

**Soil Structure Interaction**  
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**Lecture 35**  
**Beams on Elastic Foundation (Contd.,)**

In the last class I was discussing about the continuity among the foundation layers. Now, I will derive the deflection equation for the upper beam and lower beam.

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$$\frac{d^4 w_1}{dx^4} + \left( \frac{k_1}{E_1 I_1} + \frac{k_2}{E_2 I_2} + \frac{k_3}{E_3 I_3} \right) \frac{d^2 w_1}{dx^2} + \frac{k_4 k_5}{E_1 I_1 E_2 I_2} w_1 = 0$$

$$\frac{d^4 w_2}{dx^4} + A \frac{d^2 w_2}{dx^2} + B w_2 = 0$$

$$A = \frac{k_1 (E_1 I_1 + E_2 I_2) + k_2 (E_1 I_1)}{E_1 I_1 E_2 I_2}$$

$$B = \frac{k_1 k_5}{E_1 I_1 E_2 I_2}$$

$$\alpha = \frac{A}{2}, \quad \beta = \sqrt{\frac{A^2}{4} - B}$$

$$\lambda_1 = \alpha + \beta, \quad \lambda_2 = \alpha - \beta$$

$$w_1 = e^{-\lambda_1 x} (C_1 \cos \lambda_1 x + C_2 \sin \lambda_1 x) + e^{-\lambda_2 x} (C_3 \cos \lambda_2 x + C_4 \sin \lambda_2 x)$$

$$w_2 = \left[ 1 - \frac{(\alpha + \beta) E_1 I_1}{k_1} \right] e^{\lambda_1 x} (C_5 \cos \lambda_1 x + C_6 \sin \lambda_1 x) + \left[ 1 - \frac{(\alpha - \beta) E_1 I_1}{k_1} \right] e^{\lambda_2 x} (C_7 \cos \lambda_2 x + C_8 \sin \lambda_2 x)$$

The basic differential equations were formulated and solved to an extent in the last class. There are 8 constants  $C_1$  to  $C_8$  which are to be found out to calculate  $w_1$  and  $w_2$ .

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**Upper beam**  

$$w_1 = \frac{P}{16 E_1 I_1 \beta} \left[ D_1 \frac{e^{-\lambda_1 x}}{\lambda_1^3} (C_5 \cos \lambda_1 x + C_6 \sin \lambda_1 x) - D_2 \frac{e^{-\lambda_2 x}}{\lambda_2^3} (C_7 \cos \lambda_2 x + C_8 \sin \lambda_2 x) \right]$$

$$\theta_1 = \frac{P}{8 E_1 I_1 \beta} \left[ -D_1 \frac{e^{-\lambda_1 x}}{\lambda_1^2} \sin \lambda_1 x + D_2 \frac{e^{-\lambda_2 x}}{\lambda_2^2} \sin \lambda_2 x \right]$$

$$M_1 = \frac{P}{8 \beta} \left[ D_1 \frac{e^{-\lambda_1 x}}{\lambda_1} (C_5 \cos \lambda_1 x - C_6 \sin \lambda_1 x) - D_2 \frac{e^{-\lambda_2 x}}{\lambda_2} (C_7 \cos \lambda_2 x - C_8 \sin \lambda_2 x) \right]$$

$$\phi_1 = \frac{P}{4 \beta} \left[ -D_1 e^{-\lambda_1 x} \cos \lambda_1 x + D_2 e^{-\lambda_2 x} \cos \lambda_2 x \right]$$

