Mass, Momentum and Energy Balances in Engineering Analysis

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Lecture - 43
Selective Mathematical Concepts in Transport Phenomena

Welcome. So far we have studied all the theories of the heat transfer, mass transfer and energy transfer for the balance. And many a times you see that we are landing up with many kind of mathematical problems. And in those problems, we find that we are sometimes representing all those model equations, for example Navier-Stokes equation in terms of some partial differential equations, and sometimes we are reducing them to ordinary differential equation. In any case, what we find that we are sometimes having the some scalar quantities like the temperature, the concentration etcetera, and sometimes, we are having some vector quantities like the velocity, momentum, force, etcetera.

Now, whatever we are doing, ultimately we are having some kind of mathematical equations to be solved for. And whenever you are going for solving in the books in the literature, you will find that many notations are used to represent all these questions. So, in conclusion of this particular subject, where we have learned about the various model equations, how conservational laws, it is also very important for you to understand that how to read such kind of problem statements mathematically.

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So, in this lecture, I shall be giving you some brief introduction to the various ways of representation of these model equations, which you encounter in the transport phenomena. So, what we shall look into, we shall look into this scalar, vector, tensor concepts, some gradients, divergence, curl, and Laplacian. These are very common understand this, these are dealt with in great detail in the mathematics courses. So, I will not be going into detail of this, I will just touch upon only the minimum knowledge you would require in context of the transport phenomena analysis.

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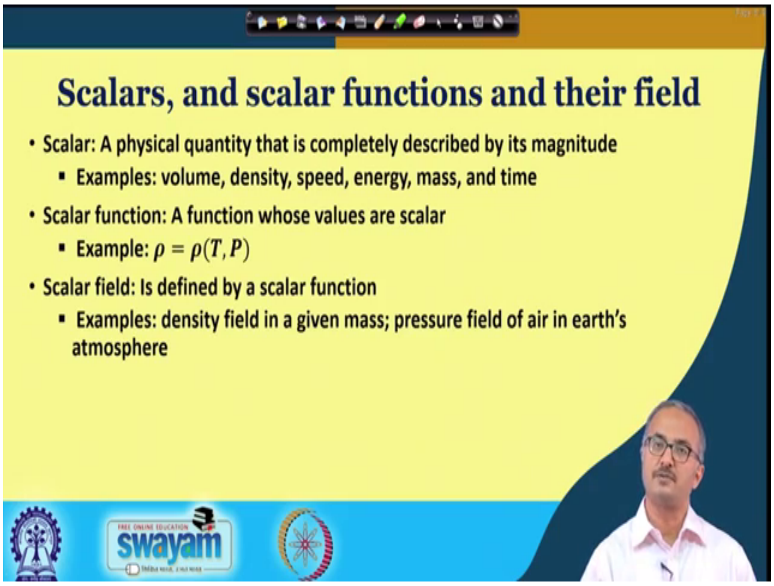

So, here we go back to our basic concepts of scalar vector. So, some things you might be knowing some things maybe new to you. Scalar perhaps all of you have studied that is scalar is any quantity or physical quantity that is given by some magnitude ok. And many examples are like volume, density, speed, energy, mass, time etcetera.

So, these are given by some kind of magnitude. Like for example, volume we say that say 2 cubic meter ok, density of water we say 1000 kg per cubic meter ok. So, speed of light is 3 into 10 to the power 8 meter per second ok. So, energy, energy can come in different ways some time many a times we talk of the latent heat of vaporisation, condensation, sometimes in terms of specific heat ok. So, a mass you know that we put in terms of say kg's or moles ok, and time you know it can be also second, minute, hour, days, years, weeks anything. So, all these things are having just some magnitude.

Then we have some scalar function. Now, scalar function what means is this, this those functions which have some scalar value, for example we talk of density. Now, we know that density in itself is a function of temperature, pressure of the particular system. So, we can represent density in terms of these variables like temperature, pressures, so it becomes a function now ok. And whenever we are plugging in the value of the temperature and pressure of the system, we get some value of the density. And we call this a density function ok, so that is how a the scalar is different from a scalar function ok, whenever it is we have putting the scalar in terms of some other variables.

Then we have scalar field. And this scalar field is defined by the scalar function. For example, if we are considering certain area some time domain in the space, it may so happen that from at different spatial locations the temperature, pressure may be different ok. So, in that case, we have to find the density at different locations with respect to the pressure, temperature. So, we will have some distribution of the density in that given region. So, in we say that this is a density field that means, the distribution of the density in the particular region, so because density is scalar. So, it is a scalar field ok. Similarly, we can have some variation of the pressure over the atmosphere, so we call it a pressure field.

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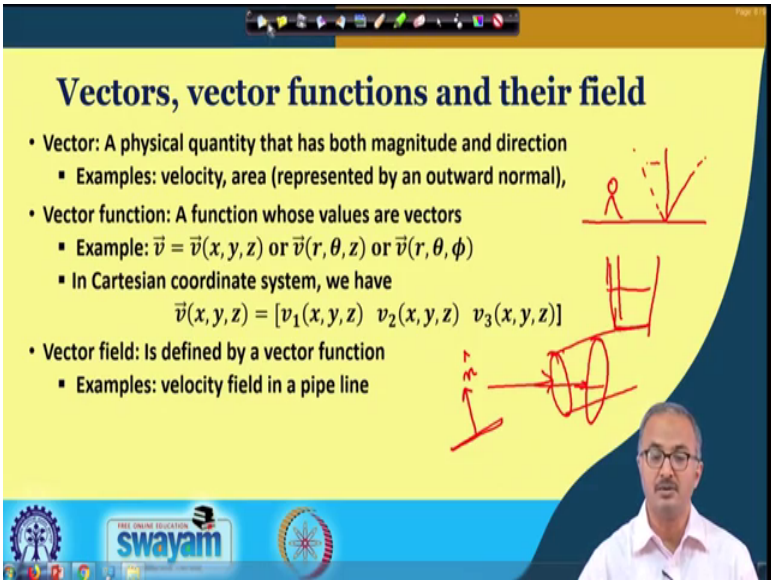

Next we come to the corresponding things in vector. In the vector, as you know that vectors has those physical quantities, which have both magnitude and direction, for example velocity. Velocity is having a magnitude, and also direction. Because, whenever even if the speed and velocity of the speed is scalar, speed has only magnitude, but velocity has both magnitude and direction that speed of a vehicle may remain same. But, if the vehicle is turning and going on the road, then it is constantly changing its velocity, even though its speed may be constant ok. So, velocity in that way becomes a vector.

Then area, area you see that area its seems many a times towards that it may be a scalar, but this is not so. Area is generally vector, because suppose with respect to some system how what is the orientation of the particular surface. So, for example if I take this particular line in this particular line, and suppose your standing here, and you have a mirror over here ok.

Now, you see that depending on how this mirror is seated, whether it is slanted or whether it is centered like this, you will find, you will having different kind of images. So, what you find that this particular even though the area of the mirror is the same, but depending on your location, you will find different images. So, these you cannot extent for any kind of say flow or some kind of race.

So, you will sign that if there is a glass here since we have a glass here full of some water some water is there in the glass, now depending on the tilt you are giving, the amount of water flow will keep changing or some fluid is entering such kind of a system ok. At suppose you have some fluid entering some system, and like this is the pipe line its entering ok.

