Soil Structure Interaction
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Lecture - 41
Plates on Elastic Foundation

In this class I will discuss about plates resting on elastic foundation. First I will discuss about the rectangular plate and then about the circular plate. I will give the basic differential equation and the boundary conditions, but I will show the solutions when I discuss about the numerical part using finite difference scheme.

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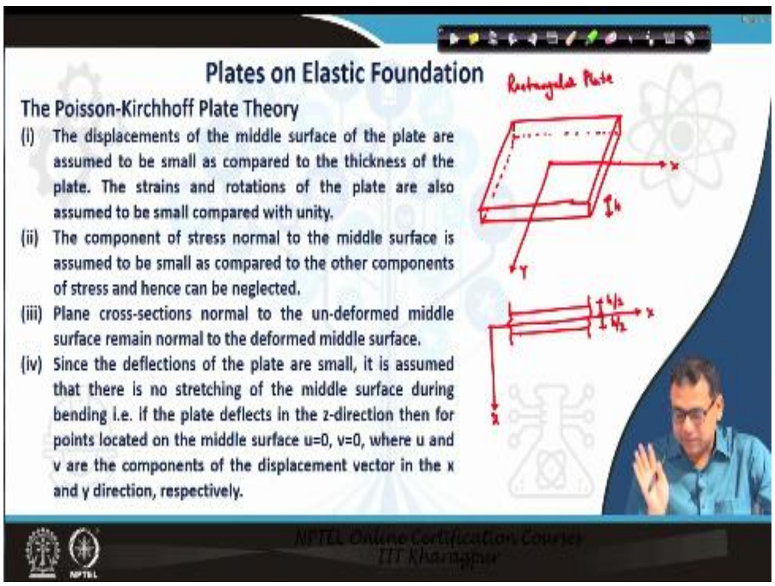

The first topic in this discussion is the Poisson-Kirchhoff plate theory but we will restrain only to thin plates part. The 4 assumptions under consideration are shown in the slide above. Consider a rectangular plate of thickness h and a section of that plate in x-z plane. The thickness h can be seen in this view and a surface can be assumed which divides the plate’s thickness to two halves.

The first assumption is that the displacements of this middle surface of the plate are assumed to be very small as compared to the thickness of the plate and the strain and rotation of the plate are also assumed to be small compared to with unity.

The component of stress normal to the middle surface is assumed to be small as compared to the other component of stress and hence can be neglected. The component of the stress normal to the middle surface means the component of the stress acting in the z direction. So, the stresses acting in the z direction are neglected because it is assumed that they are small compared to the stresses acting in other directions.

The plane cross sections normal to the un-deformed middle surface remain normal to the deformed middle surface. This means even after the deformation, the cross section normal to the un-deformed surface will remain normal to the deformed middle surface. Another assumption is that since the deflection of the plates are small, it is assumed that there is no stretching of the middle surface during the bending.

So when the beam bends, there will be no stretching of the middle surface implying that there will be no lateral deformation. When a plate deflects only in the z direction, then for the points located on the middle surface, u and v will be 0 (u and v are the components of deflection vector in the x and y directions).

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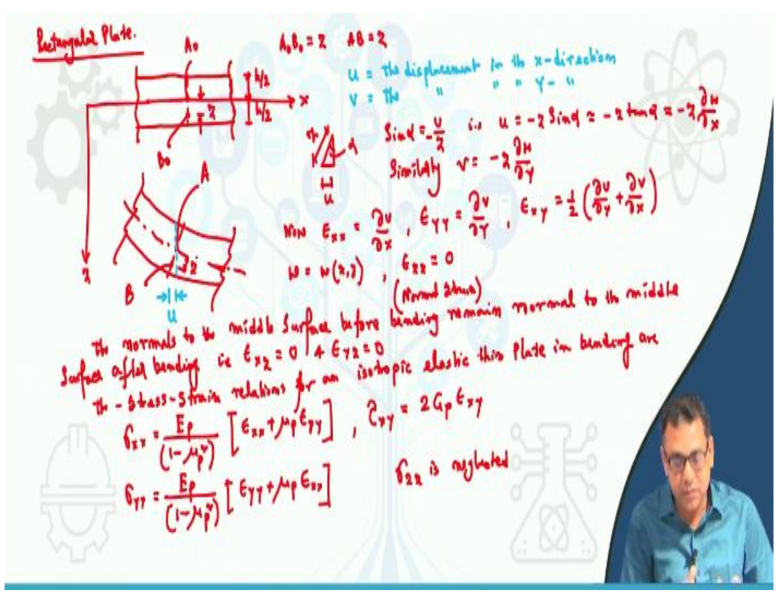

Let us start the derivation for this rectangular plate. Consider a section of the beam in the xz plane and consider 2 points A_0 and B_0. A_0 is at the middle surface and B_0 in the bottom half of this section of the plate.