**Boundary conditions:**  
 $w_1 \rightarrow 0$  if  $x \rightarrow \infty$      $C_1 = C_2 = C_5 = C_6 = 0$   
 at  $x = 0$   
 i)  $\frac{dw_1}{dx} \Big|_{x=0} = 0$     ii)  $\frac{dw_2}{dx} \Big|_{x=0} = 0$     i)  $C_3 = C_4$   
 ii)  $E_1 I_1 \frac{d^2 w_1}{dx^2} \Big|_{x=0} = \frac{P}{2}$     iii)  $E_2 I_2 \frac{d^2 w_2}{dx^2} \Big|_{x=0} = 0$     ii)  $C_7 = C_8$   
 iii)  $C_3 = \frac{P}{16 E_1 I_1 \beta \lambda_1^2} \left[ \frac{k_1}{E_1 I_1} - (\alpha - \beta) \right]$   
 iv)  $C_4 = -\frac{P}{16 E_1 I_1 \beta \lambda_2^2} \left[ \frac{k_1}{E_1 I_1} - (\alpha + \beta) \right]$

Now, let us determine these constants for infinite beam. So, if  $x$  tends to infinity, then  $w_1$  tends to 0. This is because the effect of any load will not be there at an infinite distance from the point of loading. But,  $w_1$  can be 0 only when  $C_1, C_2, C_3$  and  $C_4$  become 0. So for an infinite beam, these four constants must be zero.

Now there are still 4 more unknowns:  $C_3, C_4, C_7$  and  $C_8$ . To determine these 4 unknowns, the boundary conditions should be applied.

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The slide, titled "Continuity among the foundation soil layers", illustrates a mechanical model of two beams on a foundation. The upper beam has flexural rigidity  $E_1 I_1$  and the lower beam has  $E_2 I_2$ . They are supported by springs with stiffnesses  $K_1$  and  $K_2$  respectively. A point load  $P$  is applied to the upper beam at  $x=0$ . The diagram shows the beams, springs, and the load. Handwritten equations describe the continuity conditions at  $x=0$ :

- Upper Beam:  $E_1 I_1 \frac{d^4 w_1}{dx^4} = -p + q$
- Lower Beam:  $E_2 I_2 \frac{d^4 w_2}{dx^4} = -p + q$
- Spring reactions:  $p = K_1 w_1$  and  $q = K_2 w_2$
- Continuity of displacement:  $w_1 = w_2 = w$
- Continuity of slope:  $\frac{dw_1}{dx} = \frac{dw_2}{dx}$
- Continuity of shear force:  $E_1 I_1 \frac{d^3 w_1}{dx^3} = E_2 I_2 \frac{d^3 w_2}{dx^3}$
- Continuity of bending moment:  $E_1 I_1 \frac{d^2 w_1}{dx^2} = E_2 I_2 \frac{d^2 w_2}{dx^2}$

The slide also includes a small video inset of a presenter in the bottom right corner and the NPTEL logo in the bottom left.

This is the loading condition considered in this case and for this loading condition, at  $x = 0$ , the slope will also be zero and shear force will be equal to  $P/2$  (half the load) for the upper beam. As there are 4 unknowns, 4 boundary conditions are also needed to solve them. Two boundary conditions are identified in the upper beam alone and similar boundary conditions apply to the lower beam too making a total of 4 boundary conditions.

The boundary conditions available are (at  $x = 0$ ):

$$\begin{aligned}
 1.) \quad \left. \frac{dw_1}{dx} \right|_{x=0} &= 0; & 2.) \quad \left. \frac{dw_2}{dx} \right|_{x=0} &= 0; \\
 3.) \quad E_1 I_1 \left. \frac{d^3 w_1}{dx^3} \right|_{x=0} &= \frac{P}{2}; & 4.) \quad E_2 I_2 \left. \frac{d^3 w_2}{dx^3} \right|_{x=0} &= 0
 \end{aligned}$$

The shear force in upper beam at  $x = 0$  is half the point load but in the lower beam it is zero because any external load applied on the upper beam will act on the lower beam as UDL only because of the reaction from the upper springs. For a beam carrying a UDL, the shear force would be zero at the centre of the UDL and so for the lower beam, it is 0 at  $x = 0$ .

