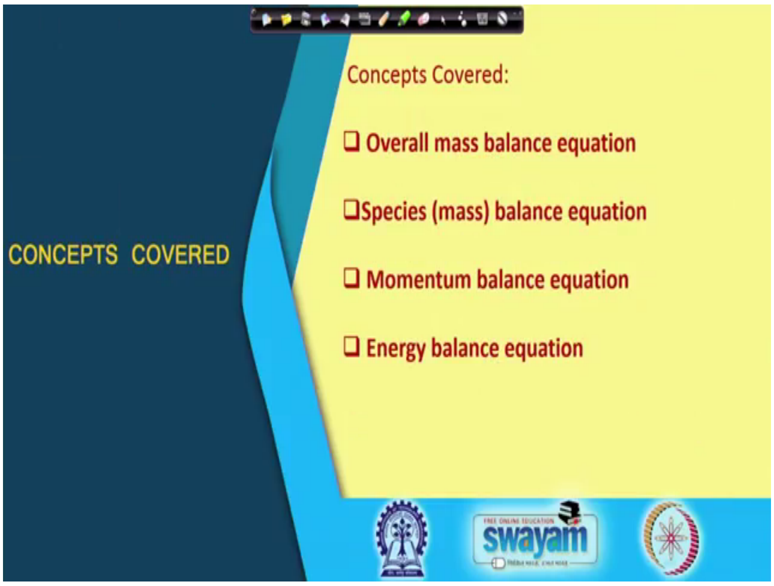
Mass, Momentum and Energy Balances in Engineering Analysis
Prof. Pavitra Sandilya
Cryogenics Engineering Center
Indian Institute of Technology, Kharagpur

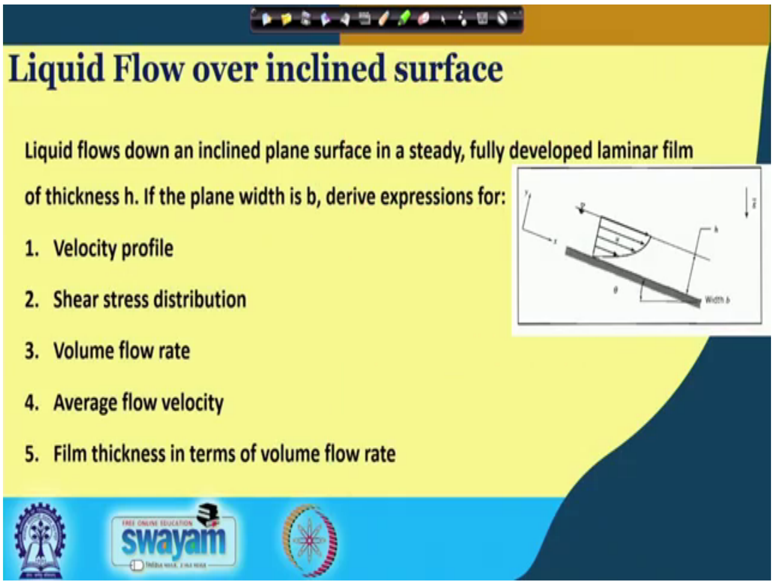
Lecture – 31
Microscopic Balances Illustrations – I

Welcome, after learning the fundamentals about the balanced equations how we can make the balanced equation, what kind of assumptions we can do and what is the importance of the various types of assumption and then how to non dimensionalize the balanced equations. We shall now take up many examples of this balanced equations so, that you can understand that from various fields like momentum transfer, heat transfer, mass transfer how you can make the balance equations and how you can non dimensionalize them also. So, first let us go with the microscopic balance equations and illustrate them.

(Refer Slide Time: 01:03)



(Refer Slide Time: 01:03)



So, here we have the liquid flow over inclined surface and this is a very common example you find that in many a times we have the situations where the liquid is flowing over the inclined surfaces, ok. And this is inclined it has been taken in general, but you understand this the angle of inclination will change the nature of the flow; that means, the angle of inclination can vary from 0, if it is, is it theta you can see this angle of inclination.

So, if it is goes to 0, it means that you are having the horizontal flow and if this goes to 90 degrees it is a vertical flow. So, in general we can take this angle of inclination somewhere in between 0 and 90 degree to a situation where it is neither horizontal nor vertical, it is somewhere in between ok, but you can extend this for any situation. So, here we have the liquid flows down on inclined plane surface in a steady fully developed laminar film of thickness h.

Now, steady you understand that the particular variable that is in this case the velocity does not vary with time at a given location, but location to location it may vary that is the meaning of steady state. And fully developed is that you know that whenever a fluid enters over a particular domain, it initially takes some time to develop, because the boundary layers are keep developing as the fluid enters over a surface. Because, before the fluid has is on a on the it goes to the particular surface it might be having some different resistance to it flow, but when it goes to another kind of surface of interest, we will find that because of the viscosity effect there will be a drag force on the fluid and that will cause the boundary layer to form over the surface. And we what we say the developing flow region is the region when the boundary layer is growing, ok. And once, the boundary layer has reached the maximum thickness after that it will remain at a constant value and we call this a fully developed flow, ok.

So, in the fully developed flow and in the entry region or the developing region, we have a result slightly different types of equations. So, in this particular example, we are considering a fully developed flow. And the flow is laminar again you know that what is laminar, what is turbulent flow? That laminar is supposed to be a kind of a very well defined flow and as you know that laminar means we assume that the flow is taking place as if layer by layer, ok.

