Mass, Momentum and Energy Balances in Engineering Analysis
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Lecture - 36
Matrix Techniques – I

Welcome. After learning some balance equations and how to set up the balance equations to analyze the various problems involving mass transfer, heat transfer, momentum transfer. We should now learn if some fundamentals about how to solve these balance equations. And these balance equations as you can see, involve some kind of differential and when you set up the balance equations for losing some kind of finite difference technique, you will find that you will land up with some set of equations which are generally linear.

If they are not linear you sometimes linearize them, because linearized equations are much easier to solve than the non-linear equations. And in all these things you find that you have different forms of the equations and sometimes you need to also interpolate something in between sometimes extrapolate. So, there are various mathematical techniques also involved, and to give it a complete shape in the few next few lectures we shall be looking into some of the fundamentals of the mathematical and numerical techniques we use to solve the model equations.

So, I shall be starting now with some metric techniques which we often use in solving these model equations. So, I shall be restricting myself only to the common ones and it will not be an exhaustive exposition on the matrix techniques to solve the various types of equations.

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So, in this particular lecture we shall be looking to two of the method; one is the LU Decomp and another is the Tridiagonal Matrix Algorithm.

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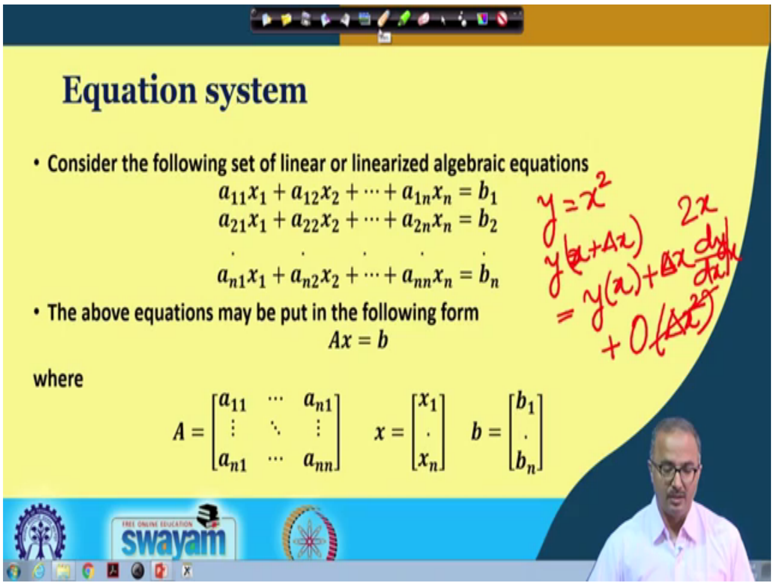

So, first let us see that what kind of equations we are targeting to solve. So, here we have a set of linear or linearized algebraic equations. Now it means that you can see that from the formal equation all the terms are linear there is no square or cube or square root or anything ok. So, that is how you are getting the linear equation and suppose you , when I say linearized equation, it means that if the parent equation is non-linear and you what you do using say Taylor series will linearize them.

For example you take that y equal to x, y equal to x you take this is a linear equation ok, this is linear equation. So, there is no problem with this, but suppose y is equal to x square in that case what you do that you linearize this equation like this that you take the Taylor series from that you find that y equal to we say x plus delta x ok. So, we take that y at x plus delta x into dy by dx at x and we are neglecting the higher order terms with the order will be this is the higher order terms ok. Now you see that in this case that this when we neglect these terms we find that we get this dy by dx will give you 2x ok. So, that is how we are able to linearize the given set of equations.

So, now you see that if you put all these equations and we shall see some examples later on that how we can get such kind of equations. We see that here we can write these equations in these terms, we put we put a 1 1 a 1 2, because we are putting try to put in a matrix form. So, we put a 1 1 a 1 2 up to a 1 n and we have x 1 2 x n number of variables then we have x 2 a 2 1 x 1 a 2 2 x 2 like this a 2 n to x n and this b 2 and then we have a n 1 x 1 to a n 2 x 2 to a nn x n b n ok. So, now, you see that with this we can put all these system club this systems into a matrix form and we put this a matrix, then x is the vector of the unknowns and b is some kind of forcing function we say.

