Soil Structure Interaction
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Lecture 18
Beams on Elastic Foundation

In last lecture I discussed about the different types of beams and plates. I also discussed about the application of those different types of structural elements in real foundation or the pavement design. In this class I will derive the basic differential equation of a beam resting on Winkler springs.

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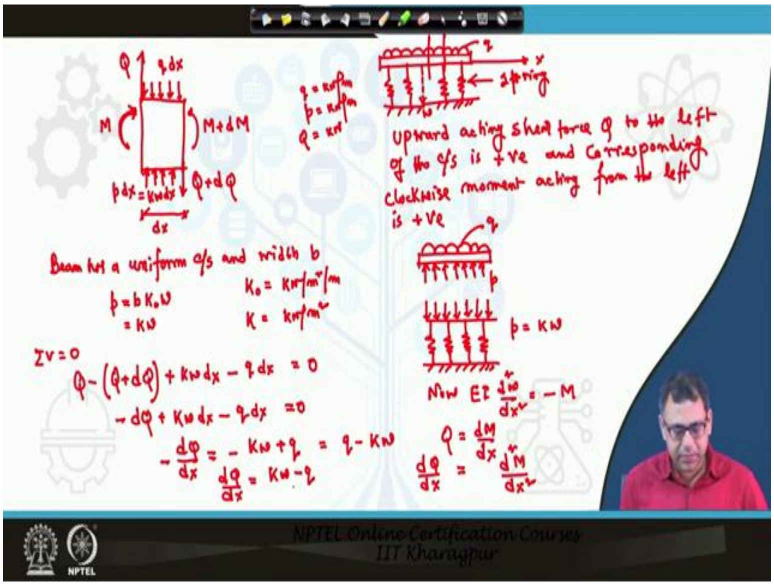

The problem already briefed in the last class will be considered here. The beam resting on Winkler springs upon which a UDL, q is acting is shown in the above figure. When a small element of the beam is considered, all the forces that act on this element are drawn and explained already. A UDL (q) on the top, a reaction of springs (p) on the bottom, an upward shear force (Q) & clockwise moment (M) on the left and a downward shear force (Q + dQ) & anti clockwise moment (M + dM) on the right side of the element.

The units of q will be kN/m as it is the intensity of UDL and so, the total load acting on the element due to the UDL is q × dx. Similarly the units of p will be kN/m and hence the total load or force due to the spring reaction will be p × dx.

According to the sign convention followed, the upward acting shear force, Q to the left of the cross section of the beam is considered positive. That means shear force acting upward and is to the left of the beam (or observation point) should be considered positive. Similarly the clockwise moment acting on the left is again positive i.e., M is positive. That means a clockwise moment acting on the left side of the beam (or observation point) should be considered positive.

The beam is considered to be having uniform cross section of width, b. As the beam is resting on Winkler springs, the reaction force from the springs is nothing but:

$$p = k \times w$$

(OR)

$$p = b \times k_{\circ} \times w$$

where, k is the stiffness of the spring or the subgrade modulus, b is the width of the beam and w is the deflection or deformation. The units of k_o are $kN/m^2/m$ and when it is multiplied by the beam width, it gives the k value (in kN/m^2) which we can use for computations. Remember that from here on, the units of k_o will be $kN/m^2/m$ and the units of k will be kN/m^2.

Considering the vertical forces summation, it should be 0 (for equilibrium), $\sum V = 0$:

$$Q - (Q + dQ) + kwdx - qdx = 0$$

$$-dQ + kwdx - qdx = 0$$

$$-\frac{dQ}{dx} = -kw + q$$

$$\Rightarrow \frac{dQ}{dx} = kw - q$$

It is known from basic definitions that:

$$EI\frac{d^2w}{dx^2} = -M$$

The rate of change of bending moment can be called as the shear force:

$$Q = \frac{dM}{dx}$$

$$\Rightarrow \frac{dQ}{dx} = \frac{d^2M}{dx^2}$$

Substituting the value of M in the above expression:

$$\Rightarrow \frac{dQ}{dx} = \frac{d^2M}{dx^2} = -EI\frac{d^4w}{dx^4}$$

Now substitute this value of dQ/dx in the equilibrium equation of vertical forces:

$$-EI\frac{d^4w}{dx^4} = kw - q$$

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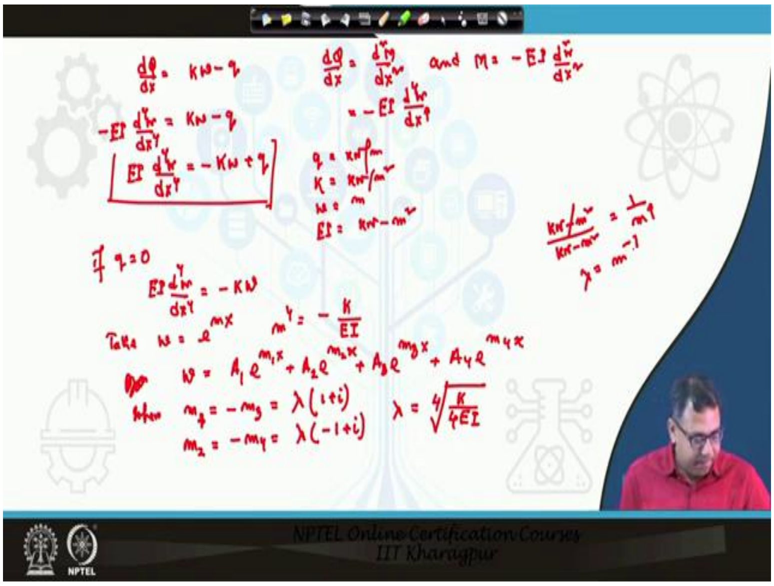

So the expression for beam will be:

$$EI\frac{d^4w}{dx^4} = -kw + q$$

The units of each term are given as: q - kN/m, k - kN/m^2, w – m, EI – kN-m^2.

If there is no UDL acting or if q = 0:

$$EI\frac{d^4w}{dx^4} = -kw$$

$$\text{Let } w = e^{mx} \quad \Rightarrow m^4 = -\frac{k}{EI}$$

$$w = A_1 \times e^{m_1 x} + A_2 \times e^{m_2 x} + A_3 \times e^{m_3 x} + A_4 \times e^{m_4 x}$$

$$\text{where, } m_1 = -m_3 = \lambda\,(1+i)$$

$$m_2 = -m_4 = \lambda\,(-1+i)$$

$$\lambda = \sqrt[4]{\frac{k}{4EI}}\ \text{m}^{-1}$$

Substitute the values of m coefficients in the equation for w.

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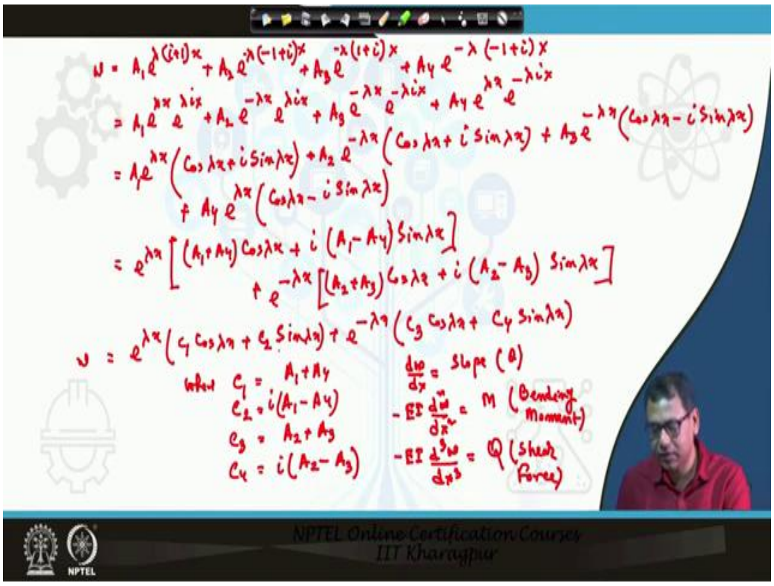

$$\Rightarrow w = A_1 \times e^{\lambda(1+i)x} + A_2 \times e^{\lambda(-1+i)x} + A_3 \times e^{-\lambda(1+i)x} + A_4 \times e^{-\lambda(-1+i)x}$$

$$\Rightarrow w = A_1 e^{\lambda x} e^{\lambda i x} + A_2 e^{-\lambda x} e^{\lambda i x} + A_3 e^{-\lambda x} e^{-\lambda i x} + A_4 e^{\lambda x} e^{-\lambda i x}$$

$$\Rightarrow w = A_1 e^{\lambda x}(\cos\lambda x + i\sin\lambda x) + A_2 e^{-\lambda x}(\cos\lambda x + i\sin\lambda x)$$

$$+ A_3 e^{-\lambda x}(\cos\lambda x - i\sin\lambda x) + A_4 e^{\lambda x}(\cos\lambda x - i\sin\lambda x)$$

$$\Rightarrow w = e^{\lambda x}\left[(A_1 + A_4)\cos\lambda x + i(A_1 - A_4)\sin\lambda x\right] - e^{-\lambda x}\left[(A_2 + A_3)\cos\lambda x + i(A_2 - A_3)\sin\lambda x\right]$$

$$\Rightarrow w = e^{\lambda x}\left[C_1\cos\lambda x + C_2\sin\lambda x\right] + e^{-\lambda x}\left[C_3\cos\lambda x + C_4\sin\lambda x\right]$$