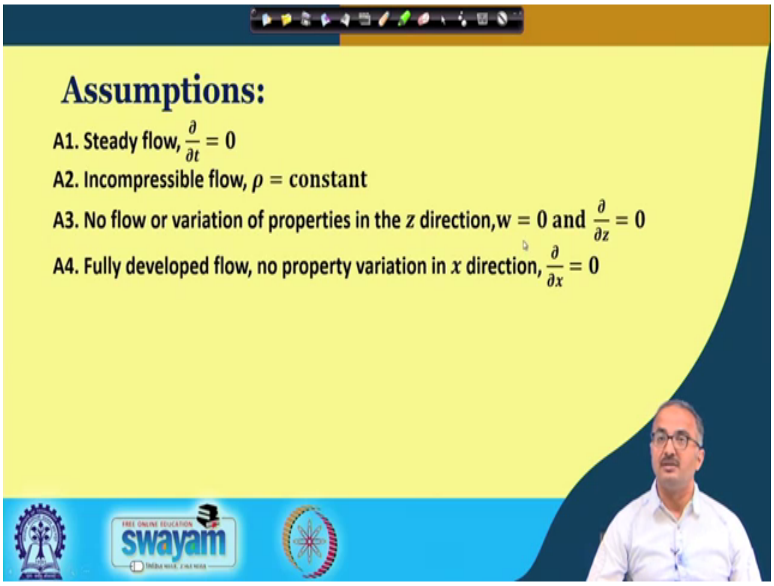
So, in case of laminar flow and we have certain values of the drag forces. And in case of turbulent so, we have some other ways of taking care of this velocity profiles through the drag forces. So, that is how we are distinguishing that nature of the flow for this particular situation. And it is assumed that the thickness of the layer is h, ok. So, as you see that is the h is a thickness of the layer, u you can see it is going in this direction and it is changing from the surface of the this and to do that as you go original surface it is more increasing, ok.

And here is the width of the per to the width of the particular surface b, ok. And here you can see that how we choose the direction this, this is the flow direction is taken to be the x direction and the perpendicular to this is taken to the y direction; that means, we are assuming on the z direction, it is a infinite extent; that means, there is no effect of the boundaries on the fluid flow in the z direction that is how we are reducing a 3 d situation to a 2 d situation over a 2 dimensional situation.

And if the this plane width as I say is b, we have to derive expressions for the velocity profile, the shear stress distribution, volumetric flow rate, average flow velocity and film thickness in terms of the volume flow rate, ok. So, these are the things we have to find out how the velocity varies as we move away from the surface then the what is the shear stress on the fluid at the surface then what is the volumetric flow rate, the average velocity even though the velocity is changing, but for the average velocity and then the film thickness, how the film thickness varies as the flow rate changes.

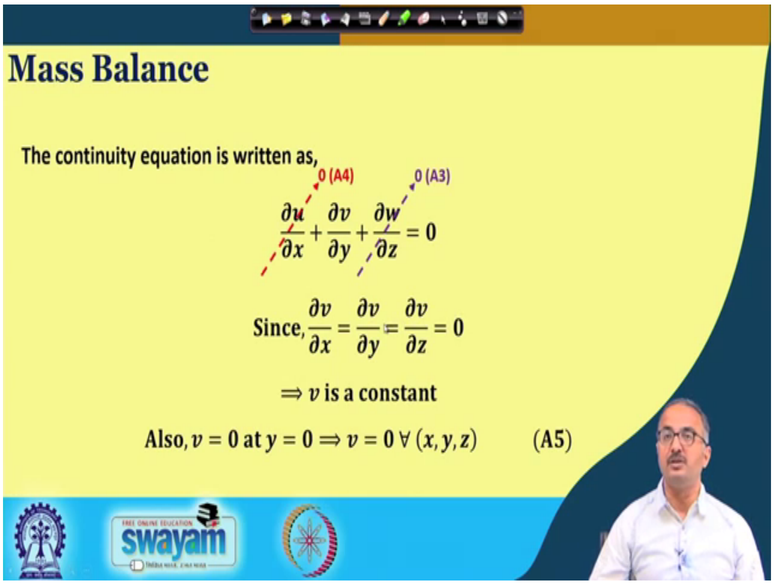
(Refer Slide Time: 06:08)

.



So, let us make some assumptions to do this, we assume that it is a steady flow. So, that all the time derivatives are taken to be 0, then we say that the fluid is incompressible that liquid for liquid, we can always safely assume it to be incompressible. So, that the density becomes constant. Then we say that there is no flow or property variations in the z direction so, that the z direction and velocity is w which is taken to be 0 and all the derivatives with respect to z are taken to be 0. And as we said that we are assuming fully developed flow and so, there is no property variation in the x direction. So, we take all the derivatives with respect to x to be 0; that means, everything is now varying to y.

(Refer Slide Time: 07:03)



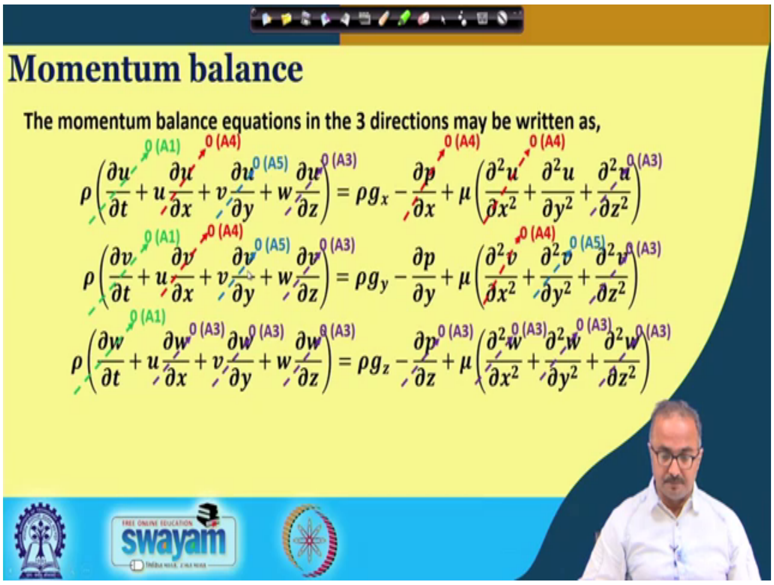
So, first we go for the continuity equation, as I told you earlier continuity represents the mass balance, and this is the total mass balance. And because we have assumed it to be incompressible so, you will find that the rho has been taken the density has been taken out of the differential, ok. And because of steady state assumption the dou rho by dou t also goes to 0. So, because of this, we find that we are having dou u by dou x plus dou v by dou y plus dou w by dou z equal to 0.

And as per the assumptions 3 and 4, we can neglect these two terms and we they are coming to 0 for the assumptions. These here we show the value of this term and this in the parenthesis, we will show the assumption which has been used to get these values, ok. And now, we have this equation the because of this we find that dou v by dou y all to 0; that means, v component is not changing where within the y direction and v is a constant, ok.

