Soil Structure Interaction
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Lecture 39
Beams on Elastic Foundation (Contd.,)

In the last class, I discussed about a beam with finite length subjected to point load at the centre and resting on two parameter soil medium.

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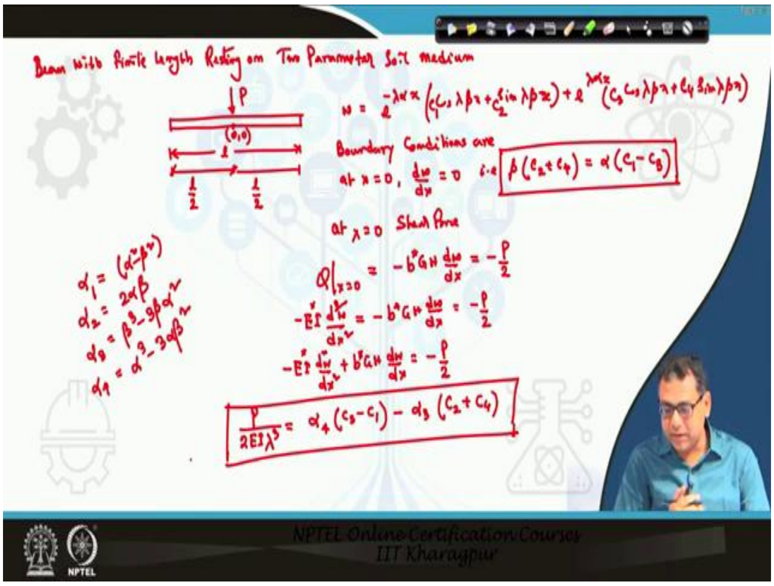

This was the problem and two boundary conditions were already applied. The basic differential equation from which the solution started is:

$$EI\frac{d^4w}{dx^4} - b^*GH\frac{d^2w}{dx^2} + b^*kw = bq$$

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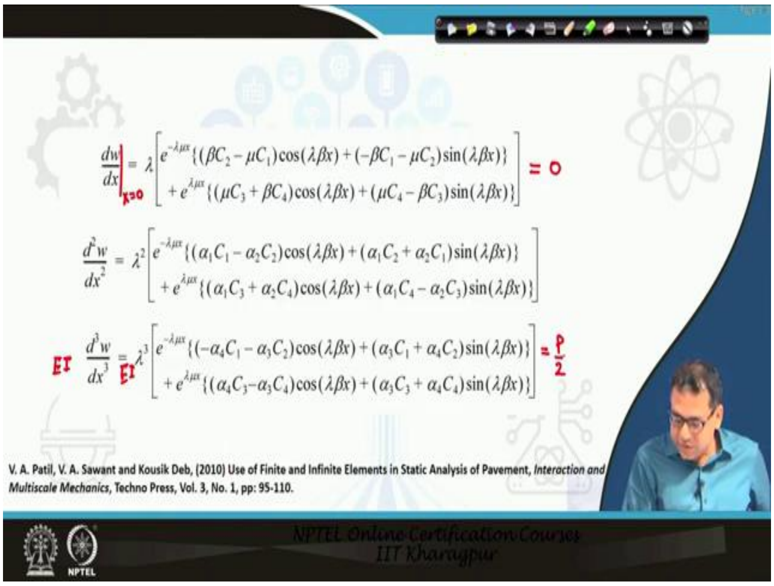