Let the distance between A_0 and B_0 be z and it is obvious that the middle surface divides the plate section into two parts, each with thickness h/2. After applying a load, the plate bends and in that deflected shape (or state), the position of A_0 is now A and the position of B_0 is B. So, the distance between A and B will be z again.

But if a normal is considered, then there is a shift in this point and that shift is u. So u is the displacement in x direction. Similarly there is another component, v which is the displacement in y direction. The formulation of u can be explained with the small right-angled triangle figure in the above slide. In that figure, the vertical straight line is the initial position of z, A_0B_0 and the hypotenuse of this triangle is the deformed distance between these points, AB. So, the distance between them in the x direction is u. So, we can write:

$$\sin\alpha = -\frac{u}{z} \quad \Rightarrow u = -z\sin\alpha \approx -z\tan\alpha \quad \Rightarrow u = -z\frac{\partial w}{\partial x}$$

$$\text{Similarly, } v = -z\frac{\partial w}{\partial y}$$

Now: $\varepsilon_{xx} = \frac{\partial u}{\partial x}$; $\varepsilon_{yy} = \frac{\partial v}{\partial y}$; $\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$; $w=w(x,y)$; $\varepsilon_{zz} = 0$ ($\because$ Normal strain)

It is also assumed that the normals to the middle surface before bending remain normal to the middle surface after bending. This is possible only when strain in xz and in yz planes is 0.

$$\varepsilon_{xz} = \varepsilon_{yz} = 0$$

The stress strain relationship for an isotropic elastic thin plate in bending can be written as:

$$\sigma_{xx} = \frac{E_p}{\left(1-\mu_p^2\right)}\left[\varepsilon_{xx} + \mu_p\varepsilon_{yy}\right]; \quad \tau_{xy} = 2G_p\varepsilon_{xy}$$

$$\sigma_{yy} = \frac{E_p}{\left(1-\mu_p^2\right)}\left[\varepsilon_{yy} + \mu_p\varepsilon_{xx}\right]; \quad \sigma_{zz} \text{ is neglected}$$

The strain expression can be re written as:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

Substituting the value of u, $u = -z\frac{\partial w}{\partial x}$ in the above expression, we get:

$$\varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2}$$

$$\text{Similarly, } \varepsilon_{yy} = -z\frac{\partial^2 w}{\partial y^2}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\varepsilon_{xy} = -\frac{1}{2}z\left(\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y}\right)$$

$$\therefore \varepsilon_{xy} = -z\frac{\partial^2 w}{\partial x \partial y}$$

By substituting all these values in the stress-strain equations, we get:

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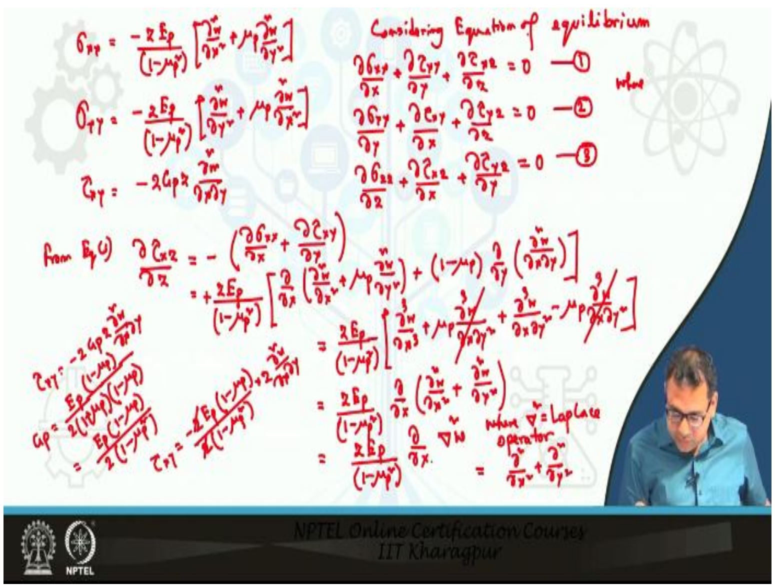

$$\sigma_{xx} = -\frac{zE_p}{\left(1-\mu_p^2\right)}\left[\frac{\partial^2 w}{\partial x^2} + \mu_p\frac{\partial^2 w}{\partial y^2}\right] \rightarrow (a)$$

$$\sigma_{yy} = -\frac{zE_p}{\left(1-\mu_p^2\right)}\left[\frac{\partial^2 w}{\partial y^2} + \mu_p\frac{\partial^2 w}{\partial x^2}\right] \rightarrow (b)$$

$$\tau_{xy} = -2G_p z\frac{\partial^2 w}{\partial x \partial y}$$

$$\Rightarrow G_p = \frac{E_p}{2\left(1+\mu_p\right)} \times \frac{\left(1-\mu_p\right)}{\left(1-\mu_p\right)} = \frac{E_p\left(1-\mu_p\right)}{2\left(1-\mu_p\right)^2}$$

$$\therefore \tau_{xy} = -\frac{E_p\left(1-\mu_p\right)}{2\left(1-\mu_p\right)^2} \times z\,\frac{\partial^2 w}{\partial x \partial y} \rightarrow (c)$$

