Soil Structure Interaction
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Lecture 37
Beams on Elastic Foundation (Contd.,)

In this class, I will solve the basic differential equation that was developed for the case where beam is resting on two parameter soil medium.

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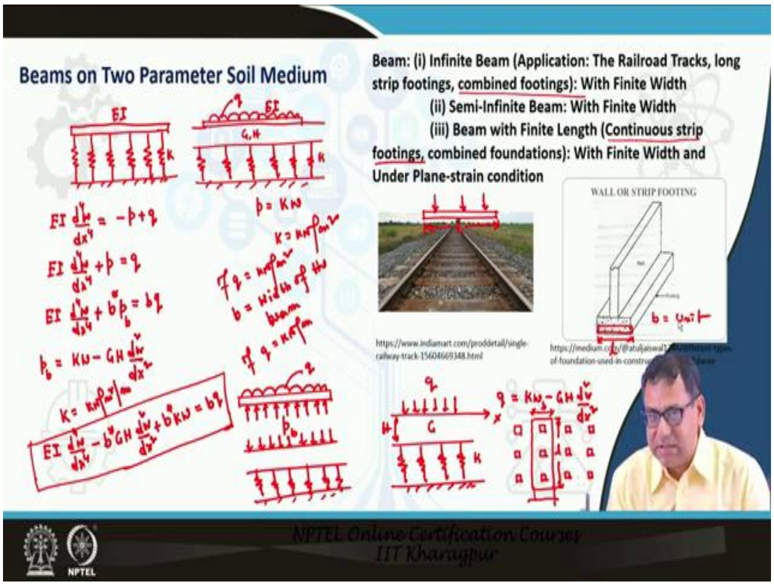

This was the differential equation derived in the last class:

$$EI\frac{d^4w}{dx^4} - b^*GH\frac{d^2w}{dx^2} + b^*kw = bq$$

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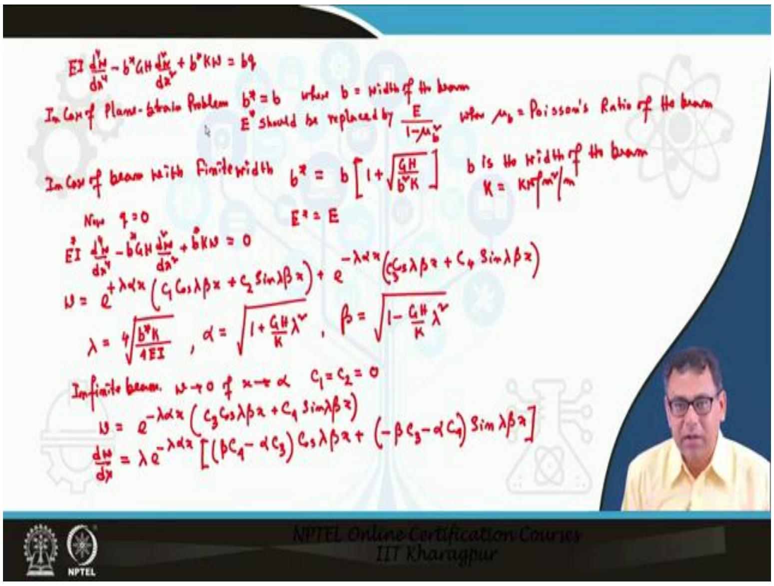

It is better to write the above equation in the below form as the elastic modulus should also be changed depending upon the condition.

$$\Rightarrow E^{*}I\frac{d^4w}{dx^4} - b^{*}GH\frac{d^2w}{dx^2} + b^{*}kw = bq$$

In case of plane strain condition: $b^{*} = b$ and $E^{*} = \dfrac{E}{1-\mu_b^{\,2}}$

In case of beam with finite width: $b^{*} = b\left[1+\sqrt{\dfrac{GH}{b^2k}}\right]$ and $E^{*} = E$

where, b is the width of the beam, μ_b is the Poisson's ratio of the beam and k is in $kN/m^2/m$.

