Soil Structure Interaction

Prof. Kousik Deb
Department of Civil Engineering
Indian Institute of Technology Kharagpur

Lecture - 42
Plates on Elastic Foundation (contd.)

In my previous lecture I have discussed about the plates resting on elastic foundation and I discussed about plates resting on winkler spring.

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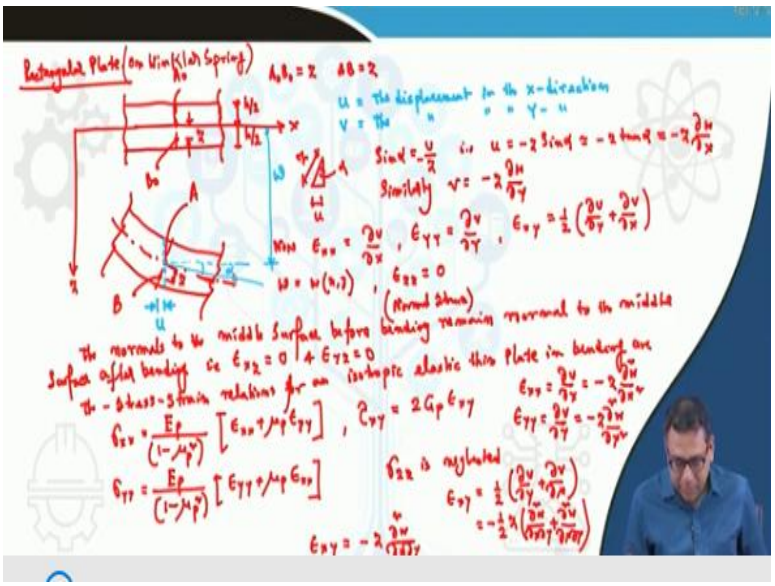

The same topic, rectangular plate resting on the winkler spring will be continued in this class. A section of the plate in xz plane was considered and then the deformation, w was formulated. The shape of the plate section after deformation is shown in the figure and α value can be obtained by taking a tangent to the middle surface at A.

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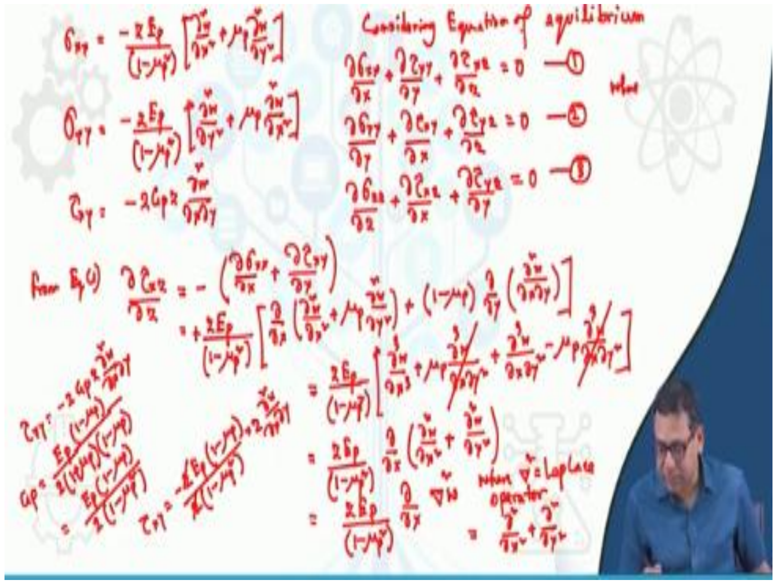

Then from the stress strain relationship, by considering equilibrium equations, the normal stresses and shear stresses were calculated.