So, there is some kind of in means angle. So, you can easily see that if this is pipeline like this or this pipeline is like this straight ok, so that we will find the different amounts of the mass will be flowing into the pipeline ok. So, this area is generally is direction is taken to be the outward normal that means, you take any area the outward normal to it is its direction ok, so that means it has the area this particular thing, and the rational normal that is how it becomes a vector. And this is very important in your analysis as as you will see in your transport phenomena.

Now, vector function is a function, whose value is a vector, for example your velocity. And you say that velocity is changing over the space in a given domain in the given region, then you can see that you have to calculate the velocity at different places ok. So, this is the velocity distribution ok. So, you that way you are getting the velocity function, as if I have shown you in terms of x, y, z in the Cartesian coordinate, in terms of r theta z in cylindrical coordinate, in terms of r theta phi in spherical coordinate whatever it may be, it is showing you the spatial distribution of the velocity.

And in the Cartesian coordinate, for example we are showing that this velocity being a vector, it will have three components. So, when I say velocity at x, y, z what it mean, the each of the components is changing with the position that means, here I say v 1, v 2, v 3 are the three components of the velocity. So, we have v 1 changing with x, y, z; v 2 changing with x, y, z; and v 3 changing with x, y, z ok. And vector field is defined an vector function. When we have distribution of velocities, we have a velocity field ok.

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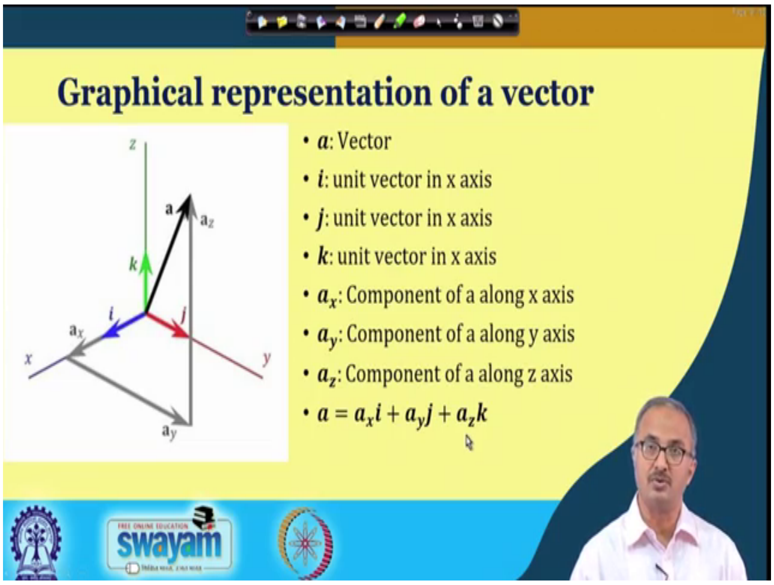

Next we go to see that how we represent a vector in a Cartesian coordinate, you can see this is the black line is showing the vector ok. And this is the vector a, and you see here we have x, y, and z coordinates. So, we are putting some unit vector that may had the magnitude of unity one ok, along each of the coordinates. So, here it is i in the x-direction, j in the y-direction, and k in the z-direction, this is the convention we follow ok.

Now, you see that in the three to reach this particular point given by this vector a, we can resolve it that means, we can make it divided into three components. So, first we can travelled in x-direction to a distance of a x ok. And then we travel the distance of a y in the y-direction. And we travel a distance of a z in the z-direction to reach this particular point ok. So, we write that a x into i that is a vector in the x-direction, then a y into j in the y-direction, and a z into a in the z-direction. So, this particular vector a is given by this three components in the Cartesian coordinate. Similar things you can also do for the cylindrical coordinate, and for the r theta phi coordinate that is the spherical coordinate.

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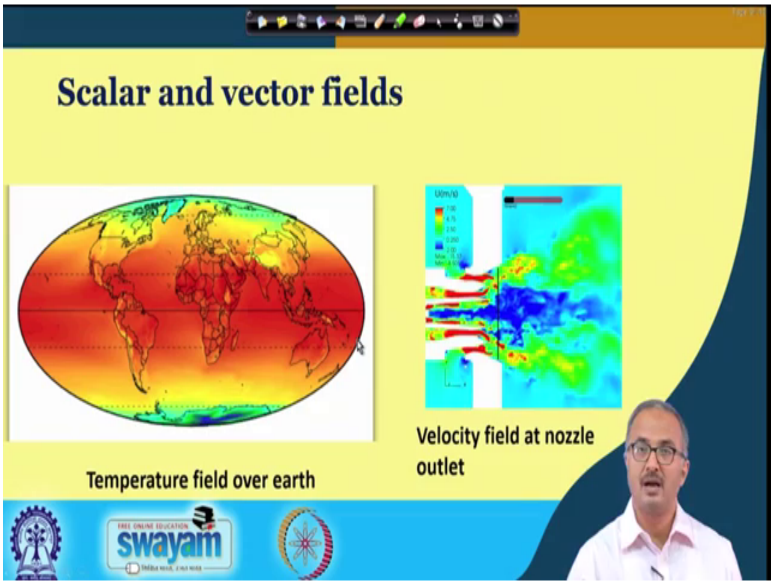

Next, I show some examples to you of the fields. So, this is the scalar field for the temperature field over the earth. So, this is earth and you can see that how the temperature differs in the along the various space that means, along various continents and the ocean ok. So, you can see that there are and these different colours are showing different magnitudes of the temperature. And this kind of colour colouring, we will see many a times in the books in the literature to tell you the distribution of the temperature.

In this particular thing, we are showing a velocity field at nozzle some exhaust gas is coming out. For example, when a satellite it is launched, the exhaust gas come out from the tail, you can see those when you see picture on a TV, you see that some gas is coming out. And that generates thrust, which propels the particular satellite in the space ok.

And if you can measure this particular velocities or model, then you find this kind of velocity distribution is there, and you can read some of this velocity values in given meter per second. And you can see that they are not in magnitude, but also the directions they are spreading ok. So, you see that these values are different, and their directions are also different ok. So, this is the typical example of the velocity field.

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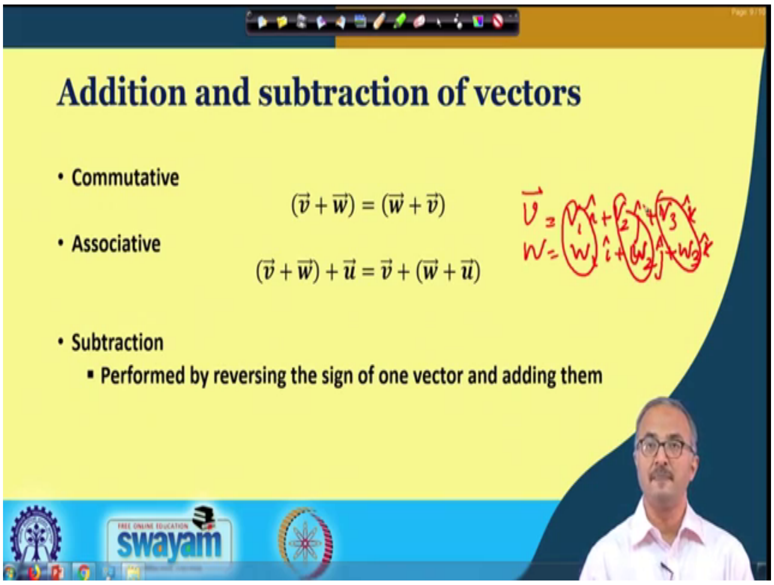

Next we come to some very basic operations involving the vectors, and some properties. First is the commutative property, it means you can commute, commute means you can exchange the things. Like if we have two vectors v vector, and w vector that means, if v plus w is equal to w plus v.