This is the basic expression for the deflection of beam.

where, $C_1 = A_1 + A_4$ $\qquad \dfrac{dw}{dx} = \theta \quad \{\text{Slope}\}$

$C_2 = i(A_1 - A_4)$ $\qquad -EI\dfrac{d^2w}{dx^2} = M \quad \{\text{Bending Moment}\}$

$C_3 = A_2 + A_3$ $\qquad -EI\dfrac{d^3w}{dx^3} = Q \quad \{\text{Shear Force}\}$

$C_4 = i(A_2 - A_3)$

The constants C_1, C_2, C_3 and C_4 should be determine by using the boundary conditions. Along with the deflection, three quantities should be determined for every case and they are: slope, bending moment and shear force. Slope is nothing but the rate of change of deflection and can be obtained by differentiating the expression for deflection with respect to x. Similarly, the bending moment and shear force are also defined above. By applying the appropriate constants and differentiating these equations, the expressions for bending moment and shear force can also be obtained.

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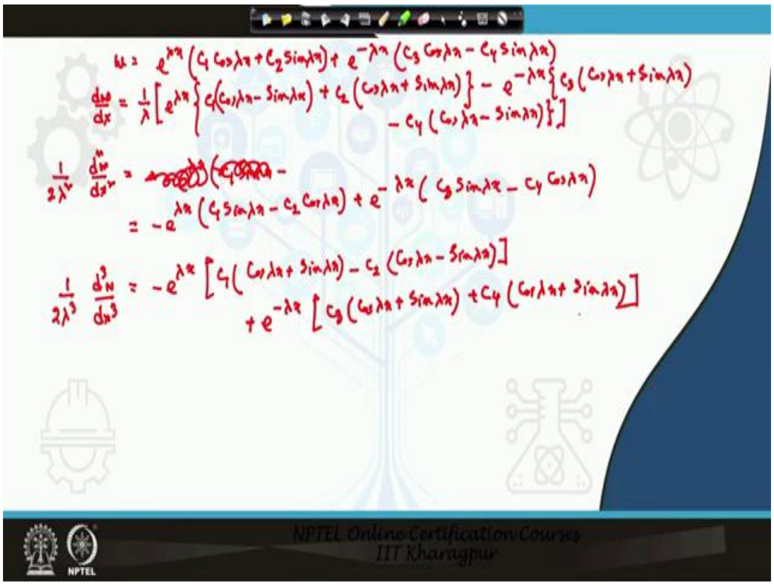

Differentiating the expression for deflection with respect to x, we get:

$$\frac{dw}{dx}=\frac{1}{\lambda}\left[e^{\lambda x}\left\{C_1(\cos\lambda x-\sin\lambda x)+C_2(\cos\lambda x+\sin\lambda x)\right\}-e^{-\lambda x}\left\{C_3(\cos\lambda x+\sin\lambda x)-C_4(\cos\lambda x-\sin\lambda x)\right\}\right]$$

Differentiating the above expression again with respect to x:

$$\frac{1}{2\lambda^2}\frac{d^2w}{dx^2}=-e^{\lambda x}\left[C_1\sin\lambda x-C_2\cos\lambda x\right]+e^{-\lambda x}\left[C_3\sin\lambda x-C_4\cos\lambda x\right]$$

Differentiating the above expression again with respect to x:

$$\frac{1}{2\lambda^3}\frac{d^3w}{dx^3}=-e^{\lambda x}\left\{C_1(\cos\lambda x+\sin\lambda x)-C_2(\cos\lambda x-\sin\lambda x)\right\}+e^{-\lambda x}\left\{C_3(\cos\lambda x+\sin\lambda x)+C_4(\cos\lambda x+\sin\lambda x)\right\}$$

From these expressions, the deflection, slope, bending moment and shear force can be determined. Note that the constants C_1, C_2, C_3 and C_4 are still unknowns which will be determined in the next class. Thank you.