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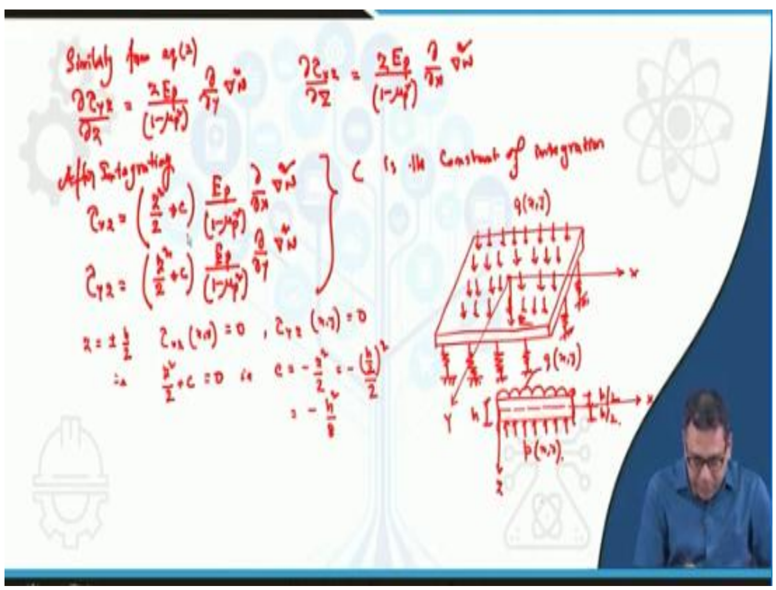

Basically the first and second equations were used to determine the shear stress in xz plane and yz plane. Consider that a UDL of intensity q(x,y) is acting over the entire region of the plate. A reaction, p will develop beneath the plate because of the soil or springs upon which the plate resting. Now consider a small segment of this plate showing the total thickness of the plate, h. In the considered section, the x axis passes through the middle surface of the plate and hence the plate is divided into two halves. As the vertical direction is z axis, we can write:

$$z = \pm\frac{h}{2}$$

At the top of the plate (z = + h/2), UDL q is acting and at the bottom of the plate (z = - h/2), the reaction from the springs, p is acting, So the shear stresses on these 2 planes will be 0.

$$\tau_{xz}(x,y) = \tau_{yz}(x,y) = 0 \text{ at } z = \pm\frac{h}{2}$$

$$\tau_{xz} = \left(\frac{z^2}{2} + c\right)\frac{E_p}{\left(1-\mu_p^2\right)}\frac{\partial}{\partial x}\nabla^2 w$$

$$\tau_{yz} = \left(\frac{z^2}{2} + c\right)\frac{E_p}{\left(1-\mu_p^2\right)}\frac{\partial}{\partial y}\nabla^2 w$$

From the above two equations, it is evident that if τxz and τyz are zero, then $\left(\frac{z^2}{2} + c\right)$ must be zero:

$$\left(\frac{z^2}{2} + c\right) = 0 \Rightarrow c = -\frac{z^2}{2}$$

$$\Rightarrow c = -\frac{\left(\frac{h}{2}\right)^2}{2} = -\frac{h^2}{8}$$

By substituting this c value, the shear stresses τ_{xz} and τ_{yz} can be calculated.

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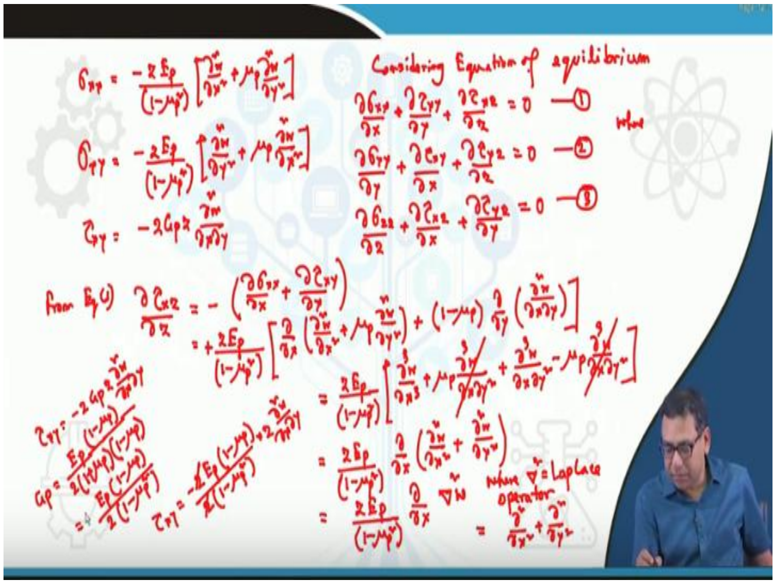