By applying the first and second conditions, we will get  $C_3 = C_4$  and  $C_7 = C_8$  respectively. Now, by applying the third condition, we get:

$$C_3 = \frac{P}{16E_1I_1\beta\lambda_1^3} \left[ \frac{k_1}{E_1I_1} - (\alpha - \beta) \right]$$

Similarly from the fourth condition:

$$C_4 = -\frac{P}{16E_1I_1\beta\lambda_2^3} \left[ \frac{k_1}{E_1I_1} - (\alpha + \beta) \right]$$

By substituting these values in the available equations, we get the expressions of required quantities in the upper beam:

$$w_1 = \frac{P}{16E_1I_1\beta} \left[ D_1 \frac{e^{-\lambda_1 x}}{\lambda_1^3} (\cos \lambda_1 x + \sin \lambda_1 x) - D_2 \frac{e^{-\lambda_2 x}}{\lambda_2^3} (\cos \lambda_2 x + \sin \lambda_2 x) \right]$$

$$\theta_1 = \frac{P}{8E_1I_1\beta} \left[ -D_1 \frac{e^{-\lambda_1 x}}{\lambda_1^2} \sin \lambda_1 x + D_2 \frac{e^{-\lambda_2 x}}{\lambda_2^2} \sin \lambda_2 x \right]$$

$$M_1 = \frac{P}{8\beta} \left[ D_1 \frac{e^{-\lambda_1 x}}{\lambda_1} (\cos \lambda_1 x - \sin \lambda_1 x) - D_2 \frac{e^{-\lambda_2 x}}{\lambda_2} (\cos \lambda_2 x - \sin \lambda_2 x) \right]$$

$$Q_1 = \frac{P}{4\beta} \left[ -D_1 e^{-\lambda_1 x} \cos \lambda_1 x + D_2 e^{-\lambda_2 x} \cos \lambda_2 x \right]$$

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Upper Beam

$$w_2 = -\frac{P}{16E_2I_2\beta} \frac{k_1}{E_1I_1} \left[ \frac{e^{-\lambda_1 x}}{\lambda_1^3} (\cos \lambda_1 x + \sin \lambda_1 x) - \frac{e^{-\lambda_2 x}}{\lambda_2^3} (\cos \lambda_2 x + \sin \lambda_2 x) \right]$$

$$\theta_2 = \frac{P}{8E_2I_2\beta} \frac{k_1}{E_1I_1} \left[ \frac{e^{-\lambda_1 x}}{\lambda_1^2} \sin \lambda_1 x - \frac{e^{-\lambda_2 x}}{\lambda_2^2} \sin \lambda_2 x \right]$$

$$m_2 = -\frac{P}{8\beta} \frac{k_1}{E_1I_1} \left[ \frac{e^{-\lambda_1 x}}{\lambda_1} (\cos \lambda_1 x - \sin \lambda_1 x) - \frac{e^{-\lambda_2 x}}{\lambda_2} (\cos \lambda_2 x - \sin \lambda_2 x) \right]$$

$$Q_2 = \frac{P}{4\beta} \frac{k_1}{E_1I_1} \left( e^{-\lambda_1 x} \cos \lambda_1 x - e^{-\lambda_2 x} \cos \lambda_2 x \right)$$

where  $D_1 = \frac{k_1}{E_1I_1} - (\alpha - \beta)$ ;  $D_2 = \frac{k_1}{E_1I_1} - (\alpha + \beta)$

$$\alpha = \frac{A}{2}; \beta = \sqrt{\frac{A^2}{4} - B}$$

$$A = \frac{k_1 (E_1I_1 + E_2I_2) + k_2 E_1I_1}{E_1I_1 E_2I_2}; \beta = \frac{k_1 k_2}{E_1I_1 E_2I_2}$$

$$\lambda_1 = \sqrt{\frac{\alpha + \beta}{4}}; \lambda_2 = \sqrt{\frac{\alpha - \beta}{4}}$$