So, A is represented like this that here we have a 1 1 to a 1 n it will be a small correction over here it will be a 1 n and then similarly a 1 2 to a 2 n and then we have this a n 1 to a n n and then this x is the vector this x 1 to x n and b is another vector this is b 1 to b n. So, this is how we set up the model equations and after we have set up this model equation we will see that, this here we have to solve for x and generally if the order of the matrix a is small we can take the take the inverse of a easily and we can solve for x, but as the dimension of a becomes more and larger and larger then it is not possible to go with the inverse because that will prove very inefficient.

So, generally we take some help of some other techniques to solve for x and we shall be looking into those techniques. So, you see that there will be some direct methods and their indirect methods, direct methods means at one go. For example, if you are able to take the inverse of a, then you will get the value of x at 1 go ok. So, that is a direct technique and if we are not able to take the inverse of a then what you do you do some iterative technique to solve for x. So, both these techniques are there and most of the times we go for the iterative techniques, because the dimension of a is very large.

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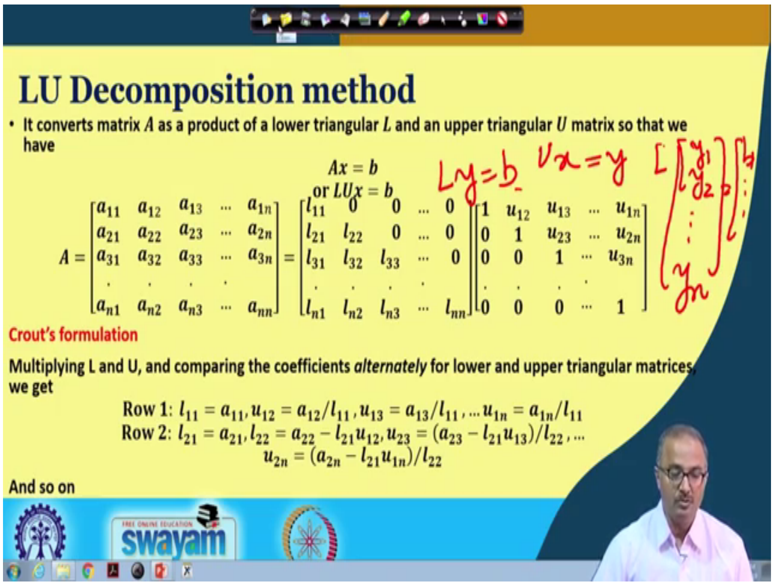

So, first let us go with one of the common method that is the LU Decomposition method or in short we call it LU Decomp method and LU means that we take we resolve the parent matrix into two matrices, one is the lower triangular matrix, another is the upper triangular matrix perhaps, but triangular matrix you know that if we have the one-half if you have the a matrix and do along the diagonal say upper half is filled up and lower half is having all 0s, then we have a upper triangular matrix on the other hand below. If we have the below the diagonal elements we have the nonzero values most of the them are nonzero values and upper elements are all 0s then we have a lower triangular matrix ok.

So, in that fashion we define we resolve this parent equation into two matrices that is lower triangular, the upper triangular matrices and you can see that you can find this is one of the ways you can see that for a lower triangular matrix we are having all these elements l 11, l 21, l 22 and other things the above the diagonal all the values are 0, on the other hand for the upper triangular matrix we are considering these diagonal elements to be 1 and a rest of them are not necessarily unity, they may or may not be unity and, but the below this diagonal they will always be 0 ok.

Now, now you can see that if you multiply this l and u matrices by comparison with the parent matrix you can find the value of all these elements of the lower triangular and upper triangular matrices. So, here I have shown you like if you consider row 1 you can get the value of l 11, u 12, u 13 etcetera then if you go to row 2 you can go to l 21, l 22 l 22 etcetera. So, this is the way you can keep going on and you can find all the values of this l and u. Now once you have got the values of l and u what you do that you know that you define that u x, u x to be say another thing y ok. Now what you do that u you want the value of x ok. So, first you solve for; that means, you are getting L y equal to b ok.

So, from this lower triangular and this b vector you can find the value of y very easily because you can set this equation like it will be lower triangular matrix and this will be y 1 y 2 up to y n and into this b 1 b 2 etcetera. So, you can see easily you can find out that you go forward swept you can find the value of y 1 will be equal to b 1 by l 11 ok. So, similarly you can make a forward swap. So, you get all the values of y once you get all the values of y now you put the value of y here. Now your way upper triangular matrix here now what you can do now you can do a backward swap so; that means, you will now find x n x n minus 1 x n minus 2 and you will go to x 1 ok.