In the above slide, the first equation shows the slope and it will be equal to 0 when x = 0. By equating this expression to 0, the first condition derived in the last class was obtained:

$$\Rightarrow (C_2 + C_4) = \alpha(C_1 - C_3)$$

If this equation is differentiated twice with respect to x and multiplied by EI on both sides, the shear force expression can be obtained. As the shear force at x = 0 is (-P/2), the last equation above should be equated to (-P/2). By doing this, the second condition derived in the last class will be obtained:

$$\frac{P}{2EI\lambda^3} = \alpha_4(C_3 - C_1) - \alpha_3(C_2 + C_4)$$

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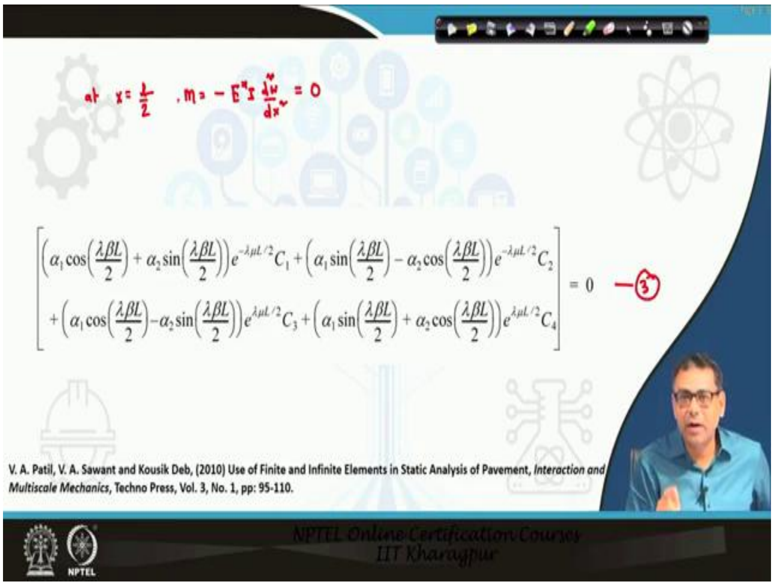

Now consider the third boundary condition which is considered at x = L/2. If the x value is L/2, it means that it is one of the edges of the beam and at the edge, the bending moment would be zero because the beam here has free ends on both edges.

$$\text{At x = L/2: } M\big|_{x=L/2} = -E^{*}I\frac{d^2w}{dx^2} = 0$$

So, the second expression shown in the second slide (of this lecture) should be multiplied by EI on both sides to make it the expression for bending moment. Then, as it would be zero at x = L/2, the expression within the bracket should be equated to zero by replacing x with L/2.

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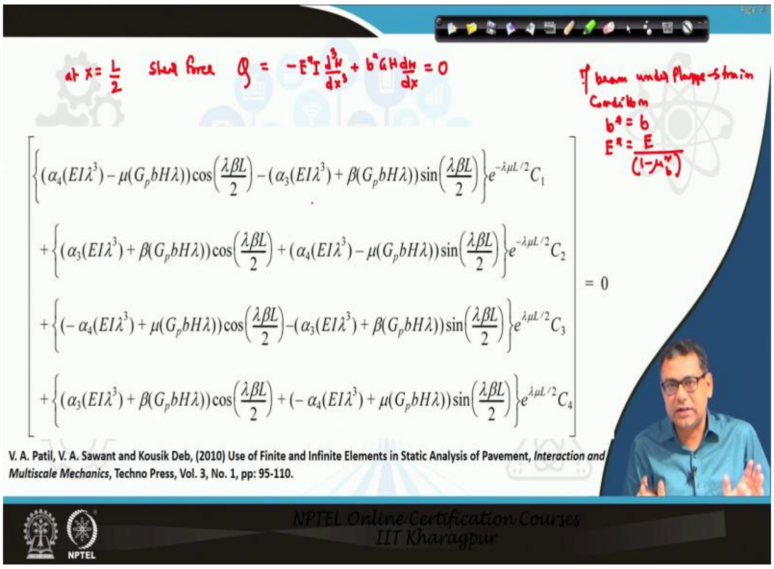

So, this would be the third condition. Now the fourth condition is that at x = L/2 the shear force q will be 0 because it is a free end.

$$Q\big|_{x=L/2} = -E^{*}I\frac{d^3w}{dx^3} + b^{*}GH\frac{dw}{dx} = 0$$

Remember that in case of plane strain condition: $b^{*} = b$ and $E^{*} = \dfrac{E}{1-\mu_b^{\,2}}$

So, this is the fourth condition. Now there are 4 equations and 4 unknowns. If these 4 equations are solved, the expressions for C_1, C_2, C_3 and C_4 can be obtained. These expressions are very lengthy and so will not be given here, but the procedure of how to determine these 4 unknowns should be known.