Till now three types of beams were discussed, one is infinite beam, semi infinite beam and beam with finite length. The applications of infinite beam were already discussed. Infinite beam generally has a finite width and the best example for this is a railway track. The length of a railway track is infinite, but it has a finite width. So, if the beam is infinite then we can say that it comes under case-2 where beam has a finite width. An infinite beam can be a strip footing also where, in the length direction it is considered infinite and in the width direction as finite. (*strip footing usually has a finite width*).

For the beam with finite length, there can be two cases as under plane strain condition and it can be under the finite width condition. So, a strip can also be modelled as a beam with finite length when the smaller dimension of the footing (usually, width) is considered as the length of the beam and the longer dimension considered as the width which is infinite. It means, usually for a strip footing, the longer dimension is length and the smaller dimension is width, but if it is considered the opposite (larger as width & smaller as length), it resembles the plane strain condition.

So, here the strip footing is modelled as a beam. The length of the footing is considered as the width of the beam and the footing width as the length for the beam. Now, for a beam with finite length under plane strain condition (i.e., infinite width), the b value will be considered as unit. So, the analysis will be done for unit width of the beam. This is the beam under plane strain condition with finite length.

If, for a structure, there is a 3 × 4 arrangement of columns as shown in the first slide and if a combined footing is provided for 3 columns that are in the same line, then the beam with finite length and width case arises. So, usually combined footings can be considered as beams with finite width and finite length. In the first case also, there was a combined footing, but in that case the length was infinite.

These are the different conditions and now, let us solve the equation. If there is no load acting on the soil (q = 0), the equation will be:

$$E^* I \frac{d^4 w}{dx^4} - b^* GH \frac{d^2 w}{dx^2} + b^* k w = 0$$

After solving the above equation, the deflection value will be:

$$w = e^{\lambda \alpha x}\left(C_1 \cos \lambda \beta x + C_2 \sin \lambda \beta x\right) + e^{-\lambda \alpha x}\left(C_3 \cos \lambda \beta x + C_4 \sin \lambda \beta x\right)$$

$$\text{where, } \lambda = \sqrt[4]{\frac{b^* k}{4EI}}\ ,\ \alpha = \sqrt{1 + \frac{GH}{k}\lambda^2}\ ,\ \beta = \sqrt{1 - \frac{GH}{k}\lambda^2}$$

Remember that if the beam is resting on the springs, b^* will be equal to b because there would be no G or H. So if beam rests on springs alone, then b^* = b for both the cases (plane strain or beam with finite width). This is the reason, the b^* term was not introduced in the beams resting on springs. When beam rests on springs, b^* = b for both the conditions. But, the E value should be replaced if the beam is under plane strain condition no matter the beam rests on springs or two parameter model.

Here the infinite beam case will be discussed as it is relatively easier compared to the finite beam case. Then the various coefficients involved in the finite beam will also be given. The condition required to satisfy the infinite beam condition is: $w \rightarrow 0$ If $x \rightarrow \infty$.

This will happen only if $e^{\lambda x}$ term disappears in the deflection equation and for that to disappear, both C_1 and C_2 should be 0. If they both are zeros, the deflection will be reduced to:

$$w = e^{-\lambda \alpha x}\left(C_3 \cos \lambda \beta x + C_4 \sin \lambda \beta x\right)$$

Differentiating the above expression with respect to x:

$$\frac{dw}{dx} = \lambda e^{-\lambda \alpha x}\left[\left(\beta C_4 - \alpha C_3\right)\cos \lambda \beta x + \left(-\beta C_3 - \alpha C_4\right)\sin \lambda \beta x\right]$$

As there are two more unknowns in the expression, two boundary conditions are required to solve the expression. As the loading condition is symmetric and for a symmetric loading condition, the slope at the centre or x = 0 will be 0. This is the first boundary condition.