Now from the third equation of equilibrium, we get:

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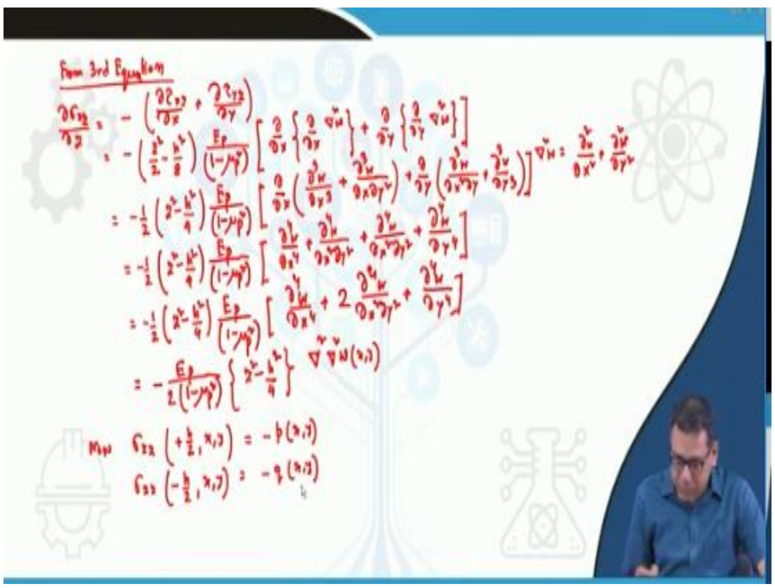

$$\frac{\partial \sigma_{zz}}{\partial z} = -\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y}\right)$$

$$\Rightarrow \frac{\partial \sigma_{zz}}{\partial z} = -\left(\frac{z^2}{2} - \frac{h^2}{8}\right)\frac{E_p}{\left(1-\mu_p^2\right)}\left[\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\nabla^2 w\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial y}\nabla^2 w\right\}\right]$$

$$\because \nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

$$\Rightarrow \frac{\partial \sigma_{zz}}{\partial z} = -\frac{1}{2}\left(z^2 - \frac{h^2}{4}\right)\frac{E_p}{\left(1-\mu_p^2\right)}\left[\frac{\partial}{\partial x}\left\{\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3}\right\}\right]$$

$$\Rightarrow \frac{\partial \sigma_{zz}}{\partial z} = -\frac{1}{2}\left(z^2 - \frac{h^2}{4}\right)\frac{E_p}{\left(1-\mu_p^2\right)}\left[\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right]$$

$$\Rightarrow \frac{\partial \sigma_{zz}}{\partial z} = -\frac{1}{2}\left(z^2 - \frac{h^2}{4}\right)\frac{E_p}{\left(1-\mu_p^2\right)}\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right]$$

$$\therefore \frac{\partial \sigma_{zz}}{\partial z} = -\frac{E_p}{2\left(1-\mu_p^2\right)}\left\{z^2 - \frac{h^2}{4}\right\}\nabla^2\nabla^2 w(x,y)$$

$$\text{Now, } \sigma_{zz}\left(+\frac{h}{2}, x, y\right) = -p(x,y)$$

$$\sigma_{zz}\left(-\frac{h}{2}, x, y\right) = -q(x,y)$$

As the z axis is taken positive in the downward direction, the z value of +h/2 indicates the bottom plane on which the spring reaction acts upon. Similarly z = -h/2, indicates the top plane upon which the UDL, q is acting. At that bottom face the shear stress is 0 but a normal stress of p, the reaction force from springs acts upon.