And when we say this addition, what we are basically adding is the component wise that means, if v is taken to be if a v is taken as some v 1 i, then v 2 j plus v 3 k. And w is taken to be w 1 i, then w 2 j plus w 3 k. The addition means, we shall be adding these two, adding these two, and adding these two in the three direction separately. And that is how we shall get the resultant vector ok.

Now, the addition is commutative. Commutative means, the order of the addition does not matter. Then associative, now you see here that v plus w three vectors we have v u, v, w. So, if we first add v and w, and then resultant vector is added to this u, then we shall be getting the same thing. If you take any other combination that means, if I have this v is outside first we add this w and u, and then we add this v. And you can see that whenever we are adding these two vectors, the resultant vector will have the in general both the magnitude as well as the direction different ok, you can check it this way.

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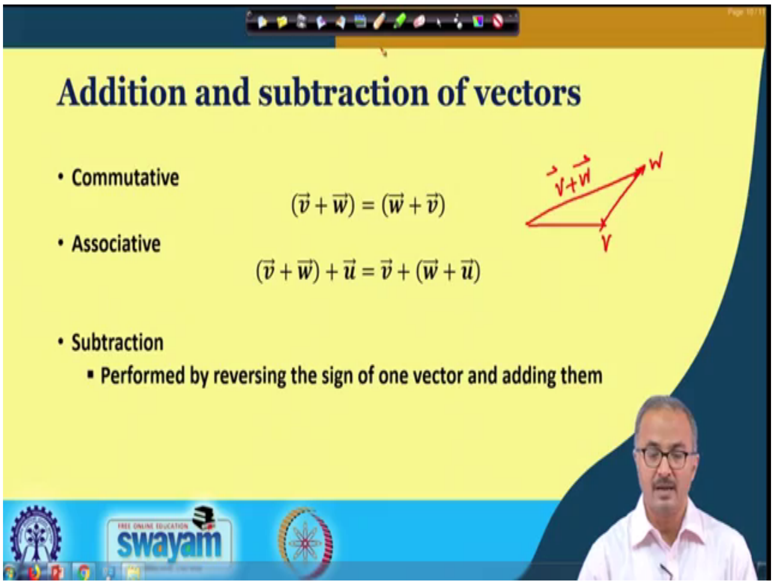

Like suppose, I may adding these two vectors, so suppose this is the v vector, and then I have the w vector ok. The resultant addition is like this ok. So, this is the v plus w ok. Now, you can see what v vector has some direction with respect to some coordinate w has some other direction, and v plus w has some other direction. So, it does not matter whether you are doing from this plus this or this plus this using that the same magnitude, and the same direction ok. Similarly, you can extend it for this kind of additions.

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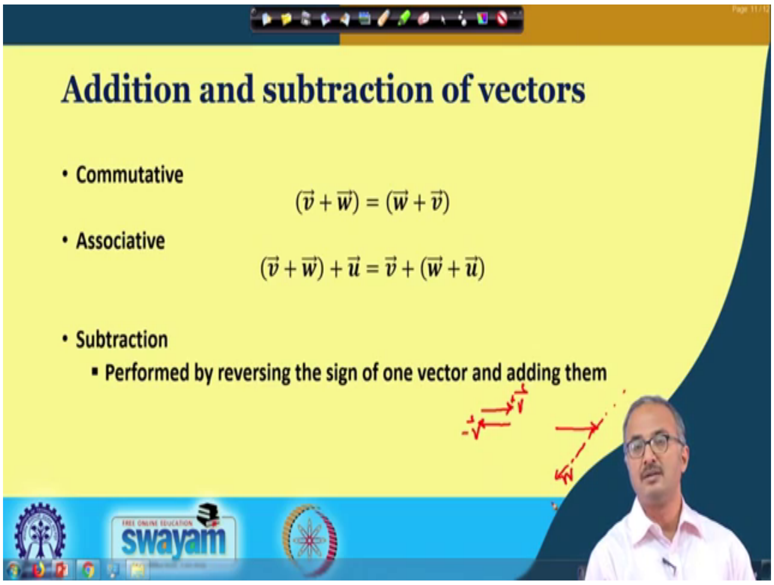

Now, when we talk of the vector subtraction, what it basically means subtract means, say v minus w. oh Negative of a vector means what that it has a same magnitude, but the direction is just opposite that is 180 turn ok. So, if I say that I have a vector like this ok.

This is if this is plus v, then just opposite in that direction, we have minus v vector ok. Just we change the direction without changing the magnitude, so that is how we say that negative sign. So, once we know the negate v minus w means, you have to draw v in the same direction as before, like I draw it like this. And w earlier using this, now we will put in this direction this is w ok. So, this is how we are making this subtraction of the two vectors.

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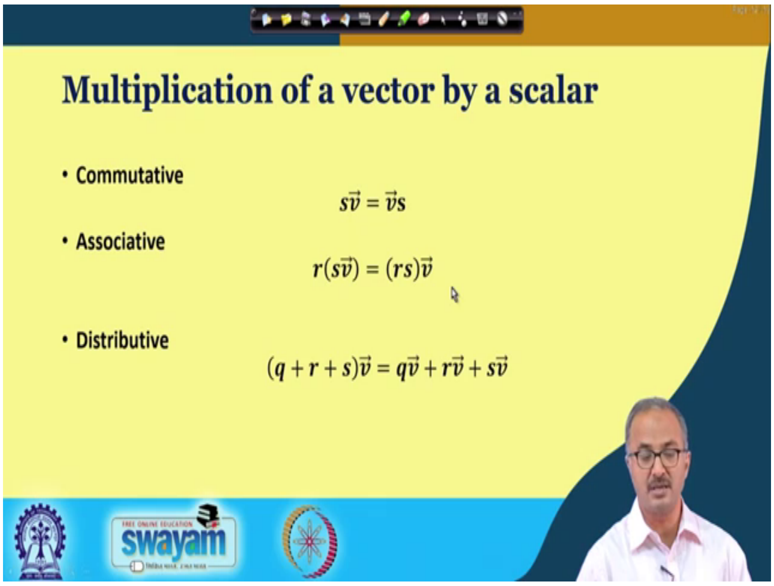

Next we come to multiplication of a vector by a scalar. So, we have a scalar s, and when we multiply this with the vector, we find this is also commutative ok. So, whether you have s first, and then v or v first, and then s, it does not matter; ultimately the result will be the same.

Associative is that suppose, we have two scalars and one vector. So, first s and v, so first you multiply s with v, and resultant vector you multiply by r that will be having same vector as if you multiply first this r and s, and then multiply by v. And you can extend it other way also, you can put r into v, and then multiply by s. So, all of them will give you the same answer, so that is now we say it is a associative.