Remember that here these boundary conditions are valid for free end beam subjected to a concentrated load at the centre. If the end conditions or the loading conditions are changed, the expressions will also change.

Here, in the solutions for beam under finite length the solutions for deflection and other quantities are being restricted to within the beam only. It is obvious that except deflection, all the other quantities will be restricted to the beam, but it is very necessary to study the deflection beyond the beam also. That means, beyond the loaded region in the soil. If it is an infinite beam, there is no beyond the loaded region as the total region is within the beam but if the beam is of finite length, then this problem arises.

Though the deflections beyond the beam region are of interest, all the solutions given till now are restricted within the beam region. So, now let us see how to determine the deflection beyond the beam region.

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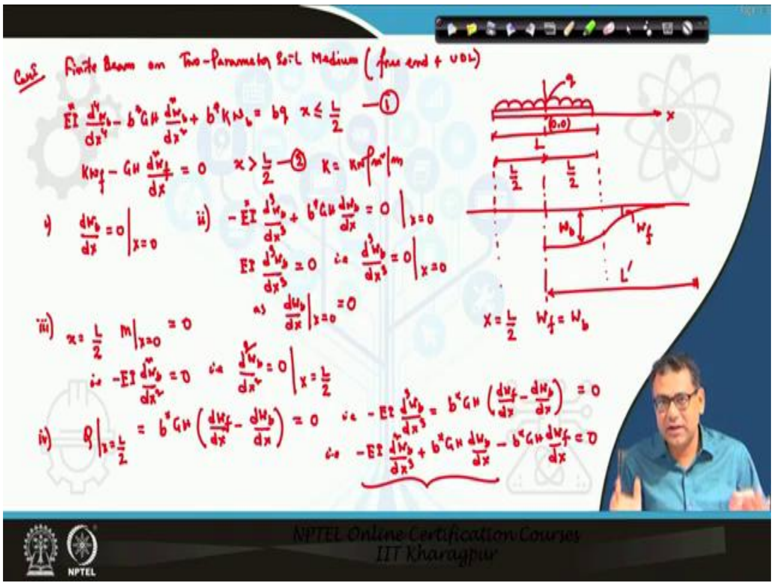

Now 2 cases will be dealt with in which the first case is where a finite beam is subjected to UDL and the second case is where the finite beam is subjected to a concentrated load at the centre. Then the equations will be applied in such a way that the deformation beyond the beam region can also be obtained. The first case or case-1 is where the finite beam of length, L resting on two parameter soil medium with free ends is subjected to UDL throughout its length. The centre of the UDL is also the centre of the beam and the intensity of the UDL is q.

Now the basic equation we know is:

$$\text{For } x \le \frac{L}{2} : E^{*}I\frac{d^4 w_b}{dx^4} - b^{*}GH\frac{d^2 w_b}{dx^2} + b^{*}kw_b = bq \rightarrow (1)$$

The probable deflection profile is shown in the figure which shows the deflection within the beam and beyond the beam region. Till now, only the deflections within the beam region were evaluated, but now as the deflection within and beyond beam are being dealt with it is better to have different terms. So, the deflections within the beam will be termed as w_b (b is for beam) and the deflections beyond the beam as w_f (f is for foundation soil).

So, at the junction or at x = L/2, one condition obviously prevails: $w_f = w_b$. This means that at the edge, the deflection of both beam and soil are same. So, beyond the loaded region, only soil will deflect, but within the loaded region or beam region, beam will deflect along with soil. This is why it was mentioned that the basic equation above is valid only if x is ≤ L/2 because this equation is valid only for the beam region. As x = 0 is at the centre, x = L/2 may either be to the left side or to the right side.

