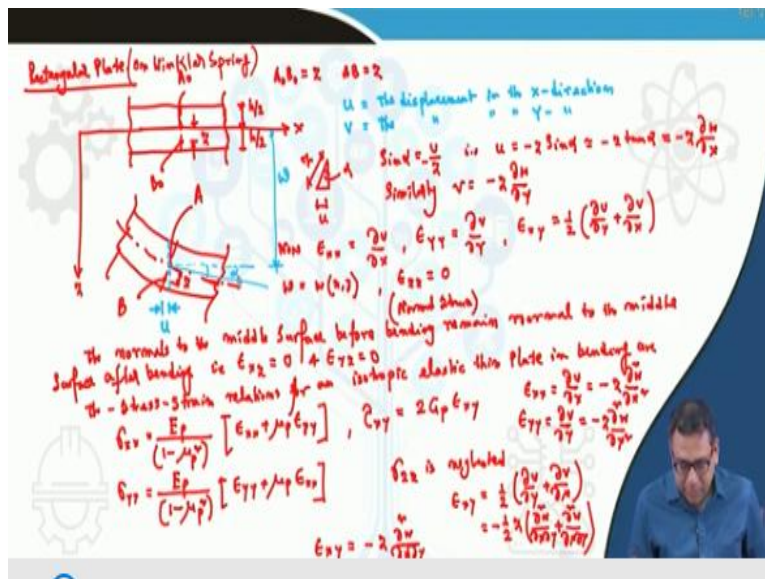


Soil Structure Interaction
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Lecture - 42
Plates on Elastic Foundation (contd.)

In my previous lecture I have discussed about the plates resting on elastic foundation and I discussed about plates resting on winkler spring.

(Refer Slide Time 00:37)



The same topic, rectangular plate resting on the winkler spring will be continued in this class. A section of the plate in xz plane was considered and then the deformation, w was formulated. The shape of the plate section after deformation is shown in the figure and α value can be obtained by taking a tangent to the middle surface at A.

(Refer Slide Time 01:39)

$\sigma_{xx} = \frac{2E\mu}{(1-\mu^2)} \left[\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right]$
 $\sigma_{yy} = \frac{2E\mu}{(1-\mu^2)} \left[\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right]$
 $\tau_{xy} = -2G\mu \frac{\partial^2 w}{\partial x \partial y}$

Considering Equation of equilibrium

$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} = 0 \quad \text{--- (1)}$
 $\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yx}}{\partial y} = 0 \quad \text{--- (2)}$
 $\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad \text{--- (3)}$

From Eq (1) $\frac{\partial \sigma_{xx}}{\partial x} = - \left(\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} \right)$

$\frac{\partial \sigma_{xx}}{\partial x} = - \left[\frac{2E\mu}{(1-\mu^2)} \left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right) + (1-\mu^2) \frac{\partial}{\partial y} \left(\frac{\partial w}{\partial x \partial y} \right) \right]$
 $= \frac{2E\mu}{(1-\mu^2)} \left[\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x \partial y} - \mu \frac{\partial^2 w}{\partial x \partial y} \right]$
 $= \frac{2E\mu}{(1-\mu^2)} \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \right)$
 $= \frac{2E\mu}{(1-\mu^2)} \frac{\partial}{\partial x} \nabla w$ where $\nabla = \text{Laplace operator} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Then from the stress strain relationship, by considering equilibrium equations, the normal stresses and shear stresses were calculated.

(Refer Slide Time: 01:48)

Similarly from eq (2) $\frac{\partial \tau_{yz}}{\partial z} = \frac{2E\mu}{(1-\mu^2)} \frac{\partial}{\partial y} \nabla w$

After integrating $\tau_{yz} = \left(\frac{2E\mu}{(1-\mu^2)} \right) \frac{\partial w}{\partial y} + C$ } C is the Constant of integration

$\tau_{xz} = \left(\frac{2E\mu}{(1-\mu^2)} \right) \frac{\partial w}{\partial x} + C$
 $\tau_{yz} = \left(\frac{2E\mu}{(1-\mu^2)} \right) \frac{\partial w}{\partial y} + C$

$x = \pm \frac{h}{2}$ $\tau_{xz}(x) = 0$, $\tau_{yz}(x) = 0$
 $\therefore \frac{\partial w}{\partial x} + C = 0$ $\therefore C = -\frac{\partial w}{\partial x} = -\left(\frac{\partial w}{\partial x}\right)$
 $= -\frac{1}{2} \frac{\partial w}{\partial x}$

Diagram showing a plate of thickness h with a uniformly distributed load $q(x,y)$ acting downwards and reaction $p(x,y)$ acting upwards. The x and z axes are shown.

