Soil Structure Interaction
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Lecture 19
Beams on Elastic Foundation (Contd.)

In the previous class I derived the expression for deflection, slope, bending moment and shear force for a beam. But those expressions are not solved for, as four constants C_1, C_2, C_3 and C_4 are still left as unknowns. In this class, I will first discuss how to determine those unknowns then I will continue with the beam problem for different cases.

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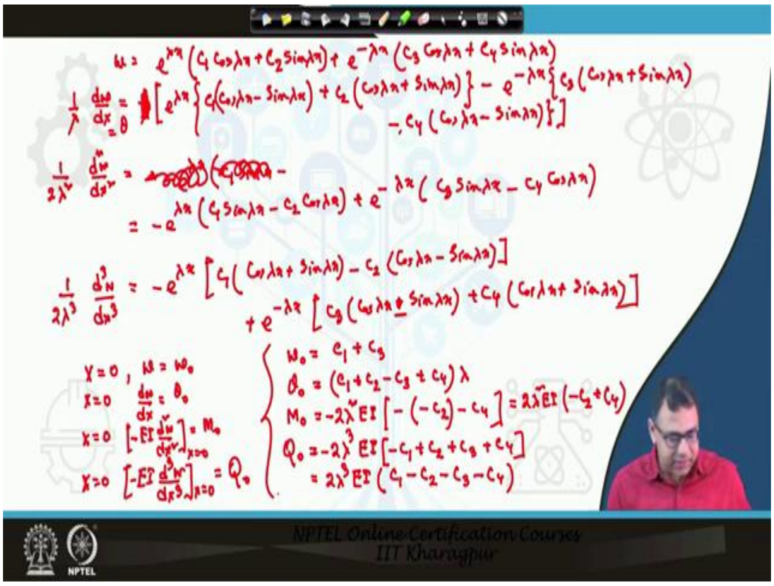

The expressions derived in the last class were:

$$w = e^{\lambda x}\left[C_1 \cos\lambda x + C_2 \sin\lambda x\right] + e^{-\lambda x}\left[C_3 \cos\lambda x + C_4 \sin\lambda x\right]$$

$$\frac{dw}{dx} = \frac{1}{\lambda}\left[e^{\lambda x}\left\{C_1(\cos\lambda x - \sin\lambda x) + C_2(\cos\lambda x + \sin\lambda x)\right\} - e^{-\lambda x}\left\{C_3(\cos\lambda x + \sin\lambda x) - C_4(\cos\lambda x - \sin\lambda x)\right\}\right]$$

$$\frac{1}{2\lambda^2}\frac{d^2w}{dx^2} = -e^{\lambda x}\left[C_1 \sin\lambda x - C_2 \cos\lambda x\right] + e^{-\lambda x}\left[C_3 \sin\lambda x - C_4 \cos\lambda x\right]$$

$$\frac{1}{2\lambda^3}\frac{d^3w}{dx^3} = -e^{\lambda x}\left\{C_1(\cos\lambda x + \sin\lambda x) - C_2(\cos\lambda x - \sin\lambda x)\right\} + e^{-\lambda x}\left\{C_3(\cos\lambda x + \sin\lambda x) + C_4(\cos\lambda x + \sin\lambda x)\right\}$$

As already mentioned, boundary conditions will be used now to solve the above equations or to determine the values of the unknown constants. To start with, the values of the four required quantities (w, θ, M and Q) can be found out at x = 0 (w_o, θ_o, M_o and Q_o) by substituting the x value in the above four expressions. If x = 0 is substituted in the above expressions, they reduce to:

$$\left\langle \begin{array}{l|l} x=0;\ w=w_{\circ} & w_{\circ}=C_1+C_3 \\ x=0;\ \dfrac{dw}{dx}=\theta_{\circ} & \theta_{\circ}=\left(C_1+C_2-C_3+C_4\right)\lambda \\ x=0;\ \left[-EI\dfrac{d^2w}{dx^2}\right]_{x=0}=M_{\circ} & M_{\circ}=-2\lambda^2 EI\left[-(-C_2)-C_4\right] \\ & \quad =2\lambda^2 EI\left(-C_2+C_4\right) \\ x=0;\ \left[-EI\dfrac{d^3w}{dx^3}\right]_{x=0}=Q_{\circ} & Q_{\circ}=-2\lambda^3 EI\left[-C_1+C_2+C_3+C_4\right] \\ & \quad =2\lambda^3 EI\left(C_1-C_2-C_3-C_4\right) \end{array} \right\rangle$$

It can be observed that only the bending moment and shear force expressions have the EI term because of their formulae. The second order and third order derivatives do not have the EI term and hence it was multiplied on both sides. So, by using the boundary condition, x = 0, four equations were obtained which can be used to determine the four unknowns.