So; that means, in the two steps you are able to get this value of the x ok. So, this is a very handy method. So, you can see that even if b has changed there and you are a matrix remains the same you do not need to do anything. If you actually all these methods come from the Knife Gauss elimination method in which you work with all the elements and if there is any change on the right hand side then you find that you again if you redo the whole problem with all the elements ok, but in this case we are able to circumvent that kind of problem.

So, this l n you will remain the same only thing is this if b changes, y value will change and y value changes; that means, x value will change. So, in case of y you go from y 1 y 2 y 2 y 3 up to y n and in case of x you go from x n x n 1 x n minus 2 to x 1.

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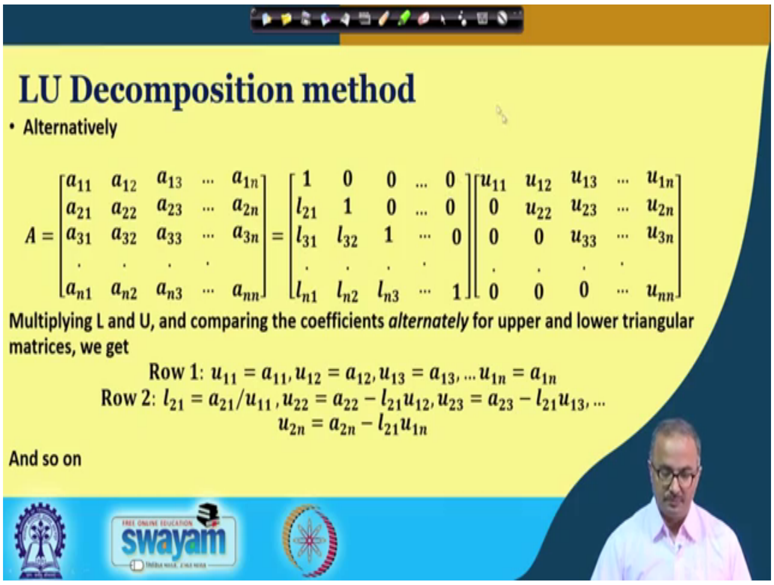

So, now this is not the only way to resolve a, you can have another way of resolution of a; that means, I can put the l u like this, but in this case for the l matrix, I put all the diagonals as 1 and for the u matrix, I put the all these diagonal elements as not necessarily 1 ok. Now again you find that by comparison of these multiplication of this two matrices with these individual elements you can again find the value of u and l ok. So, with this thing again the same methodology that you first find this l u again find the value of the y that is u x and then from that you find the value of x.

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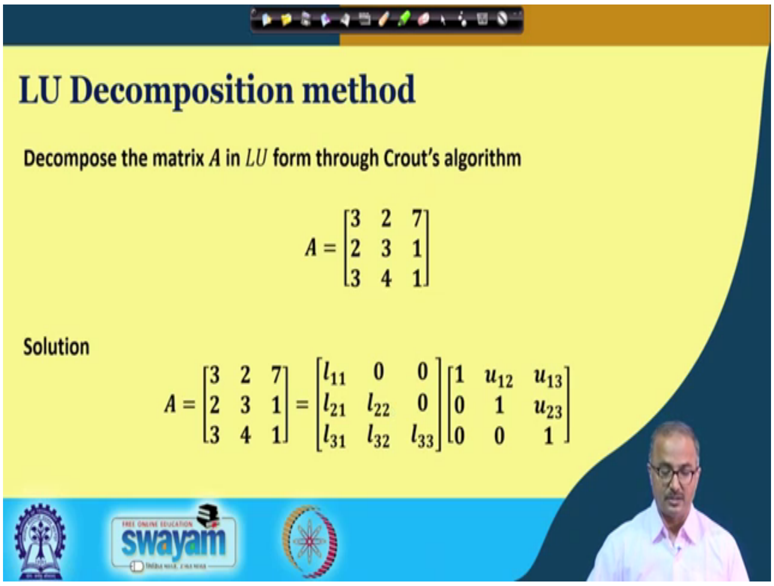

Now, we take this example problem. So, this is the A matrix given to us and this is how we are going to now find out that how we are going to decompose this A matrix in a l and u matrices with the Crout's method.