Now, considering the equations of equilibrium:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \rightarrow (1)$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} = 0 \rightarrow (2)$$

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \rightarrow (3)$$

From equation (1): $\frac{\partial \tau_{xz}}{\partial z} = -\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\right)$

Substituting the value of σ_{xx} from equation (a) and the value of τ_{xy} from equation (c):

$$\frac{\partial \tau_{xz}}{\partial z} = +\frac{zE_p}{\left(1-\mu_p^2\right)}\left[\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \mu_p \frac{\partial^2 w}{\partial y^2}\right) + \left(1-\mu_p\right)\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x \partial y}\right)\right]$$

$$\Rightarrow \frac{\partial \tau_{xz}}{\partial z} = \frac{zE_p}{\left(1-\mu_p^2\right)}\left[\frac{\partial^3 w}{\partial x^3} + \mu_p \frac{\partial^3 w}{\partial x \partial y^2} + \frac{\partial^3 w}{\partial x \partial y^2} - \mu_p \frac{\partial^3 w}{\partial x \partial y^2}\right]$$

$$\Rightarrow \frac{\partial \tau_{xz}}{\partial z} = \frac{zE_p}{\left(1-\mu_p^2\right)}\left[\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right]$$

$$\Rightarrow \frac{\partial \tau_{xz}}{\partial z} = \frac{zE_p}{\left(1-\mu_p^2\right)}\frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]$$

$$\therefore \frac{\partial \tau_{xz}}{\partial z} = \frac{zE_p}{\left(1-\mu_p^2\right)}\frac{\partial}{\partial x}\left(\nabla^2 w\right)$$

where, ∇^2 is the Laplace operator, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

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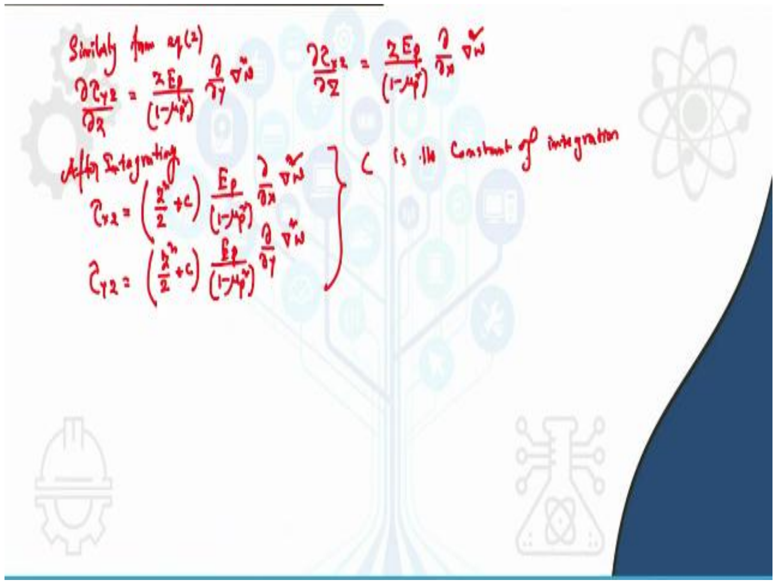

Similarly, from equation-(2):

$$\therefore \frac{\partial \tau_{yz}}{\partial z} = \frac{zE_p}{\left(1-\mu_p^2\right)} \frac{\partial}{\partial y}\left(\nabla^2 w\right)$$

After solving the last two equations, we get:

$$\tau_{xz} = \left(\frac{z^2}{2} + c\right) \frac{E_p}{\left(1-\mu_p^2\right)} \frac{\partial}{\partial x} \nabla^2 w$$

$$\tau_{yz} = \left(\frac{z^2}{2} + c\right) \frac{E_p}{\left(1-\mu_p^2\right)} \frac{\partial}{\partial y} \nabla^2 w$$

where, c is the constant of integration

In the next class I will discuss how to get the c value and by substituting the c value, how to get the basic differential equation for the plate which is resting on the elastic foundation (simply, plate resting on spring). After that I will continue with a theory for circular plate and then I will discuss about this plate resting on a 2 parameter soil medium. In the beam case also I have discussed the beam resting on spring and then on 2 parameters soil medium. So, I will follow the same pattern for plates too. Thank you.