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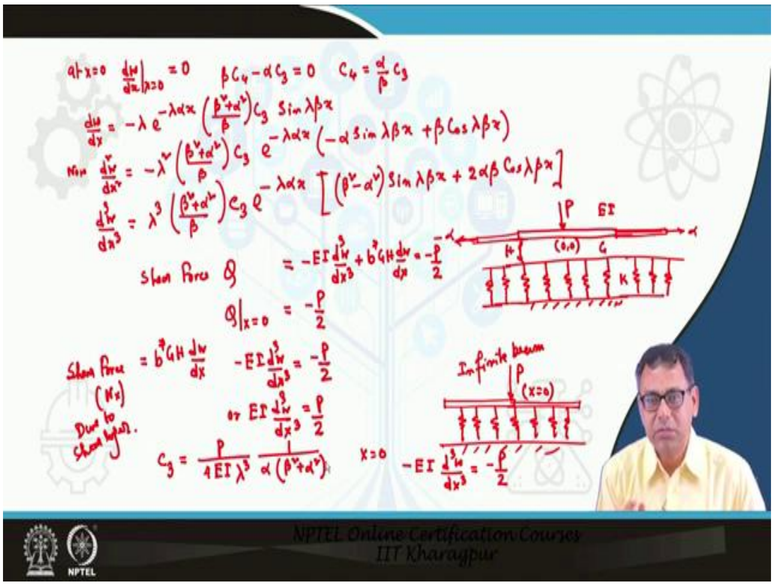

Applying the first boundary condition, at x = 0:

$$\left.\frac{dw}{dx}\right|_{x=0} = 0 \Rightarrow (\beta C_4 - \alpha C_3) = 0 \quad \Rightarrow C_4 = \frac{\alpha}{\beta} C_3$$

Substituting the value of C4 in the expression for dw/dx:

$$\frac{dw}{dx} = -\lambda^2 \left(\frac{\beta^2 + \alpha^2}{\beta} \right) C_3 \sin \lambda \beta x$$

Differentiating the above equation further:

$$\frac{d^2 w}{dx^2} = -\lambda^2 \left(\frac{\beta^2 + \alpha^2}{\beta} \right) C_3 e^{-\lambda \alpha x} (-\alpha \sin \lambda \beta x + \beta \cos \lambda \beta x)$$

$$\frac{d^3 w}{dx^3} = -\lambda^3 \left(\frac{\beta^2 + \alpha^2}{\beta} \right) C_3 e^{-\lambda \alpha x} \left[(\beta^2 - \alpha^2) \sin \lambda \beta x + 2\alpha\beta \cos \lambda \beta x \right]$$

Now considering that a point load is acting, the shear force at x = 0 will be –P/2. As the load applied here is a UDL, under a UDL centre, shear force will be zero. If a point load, P acts on an infinite beam resting over springs, the shear force would be:

$$Q = -EI \frac{d^3 w}{dx^3}$$

This value of shear force at x = 0 will be:

$$\left. Q \right|_{x=0} = -EI \frac{d^3 w}{dx^3} = -\frac{P}{2}$$

But if the beam rests on a two parameter soil medium, the additional shear force due to the shear layer presence will be:

$$b^* GH \frac{dw}{dx}$$

If a point load acts on an infinite beam resting over a two parameter soil medium, the general expression for shear force can be written as:

$$Q = -EI \frac{d^3 w}{dx^3} + b^* GH \frac{dw}{dx} = -\frac{P}{2}$$

But, at the centre or at x = 0, the slope is zero and hence the above equation again reduces to:

$$-EI \frac{d^3 w}{dx^3} = -\frac{P}{2}$$

$$\Rightarrow EI \frac{d^3 w}{dx^3} = \frac{P}{2}$$

Substituting these values in the deflection expression, the C_3 value can be determined.

$$C_3 = \frac{P}{4EI\lambda^3} \frac{1}{\alpha\left(\beta^2 + \alpha^2\right)}$$

So using one boundary condition the relation between C_4 and C_3 was established. By applying the other boundary condition, the C_3 is determined. Now, as the relation between C_4 and C_3 is known, C_4 can be found out just by mere substitution.

In the next class, after determining the value of C_4, both constants will be substituted in the equation and the equation of the deflection and bending moment for infinite beam resting on two parameter soil medium will be derived. Then I will show you the equation for beam with finite length resting on two parameter soil medium. Thank you.