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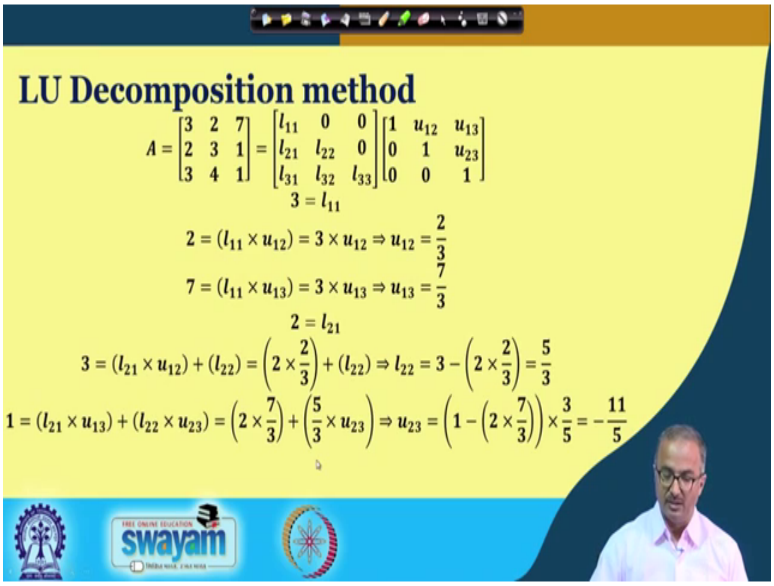

And here I have shown you that how you can get the values of the all the elements of this lower triangular and the upper triangular matrices ok.

So, you can go one by one you can see first we get this l 11 as 3 that is this 1 value then l 11 u 12 is equal to 2 that is this and this way you keep on first you take this particular row then you take this particular row, then you take this particular row and you solve alternately for l and u and you can get all the elements of l and u.

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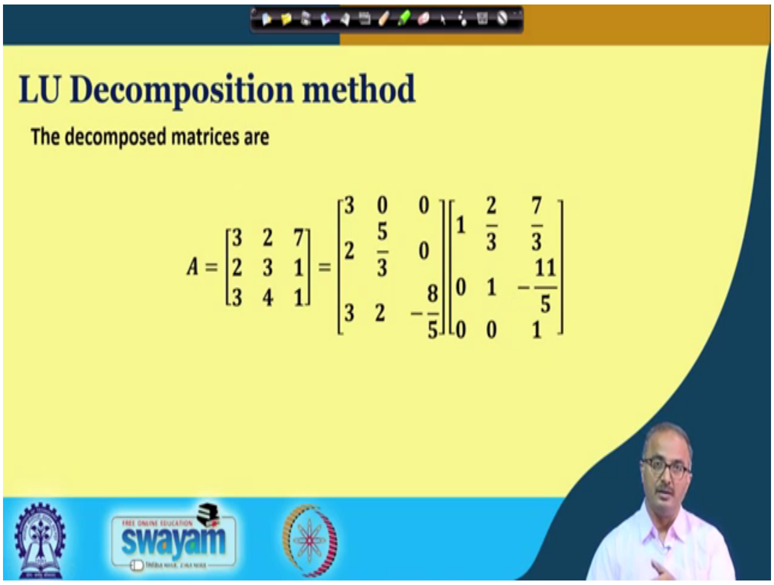

So, here you have the lower triangular matrix and the upper triangular matrix ok. So, here we have the basic thing is this we have learned how to decompose a matrix into l and u matrixes.

Now, once you have got this I can give you any other function that will be the b forcing function and now you can solve for the x.

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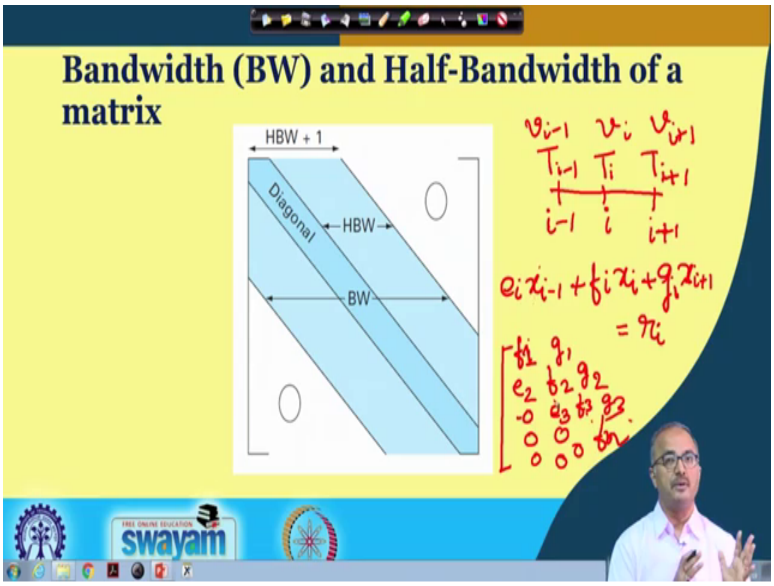