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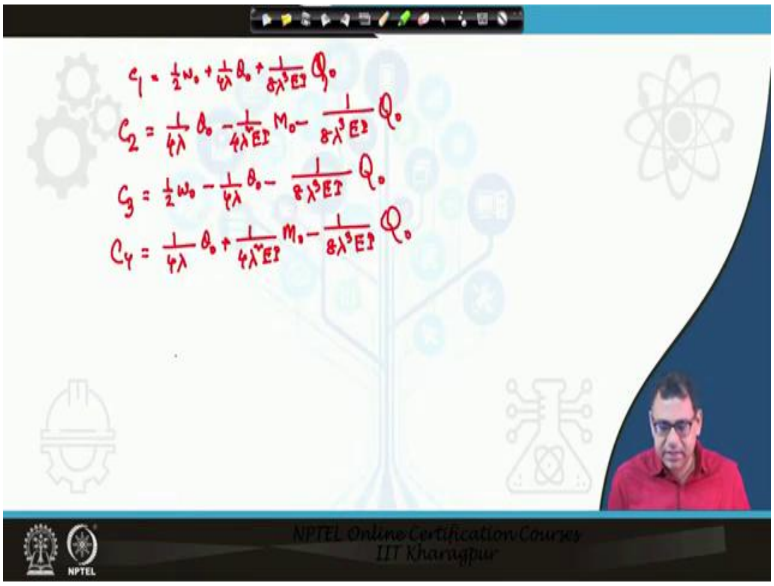

By solving the 4 equations, we get the values of the four constants:

$$C_1=\frac{1}{2}w_{\circ}+\frac{1}{4\lambda}\theta_{\circ}+\frac{1}{8\lambda^3 EI}Q_{\circ}$$

$$C_2=\frac{1}{4\lambda}\theta_{\circ}-\frac{1}{4\lambda^2 EI}M_{\circ}-\frac{1}{8\lambda^3 EI}Q_{\circ}$$

$$C_3=\frac{1}{2}w_{\circ}-\frac{1}{4\lambda}\theta_{\circ}-\frac{1}{8\lambda^3 EI}Q_{\circ}$$

$$C_4=\frac{1}{4\lambda}\theta_{\circ}+\frac{1}{4\lambda^2 EI}M_{\circ}-\frac{1}{8\lambda^3 EI}Q_{\circ}$$

So, the constants C_1, C_2, C_3 and C_4 can be calculated, if the values of w_o, θ_o, M_o and Q_o are known. Now, by substituting these constants values in the four basic equations given in the beginning of this lecture, the bending moment, shear force, slope and deflection can be determined. So these are the basic equations from that can be solved to determine any value.

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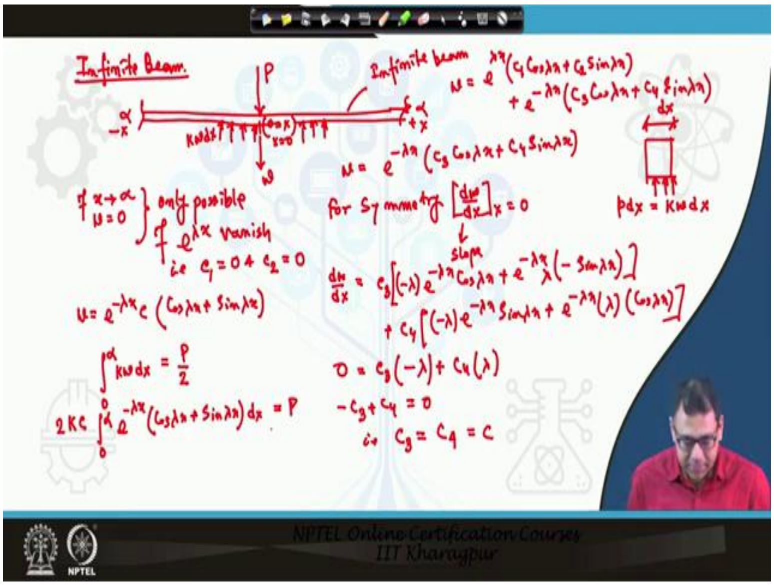

This can be extended to the infinite beam problems. An infinite beam is a beam which is extended to infinite length (or is infinite in the x direction). The positive x direction (+x) may be considered to be towards the right side of the beam and the negative x direction (-x), towards the left. Consider a point load, P acting on the beam and the point where P is acting is considered as the x = 0 point. Any point on the beam will be defined by the distance between that particular point and the point of loading (x = 0). For example, if a point A is at a distance of 'a' units from the point load towards the right, the x value is 'a' (x = a) for point A. Similarly if a point B is at a distance of 'a' units from the point load towards the left, the x value is '-a' (x = -a) for point B.

So, if the beam is extended till infinity in both the directions (+x & -x), the effects of the load will not be felt at the ends of the beam. It means that due to the infinite distance between the load and the beam ends, the deflection, slope, bending moment and shear force induced by the load will become 0 at x =∞.