So, for x > L/2 the equation will be:

$$\text{For } x > \frac{L}{2}: kw_f - GH\frac{d^2w_f}{dx^2} = 0 \rightarrow (2)$$

The right hand side of the above equation is 0 because beyond the beam region, no load will be applied. Note that the units of k are $kN/m^2/m$.

Now, there are two differential equations to solve out of which the first needs four boundary conditions to be solved and to solve the second one, two boundary conditions are required. The first boundary condition is that the slope at the centre (x = 0) because the load is UDL:

$$1.)\ \frac{dw_b}{dx} = 0\Bigg|_{x=0}$$

The second boundary condition is that the shear force also will be 0 at the centre (x = 0):

$$-EI\frac{d^3w_b}{dx^3} + b^*GH\frac{dw_b}{dx} = 0\Bigg|_{x=0}$$

As the slope at the centre is 0 (1st boundary condition), the above expression reduces to:

$$\Rightarrow -EI\frac{d^3w_b}{dx^3} = 0\Bigg|_{x=0}$$

$$2.)\ \frac{d^3w_b}{dx^3} = 0\Bigg|_{x=0}$$

The third boundary condition is that the bending moment will be 0 at the edge where, x = L/2.

$$3.)\ \text{-}\,\mathrm{EI}\frac{d^2 w_b}{dx^2}=0\bigg|_{x=\frac{L}{2}} \quad \Rightarrow 3.)\ \frac{d^2 w_b}{dx^2}=0\bigg|_{x=\frac{L}{2}}$$

The fourth condition is that the shear force is also 0 at x = L/2 (free end)

$$Q\big|_{x=\frac{L}{2}}=b^*GH\left(\frac{dw_f}{dx}-\frac{dw_b}{dx}\right)=0$$

The second boundary condition used was also for shear force, but this expression is a bit different because this is the edge of the beam. So, here both the beam and foundation soil should be considered to evaluate the shear force. So:

$$\Rightarrow \text{-}\mathrm{EI}\frac{d^3 w_b}{dx^3}=b^*GH\left(\frac{dw_f}{dx}-\frac{dw_b}{dx}\right)=0$$

$$\Rightarrow \text{-}\mathrm{EI}\frac{d^3 w_b}{dx^3}+b^*GH\frac{dw_b}{dx}-b^*GH\frac{dw_f}{dx}=0$$

Initially, only the first two terms of the above expression were in the shear force equation, but here the third term is additional because the deflections beyond the loaded region are also being considered.

These are the 4 boundary conditions which will be required to solve the equation (1). Still, two more boundary conditions are needed to solve the equation (2).

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For the second case (equation-2), the first boundary condition is already discussed.

$$w_f = w_b\Big|_{x=\frac{L}{2}}$$

By solving the first equation, the w_b value can be determined and that value can be used in the above boundary condition.

The second boundary condition can be discussed in two ways. If the region beyond the beam is considered sufficiently long then there will be a point in the soil where deflection will be zero ($w_f = 0$). Let us call that length as L′ at which the w_f becomes 0.

$$w_f = 0\Big|_{x=L'}$$

Sometimes the slope will be considered and equated to 0 at L′. If the deflection itself is 0, then the slope will also be 0.

$$\theta_f = 0\Big|_{x=L'}$$

These 2 boundary conditions can be used to solve the second equation. Four boundary conditions are derived to solve equation-(1) and the above two boundary conditions can be used to solve equation-(2). From this, the solution for the system can be obtained.

In the next class I will show you what would be the boundary condition if the beam with finite length is subjected to a concentrated load at the centre considering both within the beam region as well as the region beyond the beam. After that, I will discuss another method where the modulus of elasticity or flexural rigidity of the beam or the modulus of subgrade reaction of the soil is not constant. Thank you.