Now, before we go to the another kind of solution here is a special kind of solution that when many a times we find that we deal with sparse matrices, sparse matrix means that most of the elements are 0 in the matrix. So, for the efficiency of the computer coding we do not need to store the 0 matrixes and many times we find in our transport phenomena problem we have only in a given equation we do not have all x 1 x 2 x 3 x 4 up to x n in a given equation we have only neighboring points; that means, suppose I talk of a point i and these equations are obtained from the finite difference equations generally. So, we find suppose I have I talk of a point i suppose I talk with point i and on the right hand side I can say i plus 1 and left hand side I am doing a one dimensional problem one dimensional problem.

So, suppose i with respect to i we have a i plus 1 point and i minus 1 that point suppose we are get temperature I want to find the temperature gradient suppose ok. I suppose this is the x direction. So, I have T i then T i plus 1 and then T i minus 1. Similarly if I want to find the velocity profile I will have v i then v i plus 1 and then v i minus 1 ok. So and so forth similarly you can go for the species concentration. So, when you set up this kind of equations in the transport phenomena you find that you get an equation which will look something like this that suppose you have e i sorry x i minus 1 plus f i x i g i x i plus i plus i plus 1 is equal to sum r i ok.

So, these are typical equation you get and you see that in this equation you are not having the values at all the points; you are having only the boundary points here. So, when you write this kind of equation for whole system you find that you get a system of matrices which will look like this that you will get this you on the f 1 then f 2 up to say f n ok. Now i goes from 1 to n and on this side you will have e 2 on this side you have g 1 ok.

Then you have 0 here then e 3 f 3 then g 3. So, you will find that most of these things will be 0 and you will have three bands one is g 2 here g 3 here like this it will go. So, in a way banded structure. So, one is the main diagonal then one is the super diagonal and one is the sub diagonal. So, only these three things will be there with you and rest of the elements will be 0.

So, in this case it is not an efficient technique if we are going to use the Naive Gauss elimination. So, in this case we give some particular techniques which are only for this kind of situations and because now we have three diagonal a main diagonal, one sub diagonal, one super diagonal we call it a banded structure band. So, we have we call it that three things are there. So, we call it tridiagonal matrix. So, it is not that we have only tridiagonal we can a Penta diagonal also we I can have two sub diagonal, two super diagonal.

So, various situations may be there which will depending on that we will having this kind of banded structure and for any kind of banded structure the Naive Gauss elimination method can be modified. So, that we do not have to deal with this 0s. So, in this particular figure you can see that we have the bandwidth the total number of the rows this columns is showing the bandwidth from the sub diagonals or the super diagonal and half diagonal is the from the diagonal element said to up to the say super diagonal or the sub diagonal you call the half bandwidth and rest of the things these particular things show they are 0 the rest of these elements are all 0s.

So, in this particular lecture we shall be looking into only the tridiagonal matrix systems ok.

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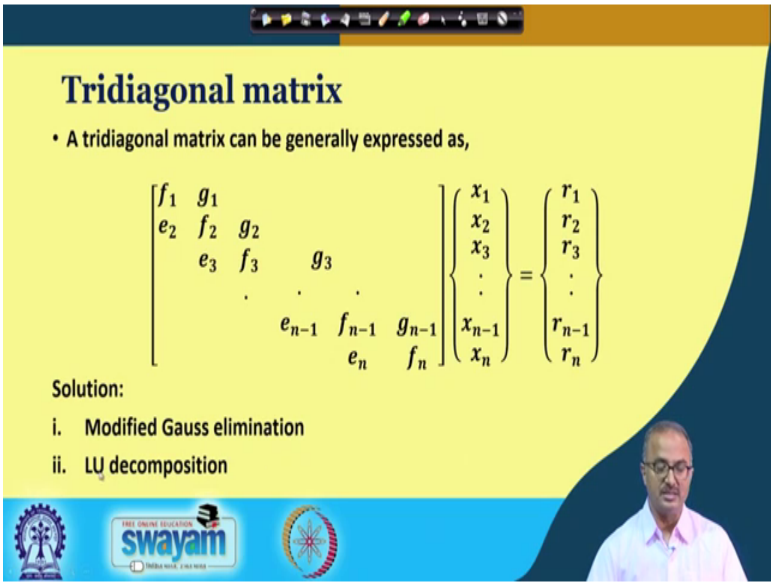

So, here as I have shown you that this is the tridiagonal matrix system. Here you have the main diagonal, this is the super diagonal, and this is the sub diagonal elements and here we have all the unknowns to be solved for and here we have the or the forcing functions. Now these can be done this can be solved by two ways, one is that we can modify that Gauss Elimination method and another is the we can use the LU Decomposition method.