So, since we know that because no slip condition, no slip means there is no relative velocity slip is the relative velocity. So, there is no slip means there is no relative velocity between the liquid and the surface at the surface; that means, because the surface is stationary. So, the liquid will also be stationary, ok. If the surface is moving with some velocity the liquid will also be moving the same velocity at the surface. So, in any case we find that the slip velocity is 0.

So, we are putting that v equal to 0 at y equal to 0. And now, because it is 0 at y equal to 0 and v does not change. So, it means v does not change be remains 0 irrespective of the position in with respect to all x y z it is having a 0 velocity. So, we are able to neglect the v component of the velocity.

(Refer Slide Time: 09:08)



So, that is how, now we, this is the 3 dimensional Navier-Stokes equation for the Newtonian fluid and here Newtonian, because here we are taking this mu that is mu as a representation for the stresses, ok.

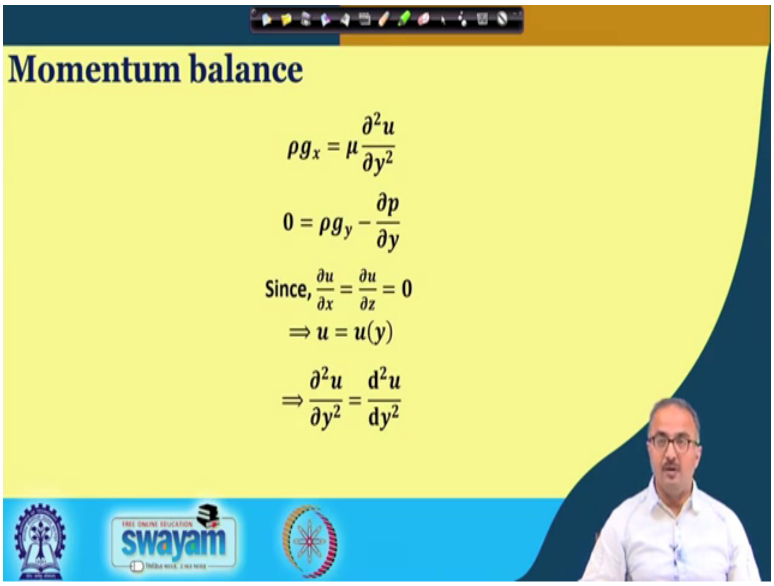
So, here we have these equations which you can find from the my earlier lecture or from any other standard book. And this particular equation, in this equation you see this is the unsteady state term, these are the convective term, these are the pressure differential pressure force, this is the shear stress term and these are nothing, but the body forces. Here the g does not necessarily mean the gravity, but it incur encompasses all kind of gravity and body forces, ok. So, that this will g is the body force per unit volume, ok.

So, now you see that because of assumption A 1, we are able to put everything 0 that is the steady state assumption, ok. And one by one we take care of each of the assumptions and we see the how we can reduce these equations. Now you can see here that because of assumption three we are able to make many of the terms to 0, ok. So, you can see that we when the terms goes to 0 and this next we come to other things also go to 0, because of assumption 4.

So, you can see that if you take care of these equations one by one then we are able to reduce many of the terms to 0 and this as I said that we also assume that anything in the z direction is 0. So, that is how we are able to put all these things as 0. Now ultimately, what we reduced with you can see that in this equation, we are having we shall be now seeing this equation that in this equation we are having this and this term, because we have we have this term and in this we are going to we have this term, ok.

Now, again you see that this particular thing goes to 0, because of the assumption 5 we have made, in the y direction that is nothing like thing and we have seen that the v was constant as we have found earlier from the continuity equation. So, this also goes to 0 and we also found that v is 0, ok. So, from that point of view this is also going to 0.

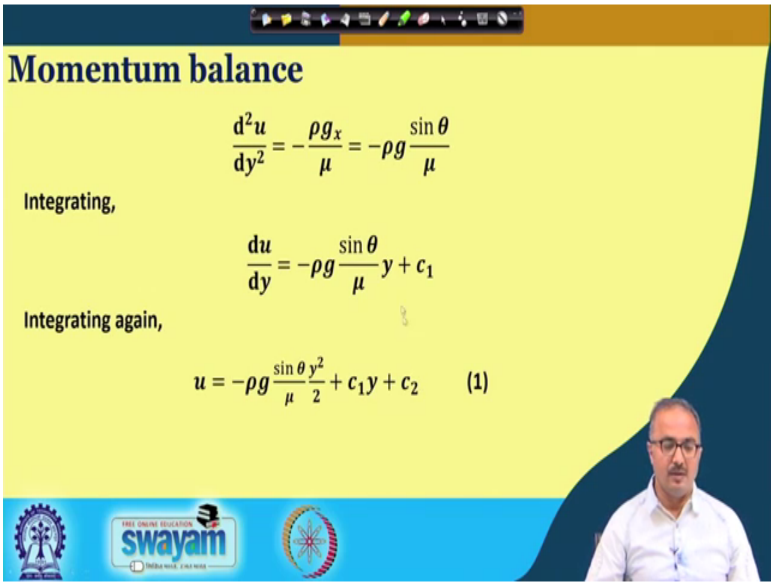
(Refer Slide Time: 11:45)



Now, after that as I just told you that you can see that these are the equation this is we are getting from the x direction momentum balance, this we are getting from the y dash momentum balance. And now, because this we found that this is to be 0 from the continuity equation, and this we have assumed 0 from our assumption that if there is no change in the z direction. So, all these things are coming to 0 it means what that u is a function of all the y; that means, u is not changing along the x axis, but at a given by it is it is just function is changing well.

And from the as we go away from the surface it is changing, ok. So, that is how you finally, deduce that use a function of on the right. So, that now this dou u by dou u square may be written in terms of the total differential that is we are putting dou to d, ok. So, that is how we are getting from a partial differential, we are getting in total differential.

