Soil Structure Interaction
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Lecture 35
Beams on Elastic Foundation (Contd.,)

In the last class I was discussing about the continuity among the foundation layers. Now, I will derive the deflection equation for the upper beam and lower beam.

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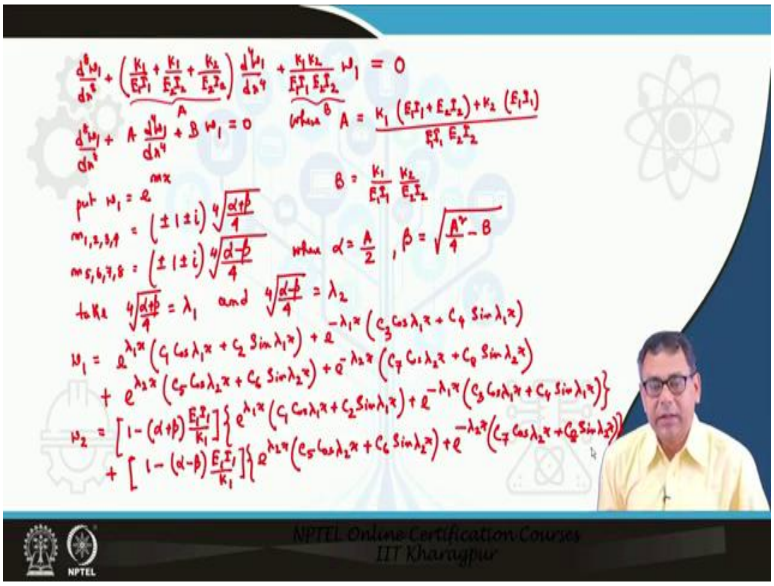

The basic differential equations were formulated and solved to an extent in the last class. There are 8 constants C_1 to C_8 which are to be found out to calculate w_1 and w_2.

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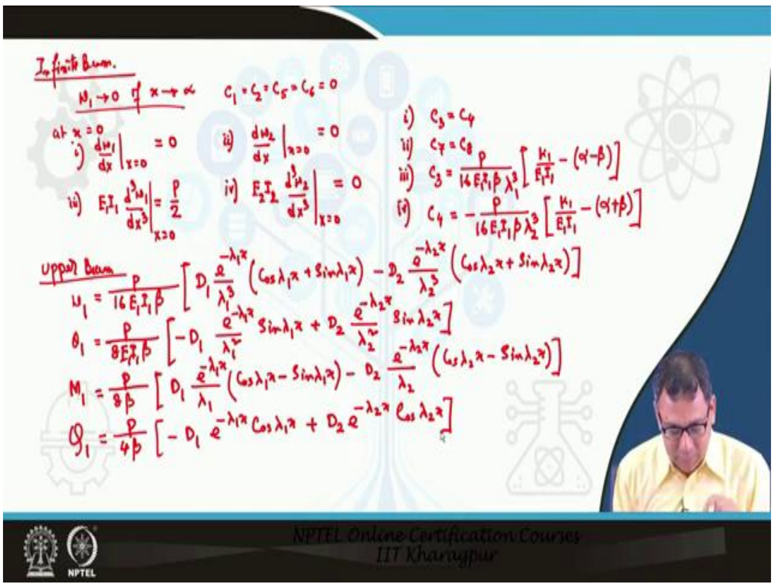

Now, let us determine these constants for infinite beam. So, if x tends to infinity, then w_1 tends to 0. This is because the effect of any load will not be there at an infinite distance from the point of loading. But, w_1 can be 0 only when C_1, C_2, C_3 and C_4 become 0. So for an infinite beam, these four constants must be zero.

Now there are still 4 more unknowns: C_3, C_4, C_7 and C_8. To determine these 4 unknowns, the boundary conditions should be applied.

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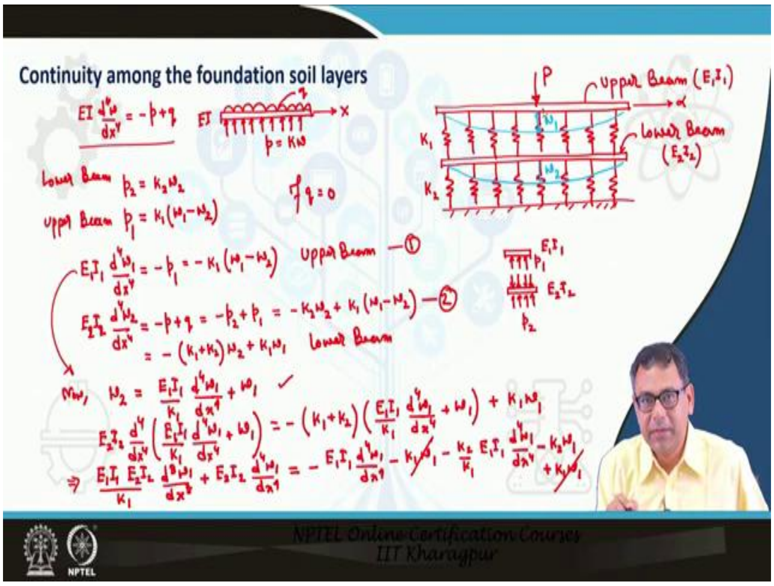