And then distributive; distributive is that first we have this three scalars q, r, s ok, and into the v. So, this is the addition of three scalars, and that an resultant we are multiplying by v. This can be same as that if you take first you multiply q with v, then r with v, s with v, and then add them up, the result will be the same. So, this is the distribution we are distributing this particular thing this distributive property.

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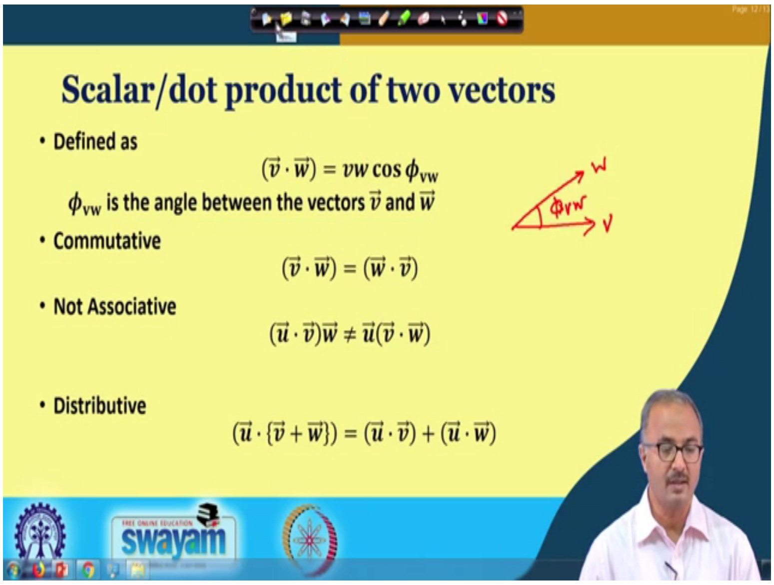

Then we have scalar product or dot product ok. In that what we have that in this case, we have for two vectors. So, if you have two vectors v and w, then v dot w is equal to v, w into cos phi v w, it means this cos phi v w is the angle between v and w that we can put in pictorially that if we have this v here, and if we have the w here, so we have this is the phi v w ok, this is the angle it representing ok. So, we are finding that this is the way we are defining the scalar product dot product, because here see the resultant will be a scalar ok.

Next we have the to see that whether this product of two vector dot product two vectors is commutative or not, here we see here it is commutative v dot w is equal to w dot v. Next we go to associative, how we are associating the various vectors. So, here we have now three vectors u, v, w. Here you can see that taking dot product first u dot v, then this is scalar now. This scalar is now multiplied by this w vector, this will not be the same. If you first multiply dot dot product of v and w that is a scalar again, and that is can take to be u. So, you find that the dot product of two vectors are not going into commutative way. And then distributive, we see the distribution is obeyed that is u dot v plus w is equal to u dot v plus u dot w ok.

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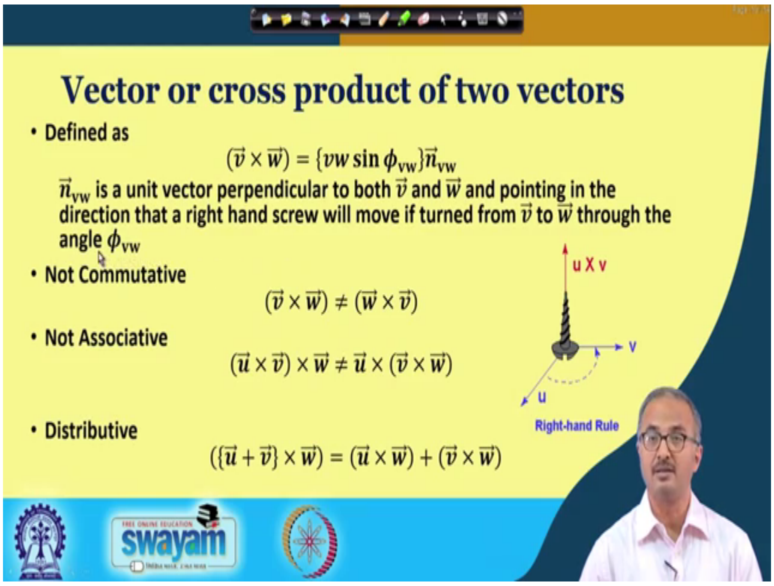

And then we have the cross products of the vector. The in this case we find that v cross w is defined like vw into sin vw into a vector n vw. Now, what is this n vw? n vw is a unit vector perpendicular to both v and w and pointing the direction that right and screw will move if turned from v to w through the angle this.

Now, you can see here that in this case, we have two vectors u u v instead of v w, we have u v. So, if you are making u cross v, then from u to v, if you put a screw your as if you are putting a screw, then you see this screw will move inside. So, this with the direction of the resultant vector u cross v ok, where magnitude will be u into v into sin this this angle sin of this particular thing ok. So, this is how we see that direction, and the magnitude of the cross product.

Then is it commutative v cross w and w cross v no, there are commutative ok. So, we should not v the order of the multiplication is important. Associative again we find now, it is the associative that means, u cross v, they first multiplied, then resultant is multiplied by w is not same as. First you multiply v and w, and then first you to the of u and cross v w. So, these are not associative.

Distributive yes we can find that u plus v, and then u multiply by w is same as u cross w plus v cross w. Here you should be aware that this w should not if you put w on the here, then it will not be the same, because w cross u, w cross v, and you find that w cross u is not equal to u cross w ok. So, this one and if you keep it at the at the front, then will not be the same.

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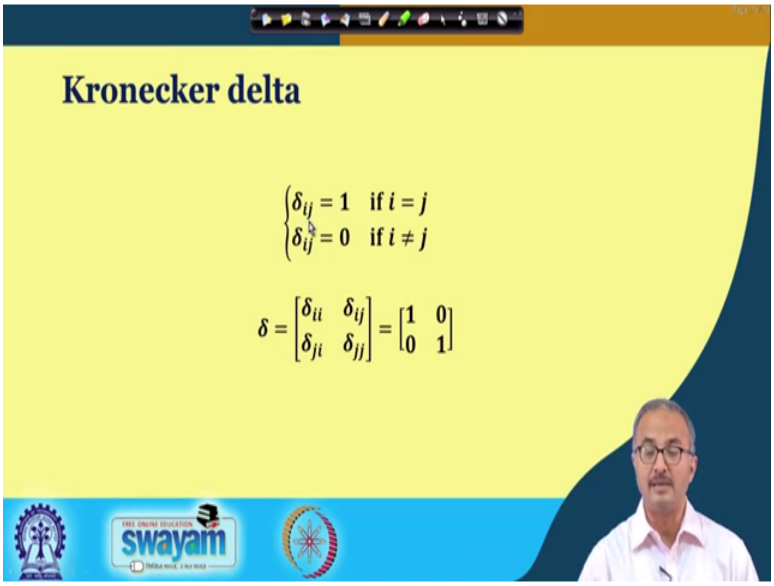

Now, here we put some notations to you, which are which must used in the vector algebra that is Kronecker delta, it is delta with i j subscript ok. And this defined as that if this i and j are the same, then we have 1. And if i is not equal to j, we have 0. And you can say here that this is given by some delta. And this Kronecker delta is like as if it is a matrix of d i i del i j del j i and del j j. Del i i same, so it is 1; del i j different 0; j i different 0; and j j same, it is 1 in a concise way.