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Similarly the expressions of the required quantities in the lower beam will be:

$$w_2 = \frac{P}{16E_2I_2\beta} \times \frac{k_1}{E_1I_1} \left[ \frac{e^{-\lambda_1 x}}{\lambda_1^3} (\cos \lambda_1 x + \sin \lambda_1 x) - \frac{e^{-\lambda_2 x}}{\lambda_2^3} (\cos \lambda_2 x + \sin \lambda_2 x) \right]$$

$$\theta_2 = \frac{P}{8E_2I_2\beta} \times \frac{k_1}{E_1I_1} \left[ \frac{e^{-\lambda_1 x}}{\lambda_1^2} \sin \lambda_1 x - \frac{e^{-\lambda_2 x}}{\lambda_2^2} \sin \lambda_2 x \right]$$

$$M_2 = \frac{P}{8\beta} \times \frac{k_1}{E_1I_1} \left[ \frac{e^{-\lambda_1 x}}{\lambda_1} (\cos \lambda_1 x - \sin \lambda_1 x) - \frac{e^{-\lambda_2 x}}{\lambda_2} (\cos \lambda_2 x - \sin \lambda_2 x) \right]$$

$$Q_2 = \frac{P}{4\beta} \times \frac{k_1}{E_1I_1} \left[ e^{-\lambda_1 x} \cos \lambda_1 x - e^{-\lambda_2 x} \cos \lambda_2 x \right]$$

where,  $D_1 = \frac{k_1}{E_1I_1} - (\alpha - \beta)$ ;  $D_2 = \frac{k_1}{E_1I_1} - (\alpha + \beta)$ ;

$$\alpha = \frac{A}{2}; \quad \beta = \sqrt{\frac{A^2}{4} - B}; \quad \lambda_1 = \sqrt[4]{\frac{\alpha + \beta}{4}}; \quad \lambda_2 = \sqrt[4]{\frac{\alpha - \beta}{4}};$$

$$A = \frac{k_1(E_1I_1 + E_2I_2) + k_2(E_1I_1)}{E_1I_1 \times E_2I_2}; \quad B = \frac{k_1k_2}{E_1I_1 \times E_2I_2}$$

These are the equations to determine all the four quantities for the lower beam and upper beam. It should be remembered that the upper beam is the real foundation and the lower beam is basically used to provide the connectivity between the springs. Now the question is how to decide the EI value for the lower beam. For upper beam, the EI value of the foundation can be used. So, what will be the EI value for the lower beam? Actually in the real case a very small EI value should be chosen for the lower beam so that it will not influence the deformation but provides connectivity among the springs.

Let us see an example to understand the effect of the EI value of lower beam on the deflection.

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**Example**

$P = 100 \text{ kN}$

$k_1 = 55000 \text{ kN/m}^2$

$k_2 = 0.25 \times 55000 = 13750 \text{ kN/m}^2$

$E_1I_1 = 1.47 \times 10^3 \text{ kN-m}^2$

$E_2I_2 = 10^7 \text{ kN-m}^2$

$I_1 = \frac{1}{12} (0.25)(0.2)^3 = 1.67 \times 10^{-9} \text{ m}^4$

$I_2 = \frac{1}{12} (0.25)(0.2)^3 = 1.67 \times 10^{-9} \text{ m}^4$

$A = \frac{k_1(E_1I_1 + E_2I_2) + k_2(E_1I_1)}{E_1I_1 \times E_2I_2} = \frac{2k_1 + k_2}{E_1I_1} = \frac{3 \times 13750}{1.47 \times 10^3} = \frac{1.8 \times 13750}{10^3} \text{ m}^{-4}$

$B = \frac{k_1k_2}{E_1I_1 \times E_2I_2} = \frac{k_1k_2}{E_1I_1} = 0.94 \left( \frac{13750}{10^3} \right) \text{ m}^{-4}$

$\alpha = \frac{A}{2} = \frac{0.9}{2} \times 13750 \text{ m}^{-4}, \quad \beta = \sqrt{\frac{A^2}{4} - B} = \frac{0.671}{10^3} \times 13750 \text{ m}^{-4}$