(Refer Slide Time: 12:51)

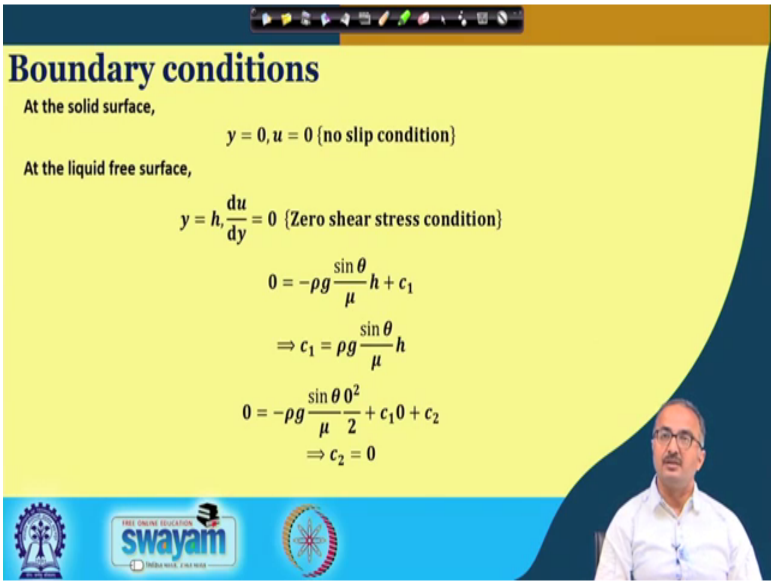


The rest of the things is like this that now, because the body the only body force we are considering is due to the gravity here. So, we, we can write that because of inclination, we take the component along the along the flow direction and the perpendicular to that. So, we take in the y direction, we take that component and that is how we are getting this sine theta over here, ok. So, g sin theta we are writing. And then we can integrate it easily and after integrating we find we are getting this particular equation c 1 and again we can integrate second time and again we get another equation.

So, this particular equation you can see it gives us that how the u is changing as we go away from the particular surface, ok. And these c 1 and c 2 values have to be found out from the boundary conditions. And because here second degree equation we need two boundary conditions, ok.

(Refer Slide Time: 13:54)

.

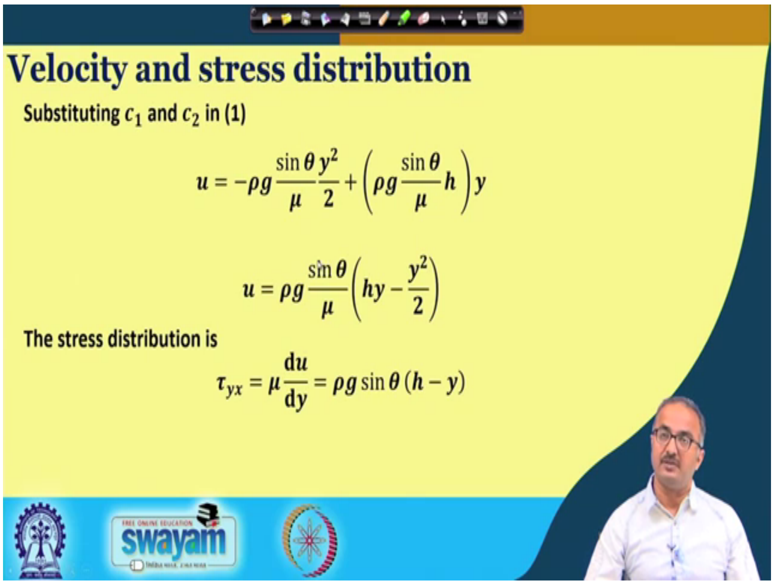


So, let us see the boundary conditions as we said that we assume no slip. So, at y equal to 0, we assume that this u is taken to be 0. And then at this liquid free surface, the free surface of liquid we say that there is no shear. Now no shear means suppose the liquid is in a in the air suppose so, air has such a low viscosity that we assume that the air does not make enough shear on the liquid. On the other hand suppose you have a situation where there are two liquids flowing one of the other, you can always have some situations like this, if you have two immiscible liquids suppose. And they are flowing together the right immiscible and with different density suppose oil and water, ok.

So, water will be at the bottom on top of that always be there and there could be situation that the two fluids are flowing and at the interface because oil and water will be having some comparable viscosities. So, because of that there could be some there a shear force at the interface between these two liquids. So, in that case then, we have to give some value to the shear force, but if we have just say a liquid and it is just a gas, gas has very-very less viscosity than a liquid. In that case we can safely assume that the shear stress at the free surface of the liquid to be 0. So, with that assumption we say that just that shear stress is nothing, but the mu into dou u by dou y. So, this mu into dou u by dou y is taken to be 0.

So, with, with that assumption we are saying that at y equal to h that is the thickness of the liquid that is h. So, we assumed that this shear stress is 0. So, with these two boundary conditions, we are now able to find out the values of c 1 and c 2. So, we see that c 2 is coming to be 0 and c 1 has this particular value, ok.

(Refer Slide Time: 16:01)

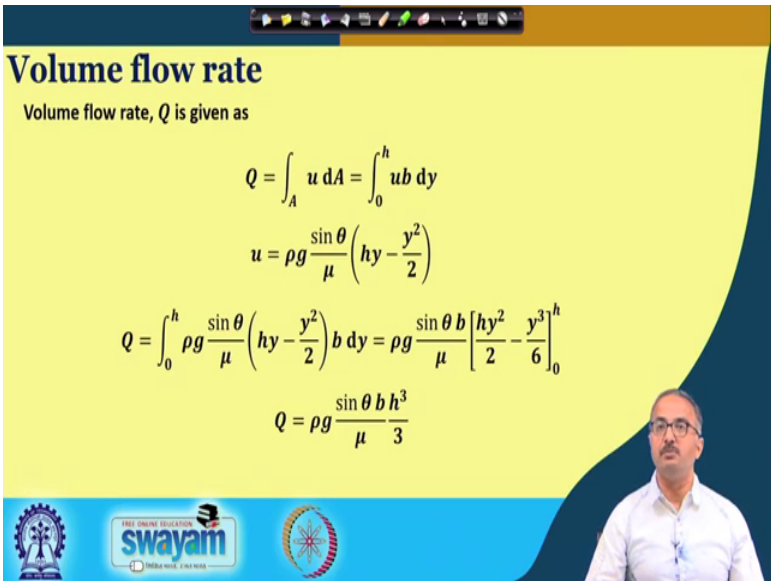


Now, putting these values of c 1 and c 2, we are now finding that this is the expression for the velocity of the liquid, ok. Now, once this is the first part of the question and this is this gives us the velocity profile over the particular surface. Next is to find out the stress distribution. For the stress distribution, you will know that we have to find out this particular value the stress is given by the Newton's law that is mu d u by d y and now we differentiate it with respect to y and we find the value of the stress, ok. And you can see that how the stress varies from the surface of the solid surface to the height of this thing, ok.