This is the loading condition considered in this case and for this loading condition, at x = 0, the slope will also be zero and shear force will be equal to P/2 (half the load) for the upper beam. As there are 4 unknowns, 4 boundary conditions are also needed to solve them. Two boundary conditions are identified in the upper beam alone and similar boundary conditions apply to the lower beam too making a total of 4 boundary conditions.

The boundary conditions available are (at x = 0):

$$1.)\ \left.\frac{dw_1}{dx}\right|_{x=0} = 0; \qquad 2.)\ \left.\frac{dw_2}{dx}\right|_{x=0} = 0;$$

$$3.)\ E_1 I_1 \left.\frac{d^3 w_1}{dx^3}\right|_{x=0} = \frac{P}{2}; \qquad 4.)\ E_2 I_2 \left.\frac{d^3 w_2}{dx^3}\right|_{x=0} = 0$$

The shear force in upper beam at x =0 is half the point load but in the lower beam it is zero because any external load applied on the upper beam will act on the lower beam as UDL only because of the reaction from the upper springs. For a beam carrying a UDL, the shear force would be zero at the centre of the UDL and so for the lower beam, it is 0 at x = 0.

By applying the first and second conditions, we will get $C_3 = C_4$ and $C_7 = C_8$ respectively. Now, by applying the third condition, we get:

$$C_3 = \frac{P}{16E_1I_1\beta\lambda_1^3}\left[\frac{k_1}{E_1I_1} - (\alpha - \beta)\right]$$

Similarly from the fourth condition:

$$C_4 = -\frac{P}{16E_1I_1\beta\lambda_2^3}\left[\frac{k_1}{E_1I_1} - (\alpha + \beta)\right]$$

By substituting these values in the available equations, we get the expressions of required quantities in the upper beam:

$$w_1 = \frac{P}{16E_1I_1\beta}\left[D_1\frac{e^{-\lambda_1 x}}{\lambda_1^3}(\cos\lambda_1 x + \sin\lambda_1 x) - D_2\frac{e^{-\lambda_2 x}}{\lambda_2^3}(\cos\lambda_2 x + \sin\lambda_2 x)\right]$$

$$\theta_1 = \frac{P}{8E_1I_1\beta}\left[-D_1\frac{e^{-\lambda_1 x}}{\lambda_1^2}\sin\lambda_1 x + D_2\frac{e^{-\lambda_2 x}}{\lambda_2^2}\sin\lambda_2 x\right]$$

$$M_1 = \frac{P}{8\beta}\left[D_1\frac{e^{-\lambda_1 x}}{\lambda_1}(\cos\lambda_1 x - \sin\lambda_1 x) - D_2\frac{e^{-\lambda_2 x}}{\lambda_2}(\cos\lambda_2 x - \sin\lambda_2 x)\right]$$

$$Q_1 = \frac{P}{4\beta}\left[-D_1 e^{-\lambda_1 x}\cos\lambda_1 x + D_2 e^{-\lambda_2 x}\cos\lambda_2 x\right]$$

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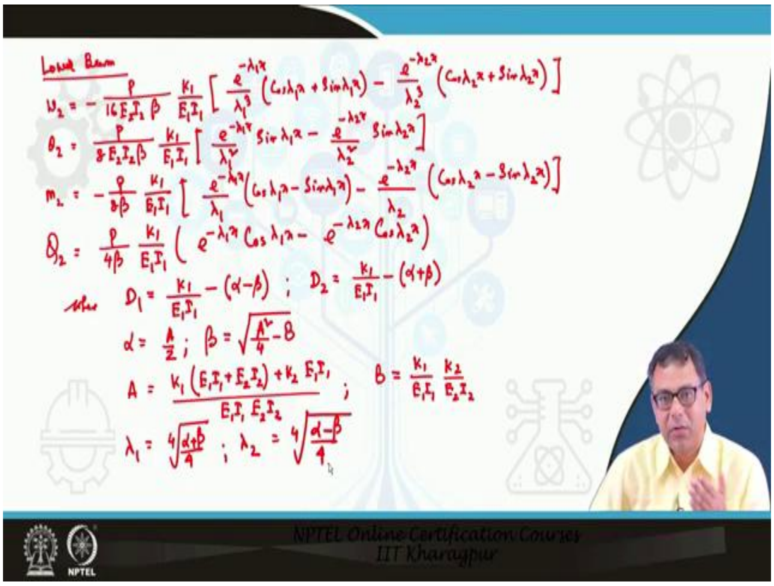