$\lambda_1 = \sqrt[4]{\frac{\alpha + \beta}{4}} = 1.524 \text{ m}^{-1}, \quad \lambda_2 = \sqrt[4]{\frac{\alpha - \beta}{4}} = 0.942 \text{ m}^{-1}$

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An infinite beam is subjected to a point load,  $P = 100$  kN at point O. The cross section of the beam is  $0.25 \text{ m} \times 0.2 \text{ m}$  (width  $\times$  height). As adopted in the previous example problems,  $k'$  value is  $55000 \text{ kN/m}^2/\text{m}$ . So, the  $k$  value will be ( $b \times k' = 0.25 \times 55000$ ) equal to  $13750 \text{ kN/m}^2$ . Flexural rigidity of the upper beam,  $E_1I_1$  is  $1.67 \times 10^3 \text{ kN/m}^2$ .

First, the deflection of the upper beam and lower beam will be determined at  $x = 0$ . The expression for  $w_1$  reduces to the following when  $x = 0$ :

$$w_{1o} = \frac{P}{16E_1I_1\beta} \left[ \frac{D_1}{\lambda_1^3} - \frac{D_2}{\lambda_2^3} \right]$$

Similarly, expression for  $w_2$  when  $x = 0$ :

$$w_{2o} = \frac{P}{16E_2I_2\beta} \frac{k_1}{E_1I_1} \left[ \frac{1}{\lambda_1^3} - \frac{1}{\lambda_2^3} \right]$$

To proceed further, the values of  $k_1$  and  $k_2$  are needed. Since this is a two layer soil system, both soil layers usually do not have the same  $k$  value. But let us see what happens if both are equal ( $k_1 = k_2 = k$ ). Consider this as case-1 where  $k_1 = k_2 = k$  and  $E_1I_1 = E_2I_2$ . So, the spring constant and flexural rigidity are same for both the layers in this case.

Case-1:  $k_1 = k_2 = k$  and  $E_1I_1 = E_2I_2 = EI$

$$A = \frac{k_1(E_1I_1 + E_2I_2) + k_2(E_1I_1)}{E_1I_1 \times E_2I_2} = \frac{2k_1 + k_2}{EI} = \frac{3k}{EI}$$

$$\Rightarrow A = \frac{3 \times 13750}{1.67 \times 10^3} = \frac{1.8}{10^3} \times 13750 m^{-4}$$

$$B = \frac{k_1k_2}{E_1I_1 \times E_2I_2} = \left( \frac{k}{EI} \right)^2 = 0.36 \times \left( \frac{13750}{10^3} \right)^2 m^{-4}$$

$$\alpha = \frac{A}{2} = \frac{0.9}{10^3} \times 13750 m^{-4}; \quad \beta = \sqrt{\frac{A^2}{4} - B} = \frac{0.671}{10^3} \times 13750 m^{-4}$$

$$\lambda_1 = \sqrt[4]{\frac{\alpha + \beta}{4}} = 1.524 m^{-1}; \quad \lambda_2 = \sqrt[4]{\frac{\alpha - \beta}{4}} = 0.942 m^{-1}$$

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$$D_1 = \frac{k_1}{E_1 I_1} - (\alpha - \beta) = \frac{0.37}{10^3} \times 13750 \text{ m}^{-4}$$

$$D_2 = \frac{k_1}{E_1 I_1} - (\alpha + \beta) = -\frac{0.972}{10^3} \times 13750 \text{ m}^{-4}$$

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Now, the  $D_1$  and  $D_2$  values can be calculated by:

$$D_1 = \frac{k_1}{E_1 I_1} - (\alpha - \beta) = \frac{0.37}{10^3} \times 13750 \text{ m}^{-4}$$

$$D_2 = \frac{k_1}{E_1 I_1} - (\alpha + \beta) = -\frac{0.972}{10^3} \times 13750 \text{ m}^{-4}$$

Till now I have determined all the coefficients and in the next class, I will put these coefficients in the expressions to determine the deflection at  $x = 0$  for lower beam as well as upper beam.

Thank you.