Now, we will then see that from here you can see here easily that when y equal to 0; that means, you are right on the surface, you find that you are getting sine theta into h; that means, we are getting the maximum shear stress on the surface. On the other hand when y becomes equal to h then you are getting this to be 0; that means, as you move away from the surface the shear the stress distribution the stress becomes less and less and it becomes 0 at the interface or the surface of the liquid. And also you can see how the variation of theta affects.

Now, if you see that if theta is 0; that means, you have a horizontal surface what we will find that they this sin 0 degree 0 degree is 0; that means, you are not getting any stress. On the other hand when this theta becomes a vertical surface your theta becomes 90 degree then sine 90 degree is 1 ok; that means, you are getting the maximum stress on the surface, ok. So, this also you can see that how the inclination will affect your stress value.

(Refer Slide Time: 18:03)



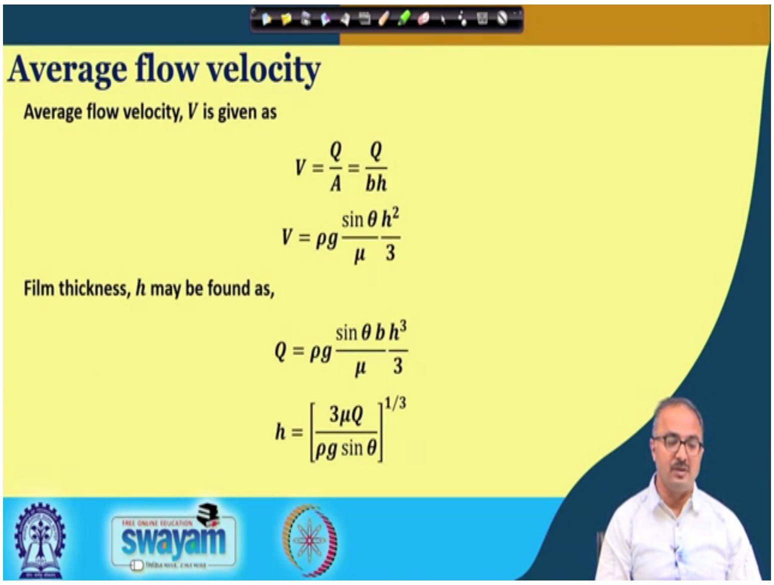
Next we come to the determination of the volumetric flow rate. Now volumetric flow rate as you know it is the product of the velocity and the cross sectional area. Now in this case the velocity is a function of the position the y. So, how do we find out this total flow rate? So, in this case what we do we first take a slice dA over that we make the u; that means, a small slice dA and in that u is entering, ok. So, u dA becomes the volumetric fluid within that small slice. And when we integrate it over the whole cross sectional area, we get the total volumetric flow rate.

So, that is how we are doing this particular analysis that first u dA is the volumetric flow rate within the small slice dA. And, then we are doing over the whole area cross section of the area we get the total volumetric flow rate. And this cross section area is nothing, but the b into dy, if you look at the particular situation the b into dy is the cross sectional area and b is a constant the width the constant.

So, the only variation is with respect to y. So, that is how you find that you get this dA is nothing, but b dy, ok. And this y is again varying from 0; that is surface to the interface that is h, height, the thickness of the liquid layer, ok. Now it becomes now easy, you simply put the expression for u here and integrate it with respect to y and with the two limits and you get this particular value of the cube, ok.

Now, again you can see here what you see here that again this Q will change with the theta it means that when you have 0 degree Q is reduces to 0, ok. When this is 100 degree you are getting the maximum flow rate, ok. So, here you can see that how the inclination also gives us the volumetric flow rate.

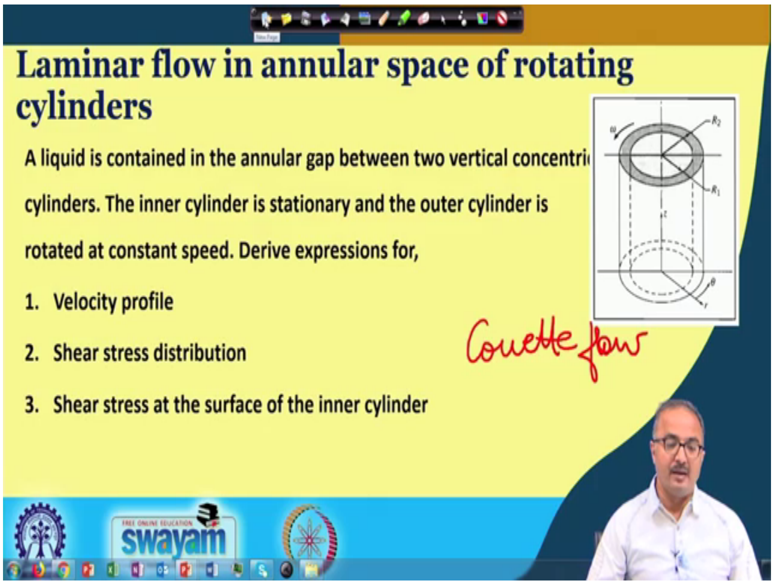
(Refer Slide Time: 20:22)



Next we come to the average flow velocity, you can average velocity is easy that total volumetric flow rate divided by the total surface area. And that we put the simply the total surface area is nothing, but b into h and the total vomited fluid we have just found out and from that we get the average flow velocity.

Again you can see the effect of the theta here and the film thickness is you can see that you just rearrange this equation, you can find that how the thickness film thickness changes with the volumetric flow rate. And you can see the thickness change as the cubic root of the volumetric flow rate, ok. So, these are some of the ways you can, you can see that how you can find many of the parameters of practical importance from the momentum balance equation.