Similarly the expressions of the required quantities in the lower beam will be:

$$w_2 = \frac{P}{16E_2I_2\beta}\times\frac{k_1}{E_1I_1}\left[\frac{e^{-\lambda_1 x}}{\lambda_1^3}(\cos\lambda_1 x + \sin\lambda_1 x) - \frac{e^{-\lambda_2 x}}{\lambda_2^3}(\cos\lambda_2 x + \sin\lambda_2 x)\right]$$

$$\theta_2 = \frac{P}{8E_2I_2\beta} \times \frac{k_1}{E_1I_1}\left[\frac{e^{-\lambda_1 x}}{\lambda_1^2}\sin\lambda_1 x - \frac{e^{-\lambda_2 x}}{\lambda_2^2}\sin\lambda_2 x\right]$$

$$M_2 = \frac{P}{8\beta} \times \frac{k_1}{E_1I_1}\left[\frac{e^{-\lambda_1 x}}{\lambda_1}(\cos\lambda_1 x - \sin\lambda_1 x) - \frac{e^{-\lambda_2 x}}{\lambda_2}(\cos\lambda_2 x - \sin\lambda_2 x)\right]$$

$$Q_2 = \frac{P}{4\beta} \times \frac{k_1}{E_1I_1}\left[e^{-\lambda_1 x}\cos\lambda_1 x - e^{-\lambda_2 x}\cos\lambda_2 x\right]$$

$$\text{where, } D_1 = \frac{k_1}{E_1I_1} - (\alpha - \beta); \quad D_2 = \frac{k_1}{E_1I_1} - (\alpha + \beta);$$

$$\alpha = \frac{A}{2}; \quad \beta = \sqrt{\frac{A^2}{4} - B}; \quad \lambda_1 = \sqrt[4]{\frac{\alpha + \beta}{4}}; \quad \lambda_2 = \sqrt[4]{\frac{\alpha - \beta}{4}};$$

$$A = \frac{k_1(E_1I_1 + E_2I_2) + k_2(E_1I_1)}{E_1I_1 \times E_2I_2}; \quad B = \frac{k_1k_2}{E_1I_1 \times E_2I_2}$$

These are the equations to determine all the four quantities for the lower beam and upper beam. It should be remembered that the upper beam is the real foundation and the lower beam is basically used to provide the connectivity between the springs. Now the question is how to decide the EI value for the lower beam. For upper beam, the EI value of the foundation can be used. So, what will be the EI value for the lower beam? Actually in the real case a very small EI value should be chosen for the lower beam so that it will not influence the deformation but provides connectivity among the springs.

Let us see an example to understand the effect of the EI value of lower beam on the deflection.

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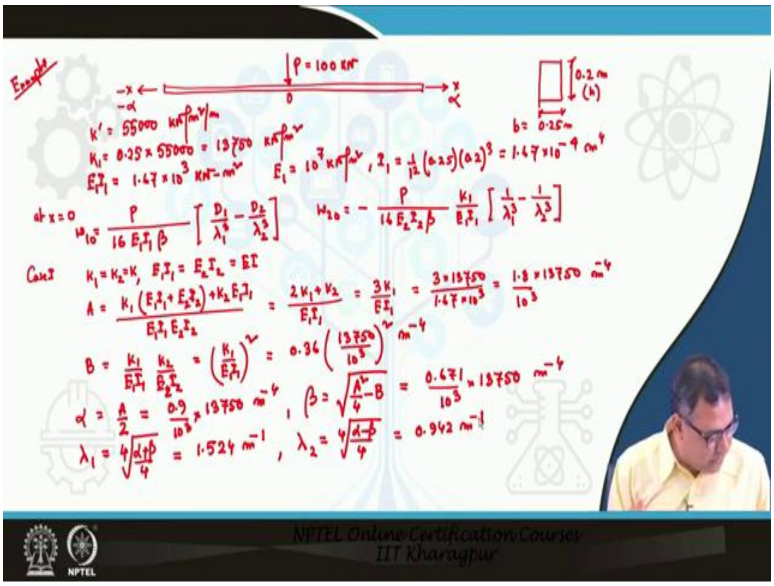

An infinite beam is subjected to a point load, P = 100 kN at point O. The cross section of the beam is 0.25 m × 0.2 m (width × height). As adopted in the previous example problems, k' value is 55000 kN/m^2/m. So, the k value will be (b × k′ = 0.25 × 55000) equal to 13750 kN/m^2. Flexural rigidity of the upper beam, E_1I_1 is 1.67 × 10^3 kN/m^2.