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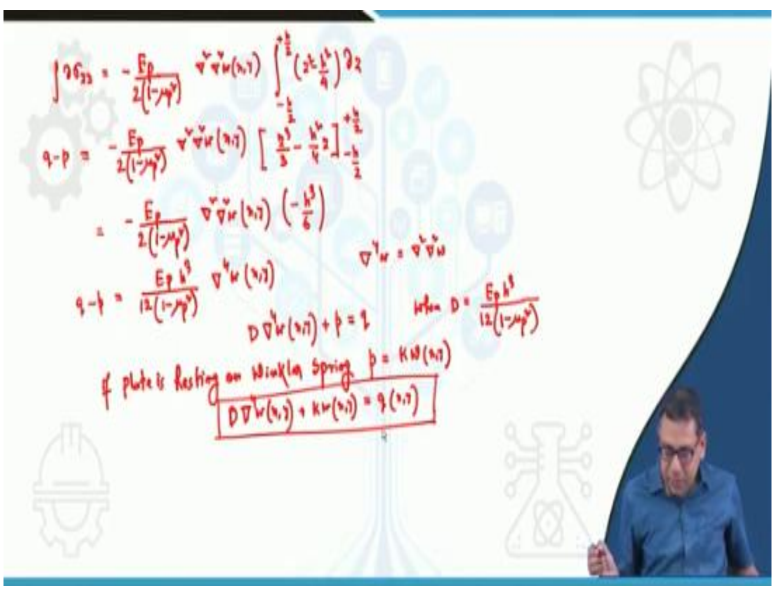

If $\frac{\partial \sigma_{zz}}{\partial z}$ is integrated:

$$\int \partial \sigma_{zz} = -\frac{E_p}{2(1-\mu_p^2)} \nabla^2 \nabla^2 w(x,y) \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(z^2 - \frac{h^2}{4} \right) \partial z$$

$$\Rightarrow q - p = -\frac{E_p}{2(1-\mu_p^2)} \nabla^2 \nabla^2 w(x,y) \left[\frac{z^3}{3} - \frac{h^2}{4} z \right]_{-\frac{h}{2}}^{\frac{h}{2}}$$

$$\Rightarrow q - p = -\frac{E_p}{2(1-\mu_p^2)} \nabla^2 \nabla^2 w(x,y) \left(-\frac{h^3}{6} \right)$$

$$\Rightarrow q - p = \frac{E_p h^3}{12(1-\mu_p^2)} \nabla^4 w(x,y)$$

$$\Rightarrow D \nabla^4 w(x,y) + p = q \; ; \quad \text{where, } D = \frac{E_p h^3}{12(1-\mu_p^2)}$$

If the plate is resting on Winkler springs, p = k × w(x,y)

$$\therefore D \nabla^4 w(x,y) + k w(x,y) = q(x,y)$$

This is the equation for a beam resting on Winkler springs.

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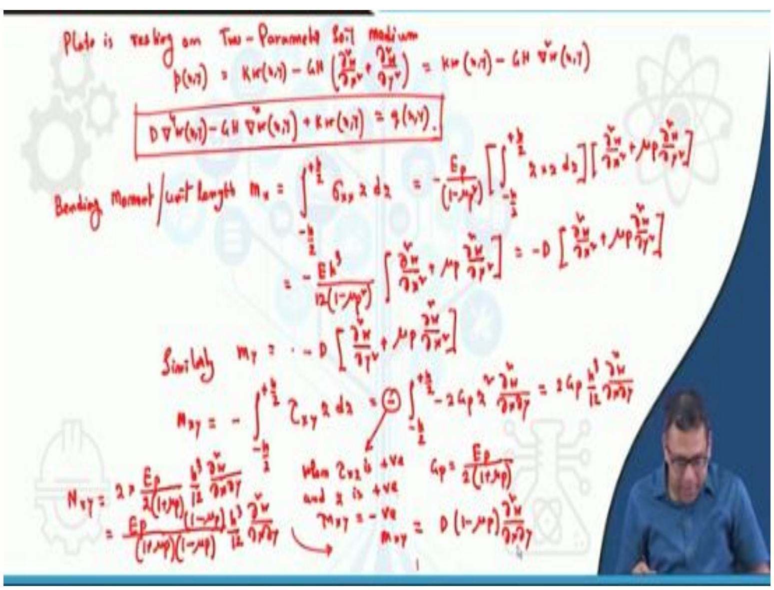

Similarly if the plate is resting on a 2 parameter soil medium, then the equation would be:

$$p(x,y) = k w(x,y) - GH \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = k w(x,y) - GH \nabla^2 w(x,y)$$

$$\Rightarrow D \nabla^4 w(x,y) - GH \nabla^2 w(x,y) + k w(x,y) = q(x,y)$$

$$\text{Bending Moment / Unit length, } M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} z dz$$