Basically the first and second equations were used to determine the shear stress in xz plane and yz plane. Consider that a UDL of intensity $q(x,y)$ is acting over the entire region of the plate. A reaction, p will develop beneath the plate because of the soil or springs upon which the plate resting. Now consider a small segment of this plate showing the total thickness of the plate, h . In the considered section, the x axis passes through the middle surface of the plate and hence the plate is divided into two halves. As the vertical direction is z axis, we can write:

$$z = \pm \frac{h}{2}$$

At the top of the plate ($z = + h/2$), UDL q is acting and at the bottom of the plate ($z = - h/2$), the reaction from the springs, p is acting, So the shear stresses on these 2 planes will be 0.

$$\tau_{xz}(x, y) = \tau_{yz}(x, y) = 0 \text{ at } z = \pm \frac{h}{2}$$

$$\tau_{xz} = \left(\frac{z^2}{2} + c \right) \frac{E_p}{(1 - \mu_p^2)} \frac{\partial}{\partial x} \nabla^2 w$$

$$\tau_{yz} = \left(\frac{z^2}{2} + c \right) \frac{E_p}{(1 - \mu_p^2)} \frac{\partial}{\partial y} \nabla^2 w$$

From the above two equations, it is evident that if τ_{xz} and τ_{yz} are zero, then $\left(\frac{z^2}{2} + c \right)$ must be

zero:

$$\left(\frac{z^2}{2} + c \right) = 0 \Rightarrow c = -\frac{z^2}{2}$$

$$\Rightarrow c = -\frac{\left(\frac{h}{2} \right)^2}{2} = -\frac{h^2}{8}$$

By substituting this c value, the shear stresses τ_{xz} and τ_{yz} can be calculated.

(Refer Slide Time 07:20)

Considering Equation of equilibrium

$$\sigma_{yy} = -\frac{2E_p}{(1-\mu_p^2)} \left[\frac{\partial^2 w}{\partial y^2} + \mu_p \frac{\partial^2 w}{\partial x^2} \right]$$

$$\sigma_{xy} = -\frac{2E_p}{(1-\mu_p^2)} \left[\frac{\partial^2 w}{\partial x \partial y} + \mu_p \frac{\partial^2 w}{\partial y^2} \right]$$

$$\tau_{xy} = -2G_p \frac{\partial w}{\partial x}$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \quad \text{--- (1)}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \quad \text{--- (2)}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \quad \text{--- (3)}$$

From Eq (1) $\frac{\partial \sigma_{xx}}{\partial x} = - \left(\frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right)$

$$= - \frac{2E_p}{(1-\mu_p^2)} \left[\frac{\partial}{\partial x} \left(\frac{\partial^2 w}{\partial x \partial y} + \mu_p \frac{\partial^2 w}{\partial y^2} \right) + (1-\mu_p) \frac{\partial}{\partial y} \left(\frac{\partial^2 w}{\partial x \partial y} \right) \right]$$

$$= \frac{2E_p}{(1-\mu_p^2)} \left[\frac{\partial^2 w}{\partial x^2} + \mu_p \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y} - \mu_p \frac{\partial^2 w}{\partial y^2} \right]$$

$$= \frac{2E_p}{(1-\mu_p^2)} \frac{\partial}{\partial x} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)$$

$$= \frac{2E_p}{(1-\mu_p^2)} \frac{\partial}{\partial x} \nabla^2 w$$

where $\nabla^2 = \text{Laplace operator} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Now from the third equation of equilibrium, we get:

(Refer Slide Time: 07:26)

From 3rd Equation

$$\frac{\partial \sigma_{xz}}{\partial x} = - \left(\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right)$$

$$= - \left(\frac{\partial}{\partial y} \left[\frac{2E_p}{(1-\mu_p^2)} \left(\frac{\partial^2 w}{\partial x \partial y} + \mu_p \frac{\partial^2 w}{\partial y^2} \right) \right] + \frac{\partial}{\partial z} \left[\frac{2E_p}{(1-\mu_p^2)} \left(\frac{\partial^2 w}{\partial x \partial y} + \mu_p \frac{\partial^2 w}{\partial y^2} \right) \right] \right) \frac{\partial w}{\partial x}$$

$$= - \frac{1}{2} \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \right) \frac{E_p}{(1-\mu_p^2)} \left[\frac{\partial}{\partial y} \left(\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y} \right) \right]$$

$$= - \frac{1}{2} \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \right) \frac{E_p}{(1-\mu_p^2)} \left[\frac{\partial^2 w}{\partial x^2} + 2 \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} \right]$$

$$= - \frac{E_p}{2(1-\mu_p^2)} \left\{ \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \right\} \nabla^2 w(x,y)$$

where $\sigma_{xz} \left(+\frac{1}{2}, x, z \right) = -p(x,z)$
 $\sigma_{xz} \left(-\frac{1}{2}, x, z \right) = -q(x,z)$

$$\frac{\partial \sigma_{zz}}{\partial z} = - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} \right)$$

$$\Rightarrow \frac{\partial \sigma_{zz}}{\partial z} = - \left(\frac{z^2}{2} - \frac{h^2}{8} \right) \frac{E_p}{(1-\mu_p^2)} \left[\frac{\partial}{\partial x} \left\{ \frac{\partial}{\partial x} \nabla^2 w \right\} + \frac{\partial}{\partial y} \left\{ \frac{\partial}{\partial y} \nabla^2 w \right\} \right]$$

$$\therefore \nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

$$\begin{aligned} \Rightarrow \frac{\partial \sigma_{zz}}{\partial z} &= -\frac{1}{2} \left(z^2 - \frac{h^2}{4} \right) \frac{E_p}{(1-\mu_p^2)} \left[\frac{\partial}{\partial x} \left\{ \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right\} + \frac{\partial}{\partial y} \left\{ \frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} \right\} \right] \\ \Rightarrow \frac{\partial \sigma_{zz}}{\partial z} &= -\frac{1}{2} \left(z^2 - \frac{h^2}{4} \right) \frac{E_p}{(1-\mu_p^2)} \left[\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] \\ \Rightarrow \frac{\partial \sigma_{zz}}{\partial z} &= -\frac{1}{2} \left(z^2 - \frac{h^2}{4} \right) \frac{E_p}{(1-\mu_p^2)} \left[\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] \\ \therefore \frac{\partial \sigma_{zz}}{\partial z} &= -\frac{E_p}{2(1-\mu_p^2)} \left\{ z^2 - \frac{h^2}{4} \right\} \nabla^2 \nabla^2 w(x, y) \end{aligned}$$

$$\text{Now, } \sigma_{zz} \left(+\frac{h}{2}, x, y \right) = -p(x, y)$$

$$\sigma_{zz} \left(-\frac{h}{2}, x, y \right) = -q(x, y)$$

As the z axis is taken positive in the downward direction, the z value of +h/2 indicates the bottom plane on which the spring reaction acts upon. Similarly z = -h/2, indicates the top plane upon which the UDL, q is acting. At that bottom face the shear stress is 0 but a normal stress of p, the reaction force from springs acts upon.