(Refer Slide Time: 21:19)

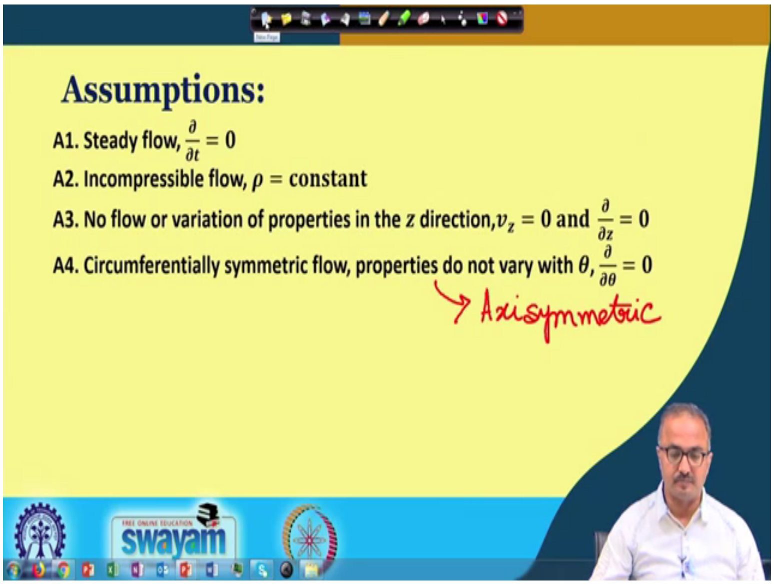


Next we go to another example. In this we are considering the laminar flow in another space of rotating cylinders. Rotating cylinders means here we have a situation, situation that a liquid is contained in the annular gap between two vertical concentric cylinders, ok. Now you can see that these two cylinders are there and in between the annular space there is some liquid which is shown by some shadow, ok. And here we have given the radius of these two cylinders. The inner, inner radius is the R1 and outer reduces R 2. And from here we are taking the origin this is a z direction this is the R direction and this is a theta direction, ok.

So, for this particular geometry, we are choosing a cylindrical coordinate and what we are to find out, we have to find out the velocity profile, the shear stress distribution and shear stress at the surface of the inner cylinder. The thing is this the inner cylinder is stationary and outer cylinder is rotated at constant speed so; that means, there is a relative motion between these two cylinders and this kind of flow is called a Couette flow, ok. So, we are having a Couette flow situation and is the Couette flow, ok.

So, this is the kind of flow whenever we have this relative motion, ok. And this kind of situations, we may find in some kind of bearings we have this kind of situation, ok. There is a relative motion. So, let us see how we are going to analyze this situation.

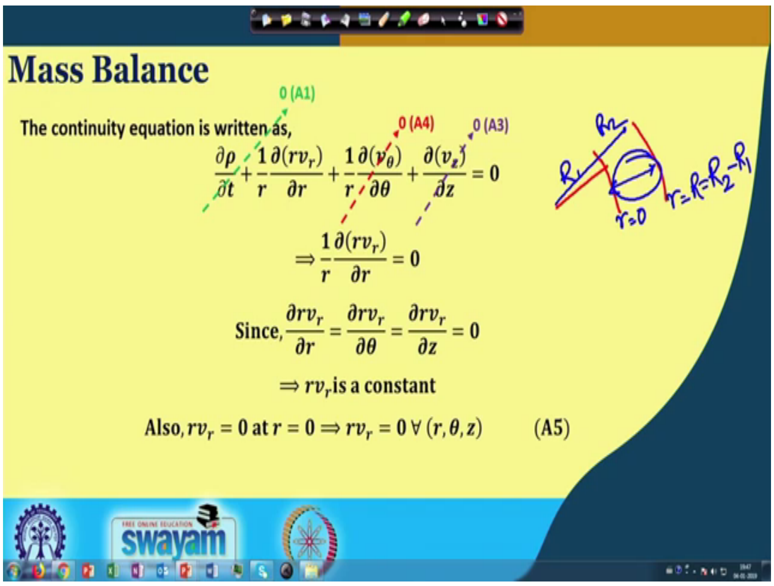
(Refer Slide Time: 23:06)



Again we make these assumptions that steady state flow in incompressible then no flow in the variation in the z direction. So, vz and dou by dou z are 0 and circumference symmetrical that is called what we call the Axisymmetric.

So, when whenever we have this kind of situation, we call it axisymmetric. So, this is symmetric along the y axis that is the no variation along the as we move throughout to the direction, ok.

(Refer Slide Time: 23:50)

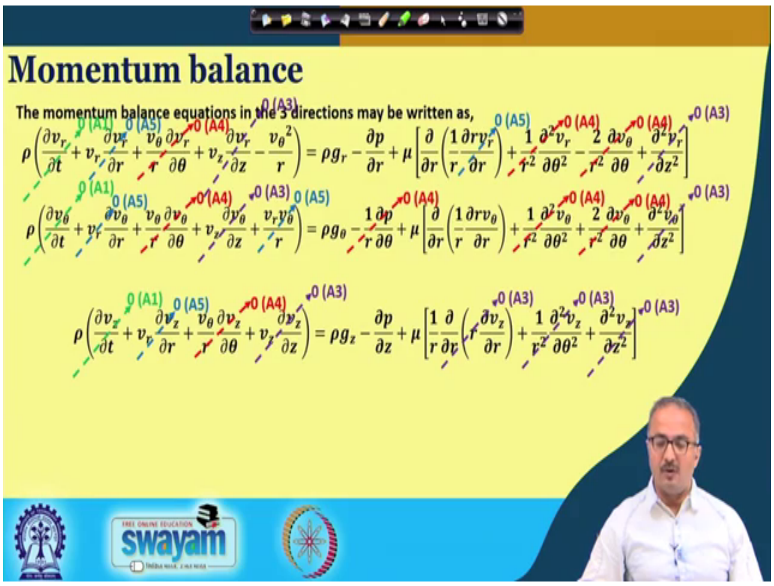


With these assumptions now we go to reduce the various equations. First again we go with the continuity equation, ok. So, now do you see that because it is a steady state so, we are taking this time derivative to be 0. And because of excess imagery we are taking the theta derivative is 0 and as we said that there is no variation in the z direction so, this is 0. So, what we reduced with we are reduced with this particular equation. And again you can see that dou rho rv by dou r is 0 from this continuity and dou vr by dou theta is 0 because of axisymmetry and dou rho rv by dou z is 0, because the assumption that there is no change in the z direction.