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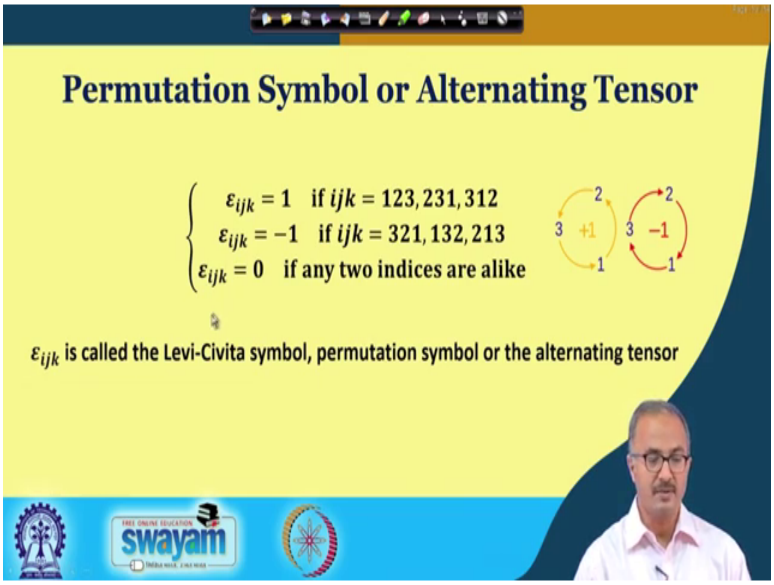

And then we have some permutation symbol or alternating tensor symbol, here you see that we are using some epsilon with i j k subscripts ok. And this is equal to 1 in this case, suppose we have 1, 2 and 3. So, if we take this direction 1 to 2, 2 to 3, 3 to 1 that means, 1 2 3, 1 2 3 or 2 3 1 or 3 1 2. So, whenever this is order of this epsilon, we get plus 1.

If the order is the other way around that means, 1 3, 3 2, and 2 1, then it will be minus 1 that is 3 2 1, 2 1 3, and 1 3 2 for this three cases, we find epsilon will be minus 1. And in case, these two of these are same that means, if 1 1 k or 2 2 i some kind of like this in that case, we will find this will be 0 ok. Sometimes this also called Levi-Civita symbol ok.

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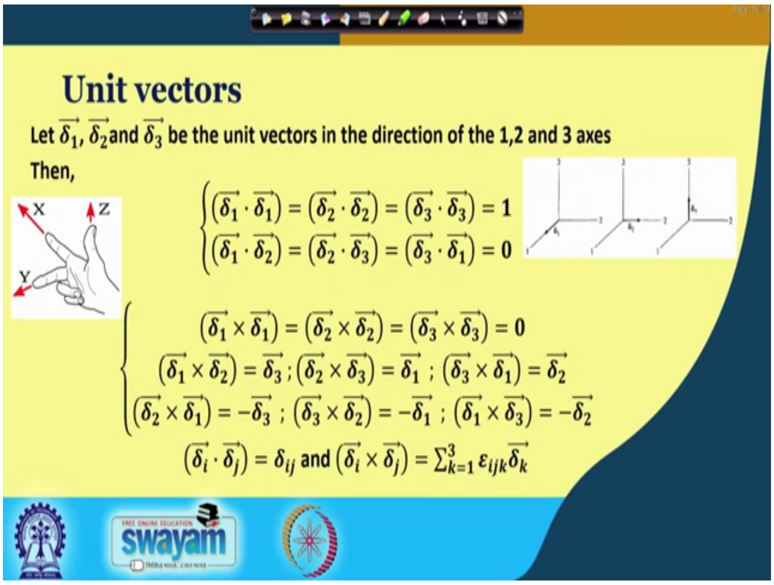

And now we shall see that, how this how this representation help us that now we have suppose, we have this dyadic, because dyadic vectors. These are nothing but i, j, k in in case of Cartesian coordinate ok. In theta coordinate also, they can be this is more generalised, it does not depend on the coordinate system. So, here we have the dou del 1, del 2, del 3 another vectors ok, there unit vectors.

So, you see that if you take the dot product of this and delta 1 dot delta 1, delta 2 dot delta 2, delta 3 dot delta 3, they are all coming to be 1 why, because the delta 1 into delta 1 means, their magnitude is 1 (Refer Time: 23:32) to 1 and cos 0 degree between the the two vectors, they are the same they are what are called co-linear, the the angle between them is 0. So, cos 0 is 1 so, we are finding they are values coming to be 1.

And if we have del 2 dot del del 1 dot del 2 del 2 del 3 or del 3 del 1, they are all having the 90 degree to each other perpendicular to each other perpendicular means, cos 90 is 1 ok. So, we find that due to cos 90 1. So, they are this dot product is coming to 0 ok. Now, here I have shown you this delta 1, delta 2, and delta 3 in the Cartesian coordinate. Now, here we show the cross 1. Delta 1 cross delta 1 delta 1 delta 2 cross delta 2, and delta 3 cross delta 3, they are coming to be 0 why, because sin 0 degree is 0 ok.

So, we are finding the cross product of the same thing will be 0. And then we go for the delta 1 and delta 2, it is giving delta 3; delta 2 delta 3 is giving delta 1; delta 3 delta 1 giving del 2, this same thing as we have just done for the rise to. Here we see that here we take the right hand thumb rule that here we have that if we put this kind of thing, if we extend our finger like this ok, like this we extend our finger. So, if this is z-direction, this x-direction, and this y-direction ok, so if you go for x to y, then you will get z. If you go y to z, it will give us x. If you go z to x, we will get this y ok.

So, if we alter these things that means, if you go from x to z, we shall be getting minus y. If you go from z to y, we shall get minus x on this side. And if you go y to x, we shall get minus z on the downward side that is how we are applying this right hand thumb rule to find out this dot product, so that is now what we see that 1 2 gives plus domain 3, then delta 3 gives plus delta 1 delta 3 delta 1 cross, this gives delta 2. And if you change the order, you see simply get the negative values. So, you can see that the order of the this placing this unit vectors is important ok. And now you can see that delta 1 dot delta j is called delta i j, and delta 1 cross this delta j is given by this permutation vector ok.