(Refer Slide Time 14:38)

Handwritten derivation on a whiteboard:

$$\int \frac{\partial \sigma_{zz}}{\partial z} dz = -\frac{E_p}{2(1-\mu_p^2)} \nabla^2 \nabla^2 w(x, y) \int \left(z^2 - \frac{h^2}{4} \right) dz$$

$$q - p = -\frac{E_p}{2(1-\mu_p^2)} \nabla^2 \nabla^2 w(x, y) \left[\frac{z^3}{3} - \frac{h^2 z}{4} \right]_{-\frac{h}{2}}^{+\frac{h}{2}}$$

$$= -\frac{E_p}{2(1-\mu_p^2)} \nabla^2 \nabla^2 w(x, y) \left(-\frac{h^3}{6} \right)$$

$$q - p = \frac{E_p h^3}{12(1-\mu_p^2)} \nabla^4 w(x, y) \quad \nabla^4 w = \nabla^2 \nabla^2 w$$

if plate is resting on Winkler Spring $p = k w(x, y)$

$$\boxed{D \nabla^4 w(x, y) + k w(x, y) = q(x, y)}$$

where $D = \frac{E_p h^3}{12(1-\mu_p^2)}$

If $\frac{\partial \sigma_{zz}}{\partial z}$ is integrated:

$$\int \partial \sigma_{zz} = -\frac{E_p}{2(1-\mu_p^2)} \nabla^2 \nabla^2 w(x, y) \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(z^2 - \frac{h^2}{4} \right) \partial z$$

$$\Rightarrow q - p = -\frac{E_p}{2(1-\mu_p^2)} \nabla^2 \nabla^2 w(x, y) \left[\frac{z^3}{3} - \frac{h^2}{4} z \right]_{-\frac{h}{2}}^{\frac{h}{2}}$$

$$\Rightarrow q - p = -\frac{E_p}{2(1-\mu_p^2)} \nabla^2 \nabla^2 w(x, y) \left(-\frac{h^3}{6} \right)$$

$$\Rightarrow q - p = \frac{E_p h^3}{12(1-\mu_p^2)} \nabla^4 w(x, y)$$

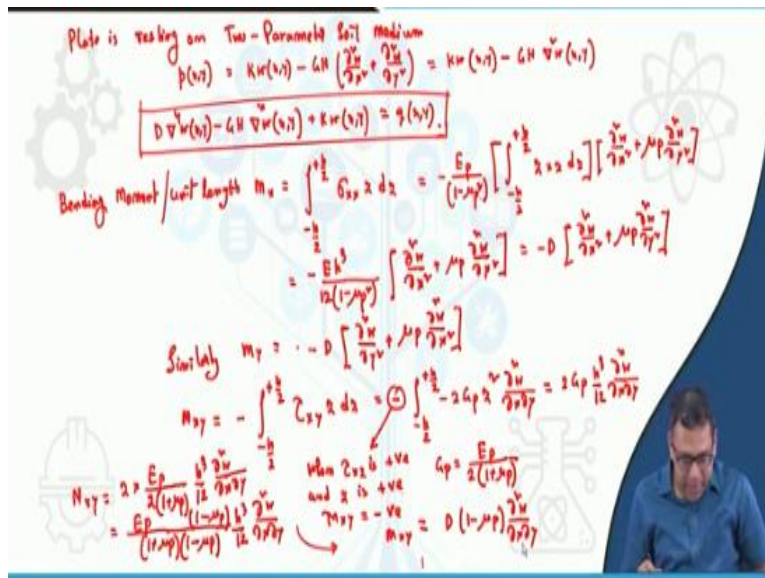
$$\Rightarrow D \nabla^4 w(x, y) + p = q; \quad \text{where, } D = \frac{E_p h^3}{12(1-\mu_p^2)}$$

If the plate is resting on Winkler springs, $p = k \times w(x, y)$

$$\therefore D \nabla^4 w(x, y) + k w(x, y) = q(x, y)$$

This is the equation for a beam resting on Winkler springs.

(Refer Slide Time 19:24)



Similarly if the plate is resting on a 2 parameter soil medium, then the equation would be:

$$p(x, y) = k w(x, y) - GH \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = k w(x, y) - GH \nabla^2 w(x, y)$$

$$\Rightarrow D \nabla^4 w(x, y) - GH \nabla^2 w(x, y) + k w(x, y) = q(x, y)$$

$$\text{Bending Moment / Unit length, } M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} z dz$$

The stress in the small segment dz is multiplied with the lever arm, z to get the moment M_x .