So, with all these things we find that r v r is a constant so; that means, is a constant means now let us see that this v r is having a at this r equal to 0 and now please understand this r is taken from the outer surface of the inner cylinder, ok. So, in this case if I, if I look at the cylinders, if I draw these cylinders, ok. So, these are two cylinders. So, we are our interest of domain of interest is this is our domain of interest, ok. This is R 1 and this is R 2. So, our we are concerned with this particular thing, ok.

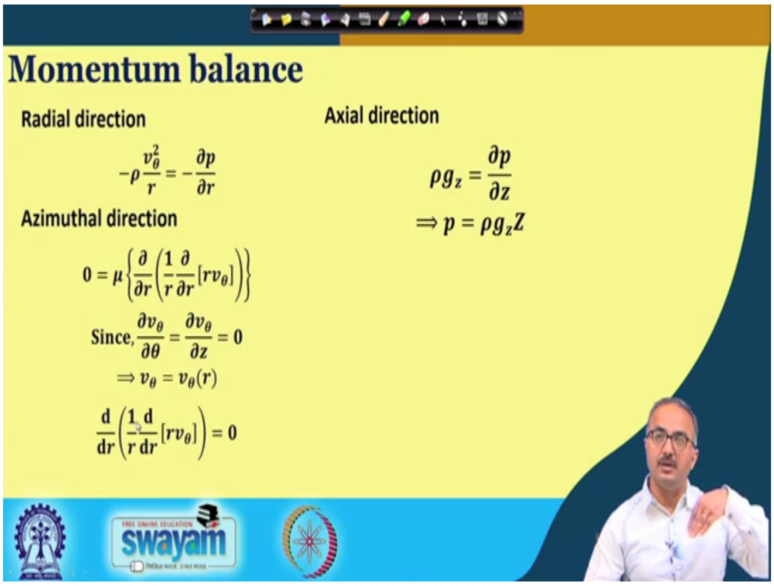
Now, in this case we find that this c suppose this is r equal to 0 and this is r equal to some r which is nothing, but the difference between these two R’s, ok. So, in this case we had r equal to 0, we are taking because this cylinder is that it is not moving, it is not rotating. So, that is why we are saying r vr is 0 that by, by the no suit condition. So, you find that this r vr is a function of only it is, it is just, it is independent of r theta and z.

(Refer Slide Time: 25:57)



Now, with this now we consider the momentum balance equation. Again we are going to reduce the various terms based on the various assumptions as we have done in our earlier problem. So, I have written that how we are reducing the various terms to 0 based on which kind of assumptions, I am not going into the explanation of these; you can easily correlate it easily. And now you can see that slowly and slowly we are able to reduce the three dimensional equation, ok.

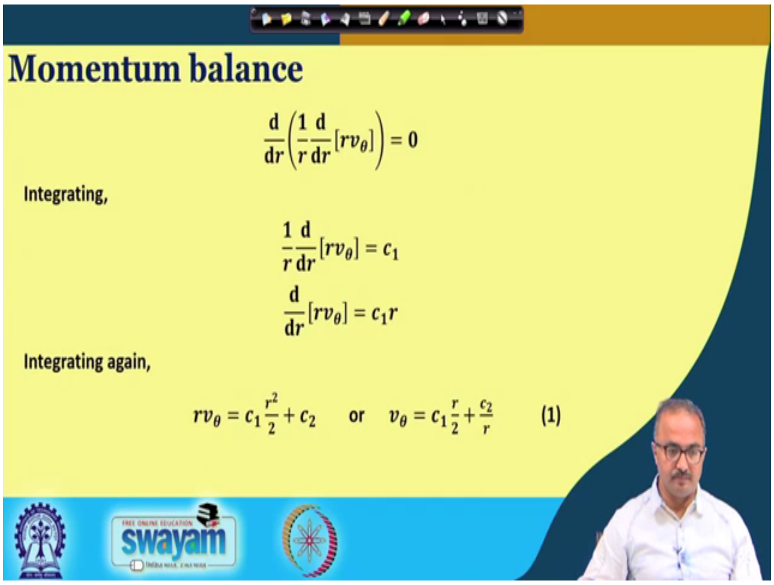
(Refer Slide Time: 26:33)



Now, all these after all doing this then what we do we find that we are getting all these equations. So, this is the Radial direction from Radial direction momentum balance, these are azimuthal direction momentum balance and this is for the Axial direction momentum balance. So, this gives us the first part. Now let us go to the second one.

And here you can see that because this incase of this where this v theta v, because theta symmetry is there. So, v theta we find that v theta is a function of only r, because dou v dou theta is 0, axisymmetric case and z now variation has been taken 0. So, v theta is function of only r so; that means, the rotational velocity with the will be changing from the inner cylinder to the outer cylinder.

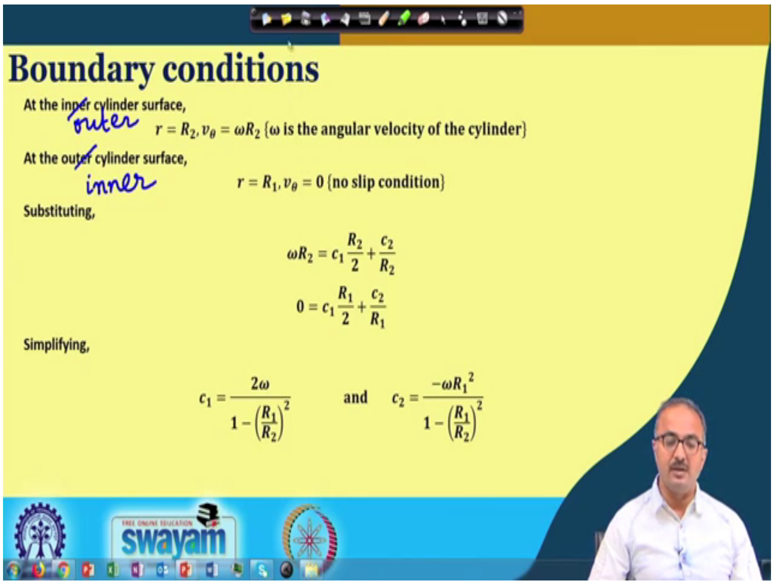
(Refer Slide Time: 27:22)



And one more thing I should tell that in this z direction what we are having this pressure is varying in the z direction. And this is due to the hydrostatic head of the particular fluid. Now coming back to momentum balance equation, we have, we have once we are considering as per the particular solid problem, we are only considering the theta direction, because we have to find a shear distribution, ok.

