetrical case where the loading and boundary conditions are independent of θ, I will show what the boundary conditions for different end conditions will be and how to solve those equations. Thank you.