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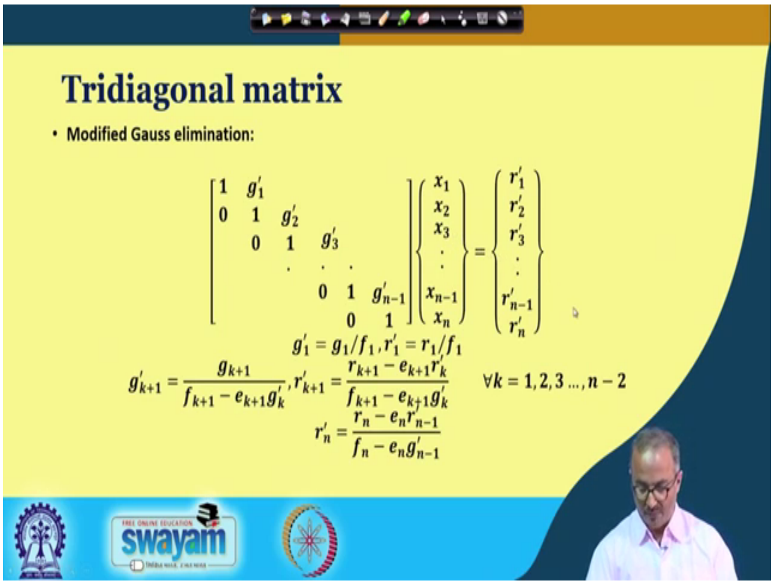

Now, if I go let us go one by one that if you go with the modified Gauss Elimination method then we will be reducing our system into this form, it is something like an upper triangular matrix with all the diagonal elements as 1 and we shall be having only 1 super diagonal ok; that means, half-bandwidth is 1 and you see that we are putting this prime because when we are converting the parent equation in this form all these values are getting changed and these values have been given like this here. You can see that you can find this g 1 prime g 2 prime g 3 prime for this from this formula ok.

So, once you get all this g 1 prime g 2 prime r 1 prime r 2 prime etcetera then you can now solve for this x. Now in this case you can see that you will go swap backward. Now why, because the x n will be equal to r n prime ok. Once you get r n prime then you see that from this you what you get from this one you get x n minus 1 plus g prime n minus 1 into x n equal to r n and prime n 1.

So, you know x n. So, you can find the x n minus 1. So, next you when you go to x n minus 2 it will involve x n minus 1 then you as you move up you will find that x 2 will be involving x 3 x 1 will be involving x 2. So, that is how you are going a backward swiping you are doing to get the values of all the x.

So, this is based on the LU modified Gaussian nation.

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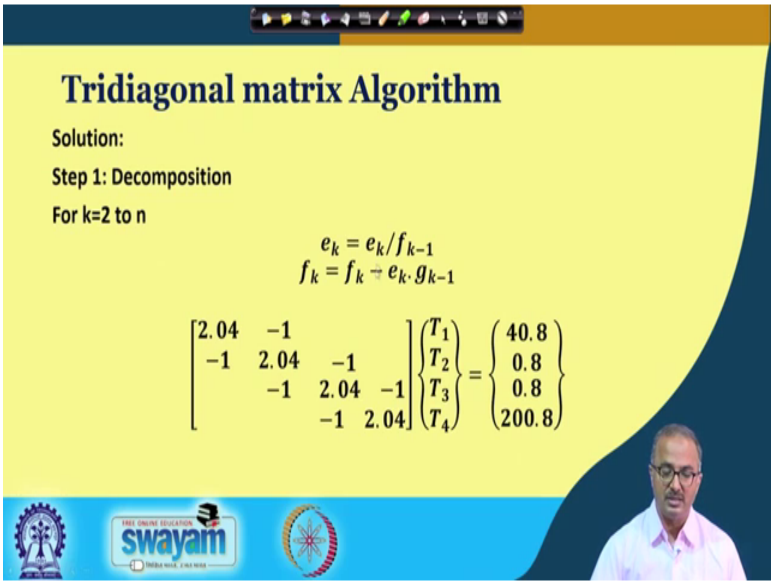

Now, we go for the LU Decomposition in LU Decomposition again you can see that this is the way you can go for the LU Decomposition.