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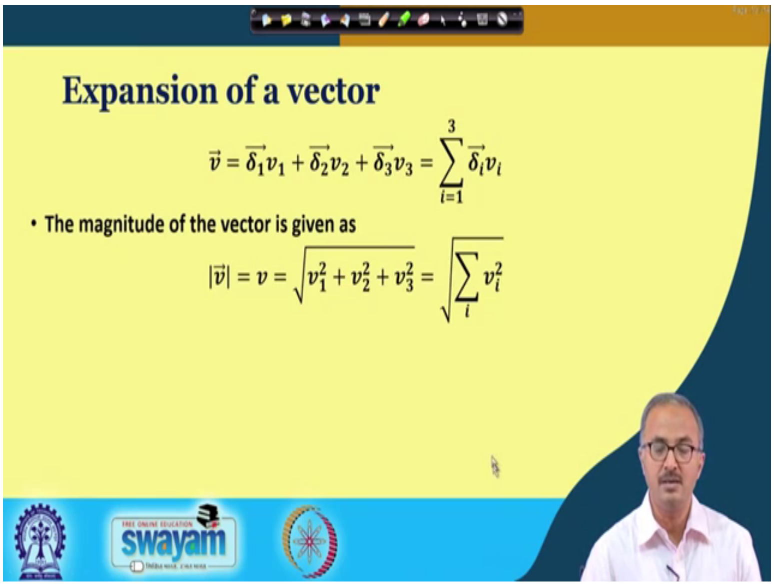

And here we are now expanding a vector in terms of this dyadic vector delta. So, here we can see that now we are writing in term dyadic vectors, and how is it helping us, it is helping us to make it very concise ok. So, we are just simply putting summation i 1, 2, 3, then delta i into v i ok. And the magnitude is like we know that we are just taking the square of each of the components, and taking the under root of that. First at the squares of each component, and take the under root we get the magnitude, and this we are putting in terms of this symbols as under square root of the v i square, the product summation of the product of the each of the components ok.

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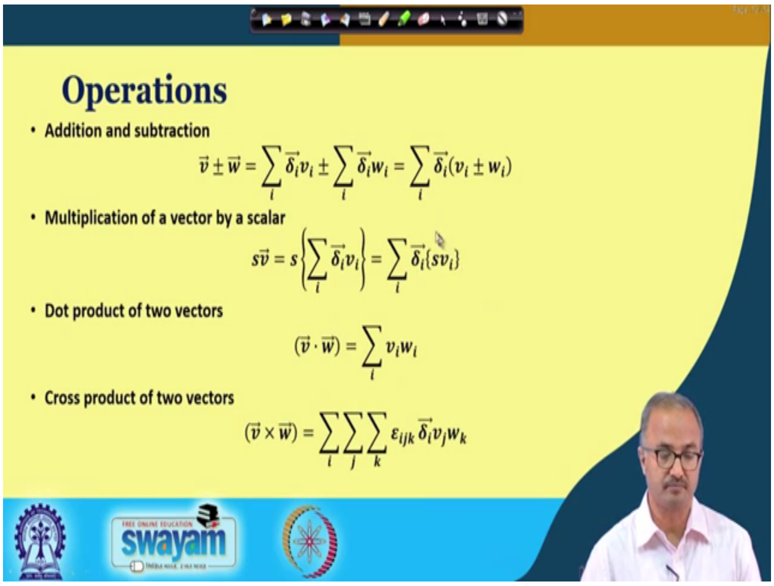

Now, we come to the operations. Here we have addition and subtraction, so it is v plus minus w as I told you that we first put in terms of this dyadic vectors. So, you can see that dyadic this delta i v i plus minus delta v i is equal to delta i v i plus minus w i. Similarly, multiplication we are putting in terms of dyadic thing, it is coming like this.

Then dot product of this is coming like v i w i and cross product coming like this epsilon i j k delta i v i v j and w k. So, we can see the putting this permutation and operator, and this chronicle delta, we are able to become a very concise way of putting this additions and summation of products.

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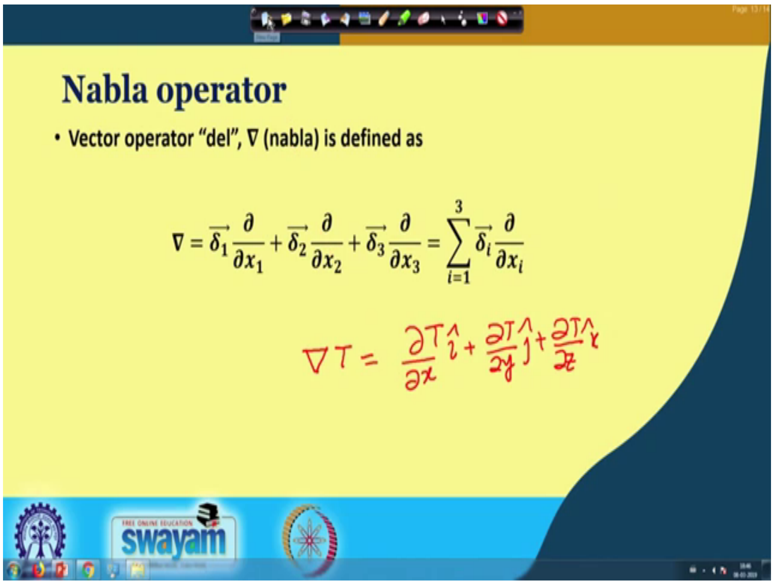

Next is nabla operator. Nabla operator is generally used to represent the partial derivatives, which we encounter many a times in the transferred phenomena equations ok. Now, you see that nabla is given is the space direction. So, we see that delta 1 dyadic product this dou by dou x 1, then delta 2, then this dou by dou x 2, and delta 3 dou by dou x 3.

And again we are putting the concise mineral like this ok, so that means if you are talking of say nabla nabla T nabla T in the Cartesian coordinate, it will look like dou by dou x say T into i, then dou by dou T dou y into j plus dou T by dou z in k ok. So, this is how we are representing this nabla operator, so from the Cartesian coordinates.

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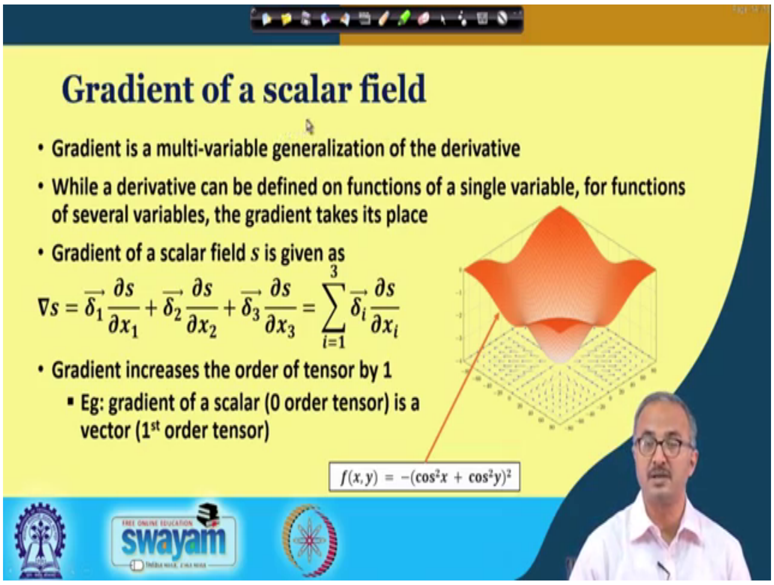

Now, understand this with change of the coordinate system, the nabla will take different forms ok, this you must remember. Now, here we have the gradient of a scalar field ok, what we understand gradient is a multi-variable generalization of a derivative. Now, while the derivative can be defined on functions of a single variable, for function of several variables, we use a gradient.

Like as I just told you that gradient of scalar field that temperature. So, here we are writing gradient of scalar field like this nabla operator into the scalar s, and this is the way we are expanding it ok. This s may be temperature, it may be mass, it can be anything, any scalar.