The expression for deflection already derived is:

$$w = e^{\lambda x}[C_1 \cos \lambda x + C_2 \sin \lambda x] + e^{-\lambda x}[C_3 \cos \lambda x + C_4 \sin \lambda x]$$

This expression is very important and is recommended to memorise. As explained, at x = ∞, w = 0 but this is possible only when the $e^{\lambda x}$ term disappears (or equal to 0). This can happen only when C_1 and C_2 are zero because for no value λx both cosine and sine terms will be zero. So, from this boundary condition, it is found that:

$$C_1 = C_2 = 0$$

So, the expression for deflection reduces to:

$$w = e^{-\lambda x}\left[C_3 \cos \lambda x + C_4 \sin \lambda x\right]$$

This is the deformation expression for the infinite beam. As there are two unknowns still, two more boundary conditions should be applied to solve this and determine the final expression for deflection. The first boundary condition that will be used here is the symmetry. As the beam is extended to infinite lengths on both sides of the load, the deflection profile will be symmetric about x = 0 (mirror image about x = 0). If the deflection profile is symmetric, the slope will be zero.

$$\left[\frac{dw}{dx}\right]_{x=0} = 0$$

Differentiating the above expression of w with respect to x, we get:

$$\frac{dw}{dx} = C_3\left[(-\lambda)e^{-\lambda x} \cos \lambda x + \lambda e^{-\lambda x}(-\sin \lambda x)\right] + C_4\left[(-\lambda)e^{-\lambda x} \sin \lambda x + \lambda e^{-\lambda x} \cos \lambda x\right]$$

As the slope is zero at x = 0, substitute dw/dx =0 and x = 0 in the above expression:

$$\Rightarrow 0 = C_3(-\lambda) + C_4(\lambda)$$

$$\Rightarrow -C_3 + C_4 = 0$$

$$\therefore C_3 = C_4 = C$$

Now there is only one unknown, C and so we need another boundary condition to determine this value. Due to the load applied on the beam, there will be a reaction developed from the soil that acts on the lower side of the beam. This reaction multiplied by the distance over which it is acting will be equal to the load applied which, in this case is P. If the reaction is considered from x = 0 to one infinity side (+∞ or -∞), it will be exactly half the load.

$$\int_0^{\infty} kw\,dx = \frac{P}{2}$$

Substituting the w value in the above expression, we get:

$$2kC\int_0^{\infty} e^{-\lambda x}\left[\cos \lambda x + \sin \lambda x\right]dx = P$$

To determine the C value from the expression, the integral function should be evaluated.

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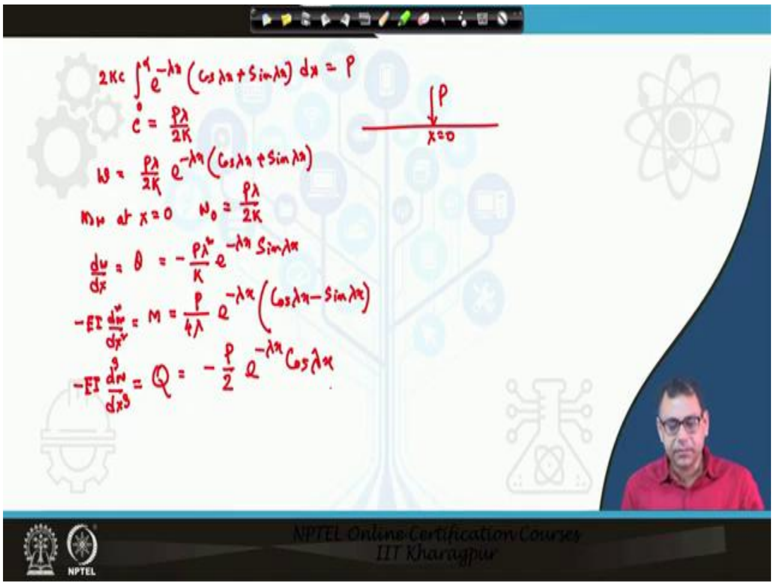

After integration, the C value will be:

$$C = \frac{P\lambda}{2k}$$

Substituting the above value of C in the expression for deflection, we get:

$$w = \frac{P\lambda}{2k} e^{-\lambda x} [\cos \lambda x + \sin \lambda x]$$

This is the final expression or the formula to determine the deflection at any point in an infinite beam, when it is subjected to a point load, P.

Now, if x = 0, w = w_o and that value of w_o is:

$$w_{\circ} = \frac{P\lambda}{2k}$$

So the deformation of the beam at the point of application of the load, P is as given above. If the above expression for deflection is differentiated with respect to x, the expression for slope (θ) can be determined.

$$\frac{dw}{dx} = \theta = -\frac{P\lambda^2}{k} e^{-\lambda x} \sin \lambda x$$

Similarly, the expression for bending moment can be determined as:

$$-EI \frac{d^2 w}{dx^2} = M = \frac{P}{4\lambda} e^{-\lambda x} [\cos \lambda x - \sin \lambda x]$$

Similarly, the expression for shear force can be determined as:

$$-EI \frac{d^3 w}{dx^3} = Q = -\frac{P}{2} e^{-\lambda x} \cos \lambda x$$

These are the expressions for the four quantities: deflection, slope, bending moment and shear force. In the next class I will show you where these quantities will be maximum and minimum or 0 in a beam. Thank you.