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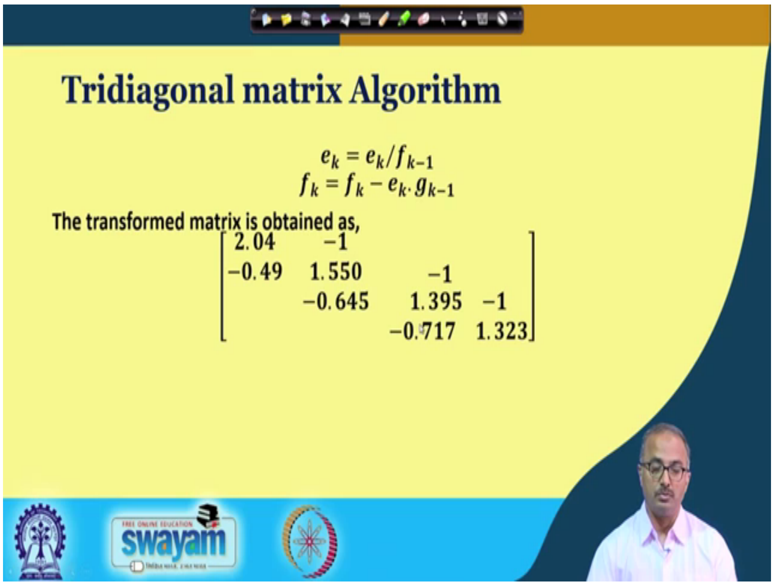

And here we have demonstrated with a example problem that how you can get the LU Decomposition method to obtain the to a solution for a Tridiagonal matrix system.

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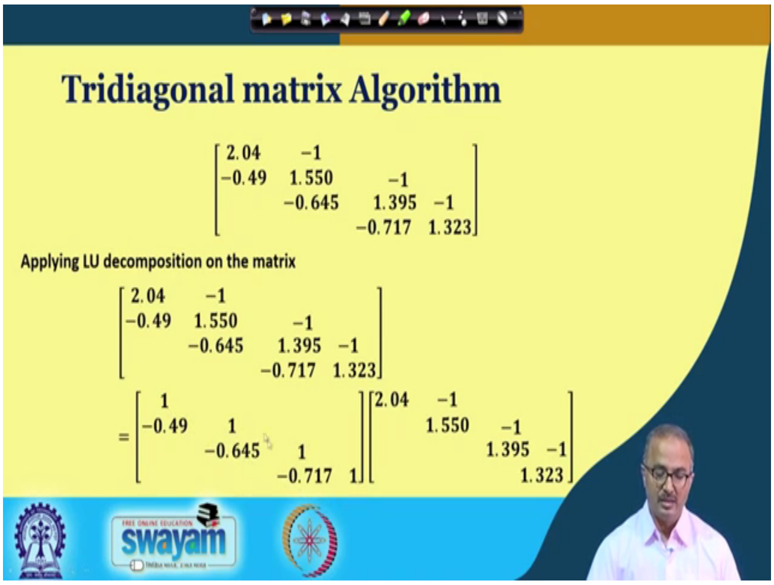

So, here you can see that as you apply the LU Decomposition algorithm you find that at each step you are able to get that how the values are getting modified and ultimately you are getting these two matrices in this is the l matrix and this is the u matrix ok.

So, you can see that in the l matrix this is something the following the Crout method you see that all these diagonal elements are 1 and you have only half-bandwidth that is 1 and in this case of u matrix, you can see these are the diagonal elements and these are the super diagonal elements.