Then gradient increases the order of tensor by 1 that means, initially it was a 0 order tensor. This s was 0 tensor, now we are getting a from a vector we have converted to sorry from a scalar we have converted to vector ok, so it we have increase the order. So, here you can see that how it looks like that f x, y is something like this. And then if you take this particular this nabla, it will take the order two by one more.

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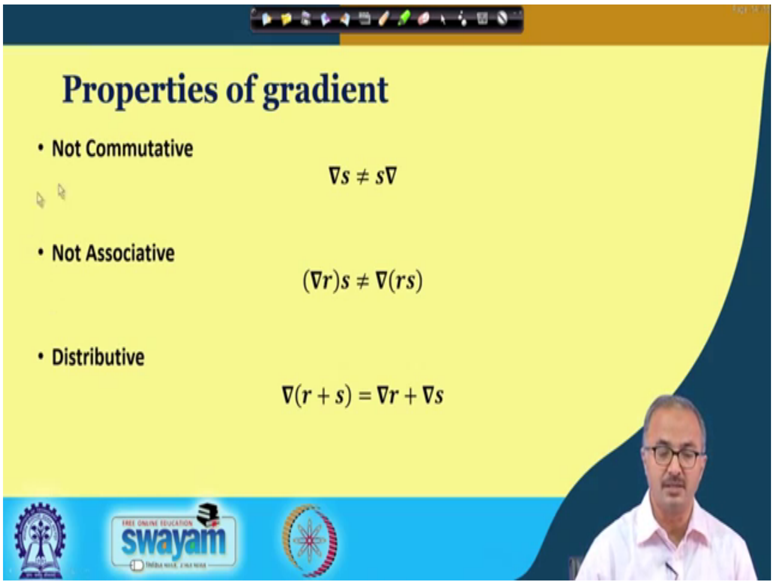

Then we have properties of this gradient is not commutative that means, nabla s is not equal to s nabla. And then not associative that means, nabla r into s is not same as nabla rs. Then distributive it is distributive nabla r plus s is equal to nabla r plus nabla s ok.

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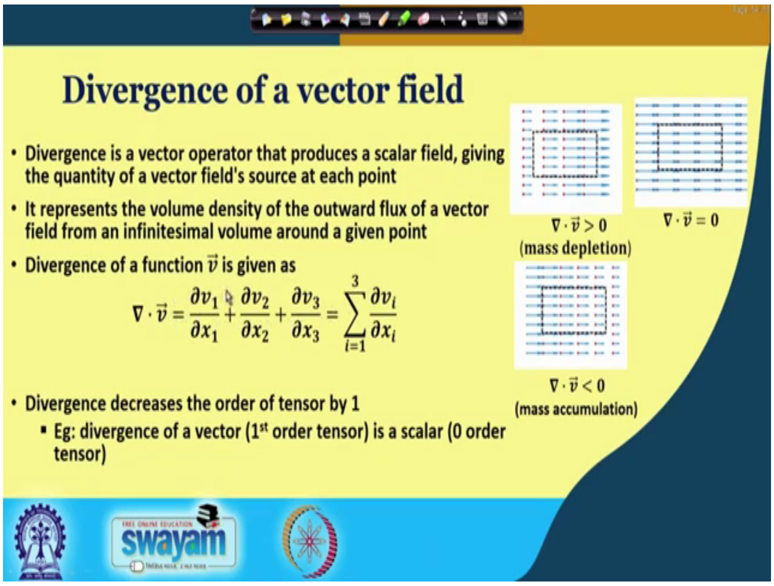

And then divergence: now, divergence of a vector field ok. Now, vector when we say divergence it means that, it is a vector operator, it produces a scalar field giving the quantity of a vector that means, in the gradient gradient was converting as scalar to a vector, whereas divergence is converting a vector to a scalar ok. And it represents the volume density of the outward flux of a vector from an infinitesimal volume around a given point ok.

So, any vector is there is a flux of a vector to a control volume, something is going out, something is coming in. So, the net of this the out minus thing that is the thing is given by the gradient. For example, divergence of the velocity if you take this, this we have seen in your in case of your mass balance, you get the divergence of velocity.

And you can see that here we are representing these divergences. If we find that this nabla dot v is more than 0 that means, we are getting more output than input that means, it is a depletion. If nabla dot v is equal to 0 that means, output is equal to input ok. And here you see that if nabla dot v is less than 0 that means, input is more than output, in that case this is the accumulation within the particular control volume. So, this is how we are interpreting this divergence of the velocity. Divergence decreases the order of tensor by 1 that means, from a vector we are getting to a scalar.

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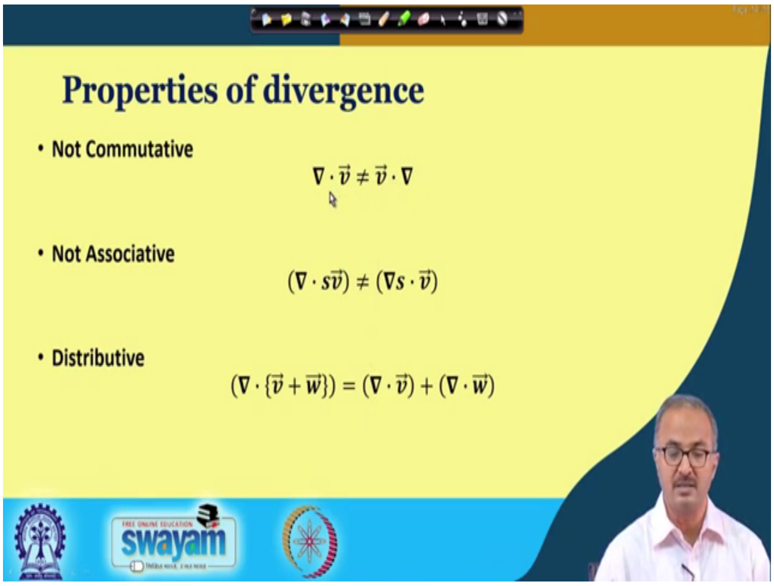

Now, properties of divergence are this that this nabla dot v is not equal to v dot nabla. And it is not associative that means, nabla dot s v is not equal to nabla s dot v. It is not distributive, it is distributive like nabla dot v plus w is equal to nabla dot v plus nabla dot w.

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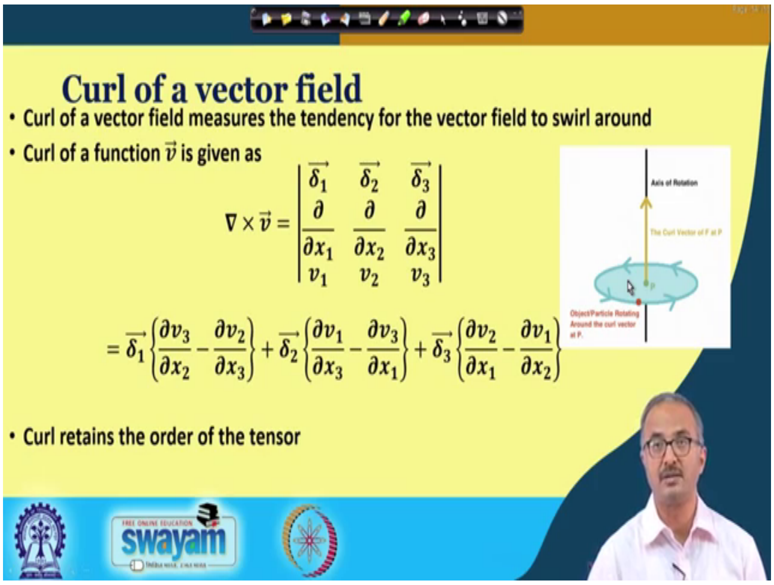

And lastly we have the curl of a vector. Curl is generally gives the tendency of some vector field. For example, if we have talk of the velocity, then if we have we curl of velocity that decides that whether we have a swirl in the flow or not in the flow remain or not, so we call it rotational flow or irrotational flow depending on whether we have the curl of the velocity to be 0 or not 0 ok.