The stress in the small segment dz is multiplied with the lever arm, z to get the moment M_x. Substituting the value of σ_{xx} from equation-(a) {Ref. last class} in the above equation, we get:

$$\Rightarrow M_x = -\frac{E_p}{(1-\mu_p^2)} \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} z \times z dz \right] \left[\frac{\partial^2 w}{\partial x^2} + \mu_p \frac{\partial^2 w}{\partial y^2} \right]$$

$$\Rightarrow M_x = -\frac{Eh^3}{12(1-\mu_p^2)} \left[\frac{\partial^2 w}{\partial x^2} + \mu_p \frac{\partial^2 w}{\partial y^2} \right] = -D \left[\frac{\partial^2 w}{\partial x^2} + \mu_p \frac{\partial^2 w}{\partial y^2} \right]$$

$$\text{Similarly, } M_y = -D \left[\frac{\partial^2 w}{\partial y^2} + \mu_p \frac{\partial^2 w}{\partial x^2} \right]$$

$$\Rightarrow M_{xy} = -\left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} \times z dz \right] = -\int_{-\frac{h}{2}}^{\frac{h}{2}} -2G_p \times z^2 \frac{\partial^2 w}{\partial x \partial y} = 2G_p \frac{h^3}{12} \frac{\partial^2 w}{\partial x \partial y}$$

$$\text{Substituting } G_p = \frac{E_p}{(1+\mu_p)} \text{ in the above equation:}$$

$$\Rightarrow M_{xy} = 2 \times \frac{E_p}{(1+\mu_p)} \times \frac{h^3}{12} \times \frac{\partial^2 w}{\partial x \partial y}$$

$$\Rightarrow M_{xy} = 2 \times \frac{E_p (1-\mu_p)}{(1+\mu_p)(1-\mu_p)} \times \frac{h^3}{12} \times \frac{\partial^2 w}{\partial x \partial y}$$

$$\Rightarrow M_{xy} = D(1-\mu_p) \frac{\partial^2 w}{\partial x \partial y}$$

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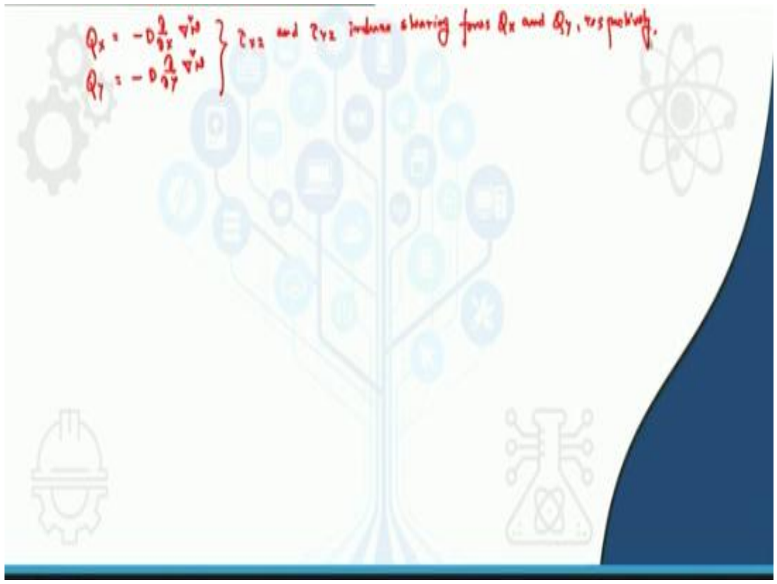

Finally, the expressions for shear force, Q_x and Q_y can be given by:

$$Q_x = -D\frac{\partial}{\partial x}\nabla^2 w$$

$$Q_y = -D\frac{\partial}{\partial y}\nabla^2 w$$

Q_x and Q_y are the sjear forces that are induced due to τ_{xz} and τ_{yz}.

I have derived the basic or governing differential equation for a beam resting on spring and then beam resting on 2 parameter soil medium. Then, I have derived the moments M_x, M_y and M_{xy} and shear forces induced Q_x and Q_y.

In the next class, I will show you the boundary conditions to solve this basic differential or governing differential equation when plate is resting on spring and also on 2 parameters soil medium. Thank you.