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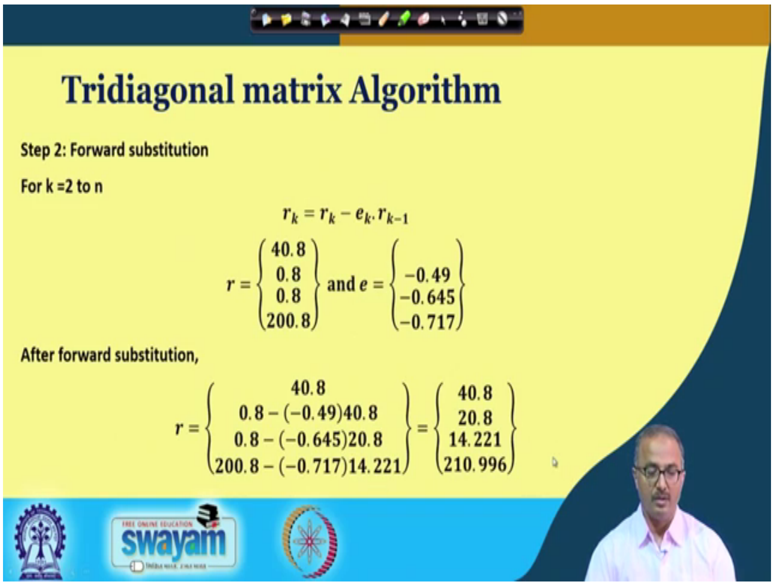

Now, these are the r; that means, as you are going for the, you are changing the l and u you know that you have to also change the r values. So, this is how you are getting the r values.

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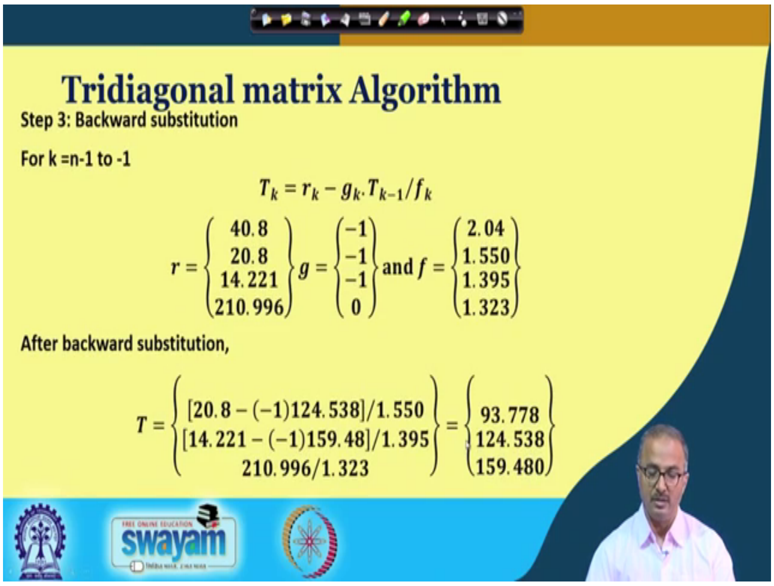

And here you are solving for the T as I told that you go with a backward sweep you go. So, you are getting the value of the T n first. So, then you get the value of the T n minus 1 then you have go to n minus 2 and then ultimately you reach the value of n minus; that means, this is a T 4 then T 3 then T 2 and T 1 ok.

So, this is the way you can see that once you have resolved the particular equation into the l and u you are able to solve for this two.

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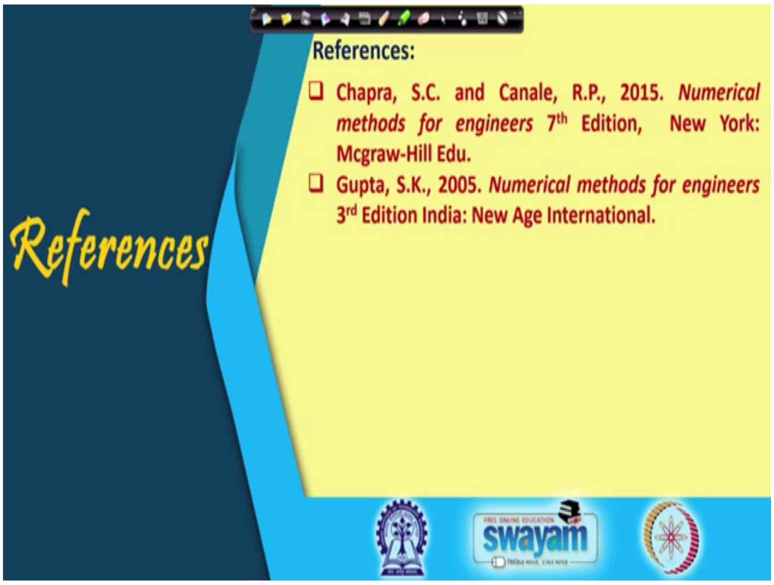

Now, these are the books which you can refer to for more examples and illustration explanations of this techniques.

Thank you.