Substituting the value of σ_{xx} from equation-(a) {Ref. last class} in the above equation, we get:

$$\Rightarrow M_x = -\frac{E_p}{(1-\mu_p^2)} \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} z \times z dz \right] \left[\frac{\partial^2 w}{\partial x^2} + \mu_p \frac{\partial^2 w}{\partial y^2} \right]$$

$$\Rightarrow M_x = -\frac{Eh^3}{12(1-\mu_p^2)} \left[\frac{\partial^2 w}{\partial x^2} + \mu_p \frac{\partial^2 w}{\partial y^2} \right] = -D \left[\frac{\partial^2 w}{\partial x^2} + \mu_p \frac{\partial^2 w}{\partial y^2} \right]$$

Similarly, $M_y = -D \left[\frac{\partial^2 w}{\partial y^2} + \mu_p \frac{\partial^2 w}{\partial x^2} \right]$

$$\Rightarrow M_{xy} = -\left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} \times z dz \right] = -\int_{-\frac{h}{2}}^{\frac{h}{2}} -2G_p \times z^2 \frac{\partial^2 w}{\partial x \partial y} = 2G_p \frac{h^3}{12} \frac{\partial^2 w}{\partial x \partial y}$$

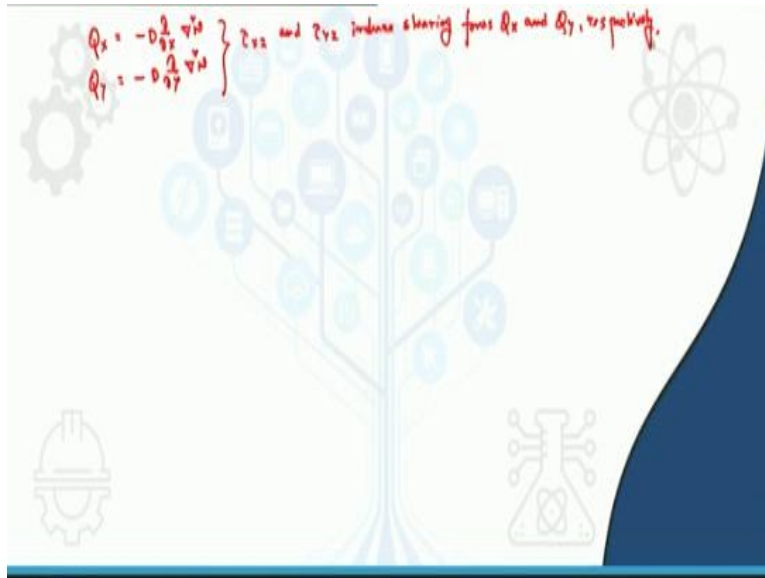
Substituting $G_p = \frac{E_p}{(1+\mu_p)}$ in the above equation:

$$\Rightarrow M_{xy} = 2 \times \frac{E_p}{(1+\mu_p)} \times \frac{h^3}{12} \times \frac{\partial^2 w}{\partial x \partial y}$$

$$\Rightarrow M_{xy} = 2 \times \frac{E_p(1-\mu_p)}{(1+\mu_p)(1-\mu_p)} \times \frac{h^3}{12} \times \frac{\partial^2 w}{\partial x \partial y}$$

$$\Rightarrow M_{xy} = D(1-\mu_p) \frac{\partial^2 w}{\partial x \partial y}$$

(Refer Slide time 28:34)



Finally, the expressions for shear force, Q_x and Q_y can be given by:

$$Q_x = -D \frac{\partial}{\partial x} \nabla^2 w$$

$$Q_y = -D \frac{\partial}{\partial y} \nabla^2 w$$

Q_x and Q_y are the shear forces that are induced due to τ_{xz} and τ_{yz} .

I have derived the basic or governing differential equation for a beam resting on spring and then beam resting on 2 parameter soil medium. Then, I have derived the moments M_x , M_y and M_{xy} and shear forces induced Q_x and Q_y .

In the next class, I will show you the boundary conditions to solve this basic differential or governing differential equation when plate is resting on spring and also on 2 parameters soil medium. Thank you.