So, here we are doing the mathematics for this that we know that this is 0; that means, this particular thing is a constant c 1 and then we find that we are doing mathematics, here we are just multiplying by r. And then we can integrate it and integrating we are getting this particular equation. Here again we have two constants of integration which may be obtained by the, by the boundary conditions.

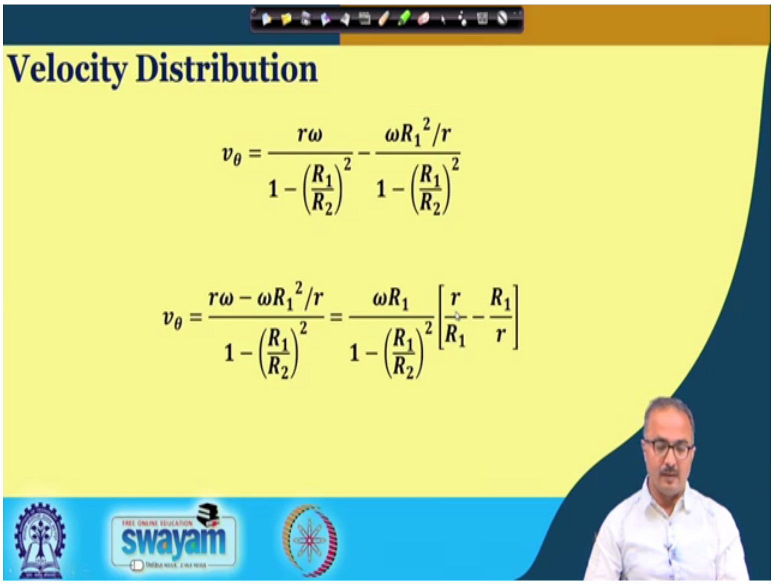
(Refer Slide Time: 28:17)



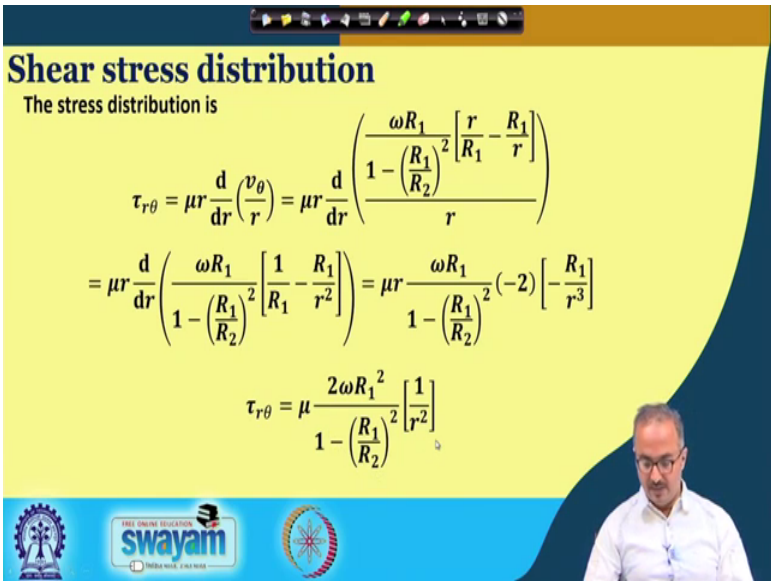
So, here we have the boundary conditions that at the inner cylinder that that is the inner surface the outer cylinder, we have the omega R 2 is the velocity linear velocity at the inners of the outer cylinder and this is the outer surface of the inner cylinder, we have the v theta to be 0 no slip boundary condition. So, this will be there is a slight correction over here, it is basically the inner outer cylinder and this is the inner cylinder, ok.

So, these corrections have to be made. So, with these boundary conditions, we can now easily find out the values of the c 1 and c 2. Once, you find the values of the c 1 and c 2 it is now very easy to determine the other variables that have been asked in the problem.

(Refer Slide Time: 29:23)

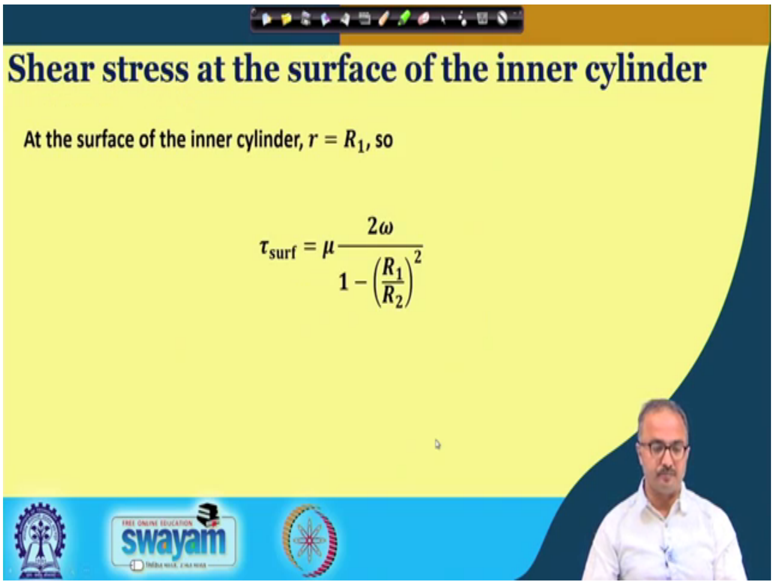


(Refer Slide Time: 29:29)



So, here is the variation of v theta with r. So, here is the variation of v theta with R and here is a stress distribution that dou tau R theta is nothing, but this particular expression. So, you get the, differential of v theta by r and by mathematical manipulation you get the stress distribution along the Radial direction.

(Refer Slide Time: 29:48)



And this is that at the surface of the inner cylinder r equal to R 1. So, that you get at the surface of the inner cylinder this is the stress you obtain. So, this is quite now simple once we found the velocity distribution after that it becomes really simple to find out the shear stress distribution.

(Refer Slide Time: 30:09)

So, these are the references which you may refer to get more explanation of these problems.

Thank you.