First, the deflection of the upper beam and lower beam will be determined at x = 0. The expression for w_1 reduces to the following when x = 0:

$$w_{1O} = \frac{P}{16E_1I_1\beta}\left[\frac{D_1}{\lambda_1^3} - \frac{D_2}{\lambda_2^3}\right]$$

Similarly, expression for w_2 when x = 0:

$$w_{2O} = \frac{P}{16E_2I_2\beta}\frac{k_1}{E_1I_1}\left[\frac{1}{\lambda_1^3} - \frac{1}{\lambda_2^3}\right]$$

To proceed further, the values of k_1 and k_2 are needed. Since this is a two layer soil system, both soil layers usually do not have the same k value. But let us see what happens if both are equal ($k_1 = k_2 = k$). Consider this as case-1 where $k_1 = k_2 = k$ and $E_1I_1 = E_2I_2$. So, the spring constant and flexural rigidity are same for both the layers in this case.

Case-1: $k_1 = k_2 = k$ and $E_1I_1 = E_2I_2 = EI$

$$A = \frac{k_1(E_1I_1 + E_2I_2) + k_2(E_1I_1)}{E_1I_1 \times E_2I_2} = \frac{2k_1 + k_2}{EI} = \frac{3k}{EI}$$

$$\Rightarrow A = \frac{3 \times 13750}{1.67 \times 10^3} = \frac{1.8}{10^3} \times 13750 m^{-4}$$

$$B = \frac{k_1k_2}{E_1I_1 \times E_2I_2} = \left(\frac{k}{EI}\right)^2 = 0.36 \times \left(\frac{13750}{10^3}\right)^2 m^{-4}$$

$$\alpha = \frac{A}{2} = \frac{0.9}{10^3} \times 13750 m^{-4}; \quad \beta = \sqrt{\frac{A^2}{4} - B} = \frac{0.671}{10^3} \times 13750 m^{-4}$$

$$\lambda_1 = \sqrt[4]{\frac{\alpha + \beta}{4}} = 1.524 m^{-1}; \quad \lambda_2 = \sqrt[4]{\frac{\alpha - \beta}{4}} = 0.942 m^{-1}$$

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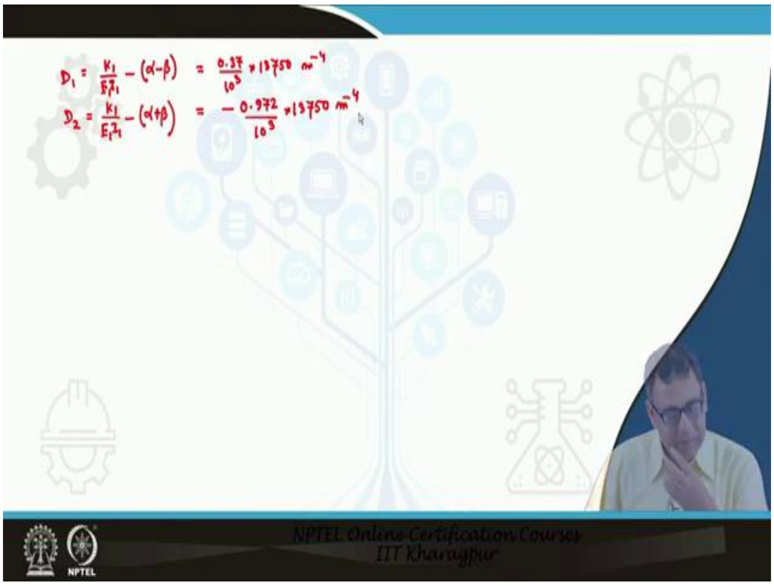

Now, the D_1 and D_2 values can be calculated by:

$$D_1 = \frac{k_1}{E_1 I_1} - (\alpha - \beta) = \frac{0.37}{10^3} \times 13750 m^{-4}$$

$$D_1 = \frac{k_1}{E_1 I_1} - (\alpha + \beta) = -\frac{0.972}{10^3} \times 13750 m^{-4}$$

Till now I have determined all the coefficients and in the next class, I will put these coefficients in the expressions to determine the deflection at x = 0 for lower beam as well as upper beam. Thank you.