Now, curl is defined like this, this nabla cross v. And this is the way, we are putting in a matrix form ok. Now, you can see if we expand it, it will come like this. Now, in the curl we are retaining the order tensor is third order, since it is very out of tensor ok. And here we have shown the interpretation of curl that if there is can rotation ok, then this is the axis of rotation so rotating. So, this is a rotational thing, this is given by the curl of the velocity.

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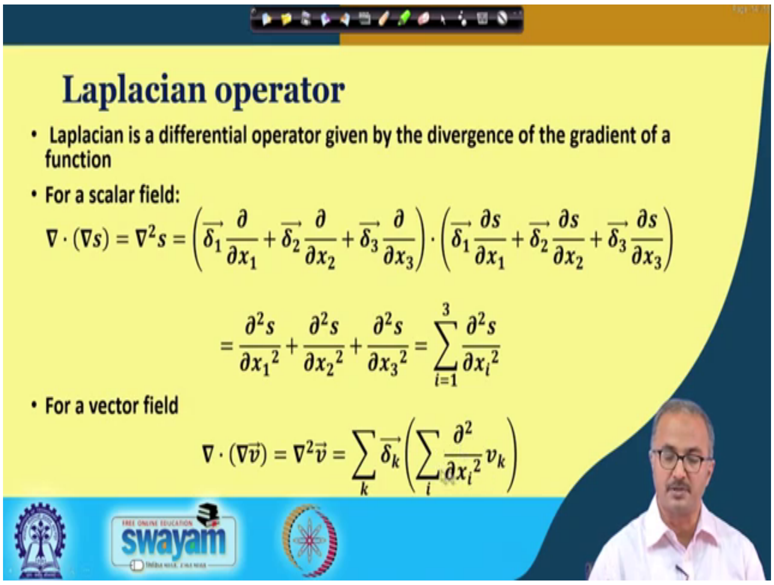

And then we have the Laplacian. Laplacian is a particular operator, it generally operates like this that nabla dot nabla ok. And given by the divergence of the gradient that means, you are we are combining the both the divergence, and the gradient. Now, when we put a gradient of a scalar, it is making it a vector. And then we are once we have vector that means, we are basically taking a dot product of the nabla vector with the gradient vector.

So, here we find that nabla dot nabla s is equal to of nabla square s, and here we are getting this particular expanded form for the Laplacian. So, these particular thing represent the Laplacian. So, this is for the scalar field of for nabla square, and where for vector field again we find that it will have a vector thing. Now, this particular thing nabla v, this is neither scalar nor vector is a tensor ok. So, this is the tensorial dot product we have to take with a particular vector ok. So, this is the particular expression for this nabla dot nabla v.

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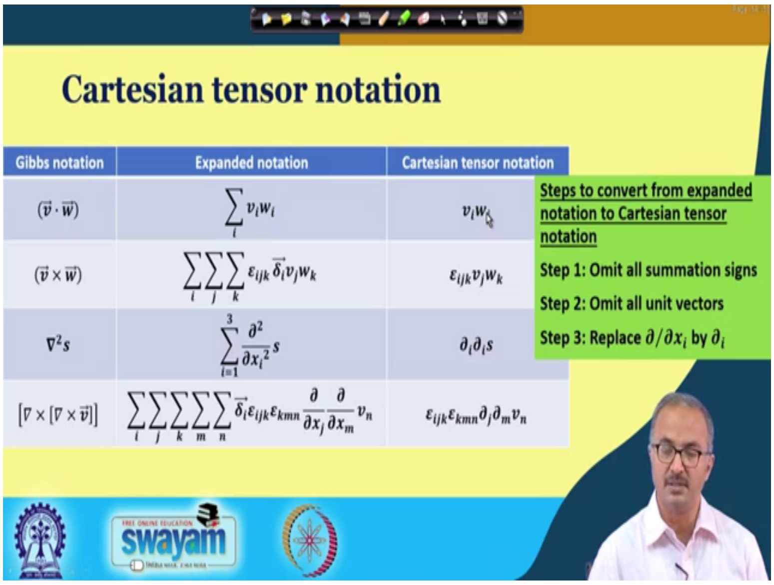

And here I have just summarized all these that if we have this is a Gibbs notation, this expanded notation, because Gibbs notation like this. So, we have expanded notation that v dot w, it is like summation of v i w i into summation of this, then in the tensor form we drop this particular summation. So, here what we are doing that omit all summation signs, omit all unit vectors, and Laplace replace dou by dou x by dou i only. So, dou i means dou by dou x i.

So, this how we are making this particular table. So, this way you find that in many books in in the many literature, they are putting this kind of notation, and so that they can economise on the space requirement. You can see this is the this particular thing requires lot of space, but same thing we are making a very, very concise form for representation ok, so these are the things which will find.

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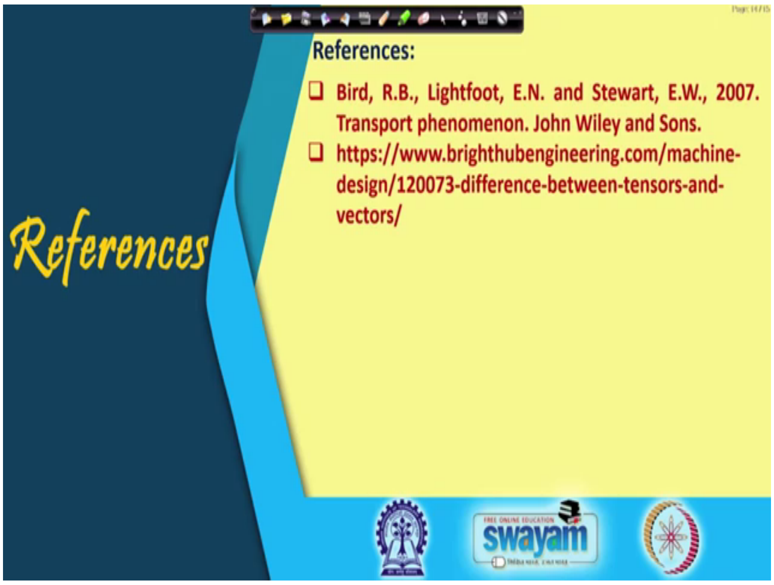

These are the books you can consult for knowing more about this. And I have put this particular lecture to you, so that you can read the various transport phenomena based literature which will be having such kind of operators.

Thank you.