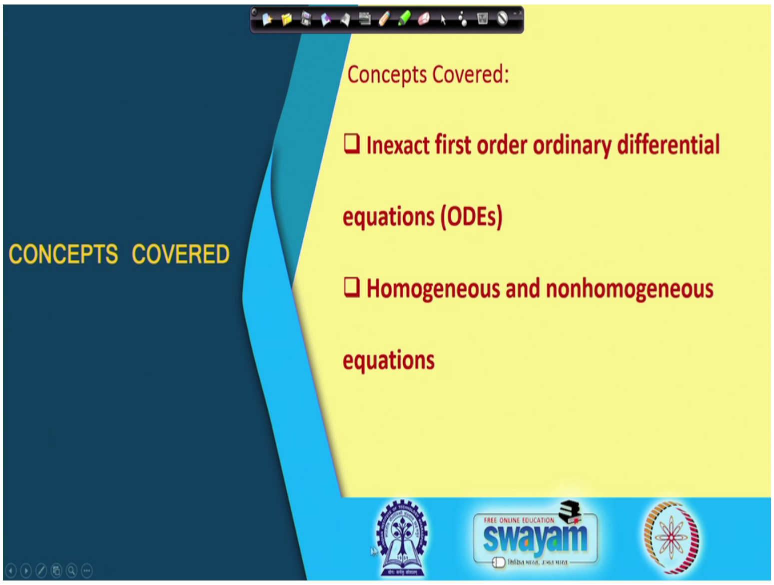
Mass, Momentum and Energy Balances in Engineering Analysis
Prof. Pavitra Sandilya
Department of Cryogenics Engineering Center
Indian Institute of Engineering, Kharagpur

Lecture – 21
Mathematical Solution of Macroscopic Balance Equations (Contd.)

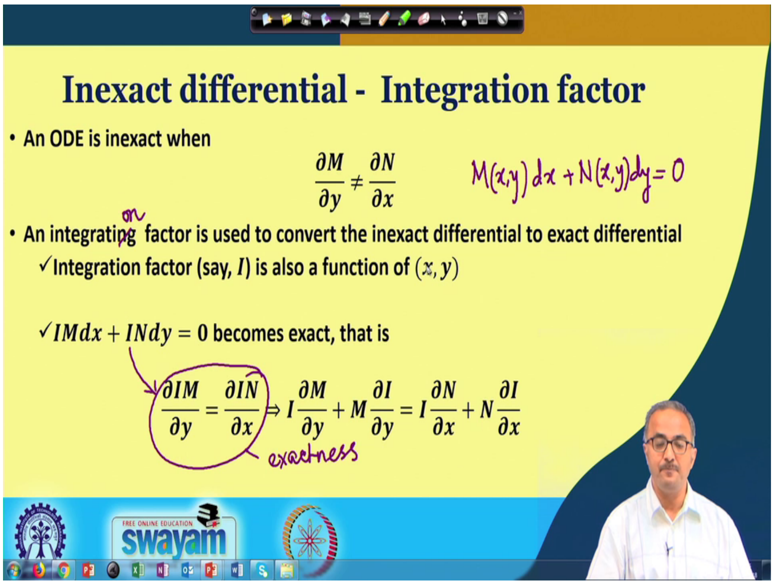
Welcome, today in this lecture we shall be continuing the Mathematical Solutions of some other kinds of Macroscopic Balance Equations.

(Refer Slide Time: 00:29)



And in this lecture we shall be looking into the inexact the first order ordinary differential equations and homogenous and heterogeneous equations.

(Refer Slide Time: 00:38)



So, first let us take out this inexact differential and in this case we shall be using some integration factor.

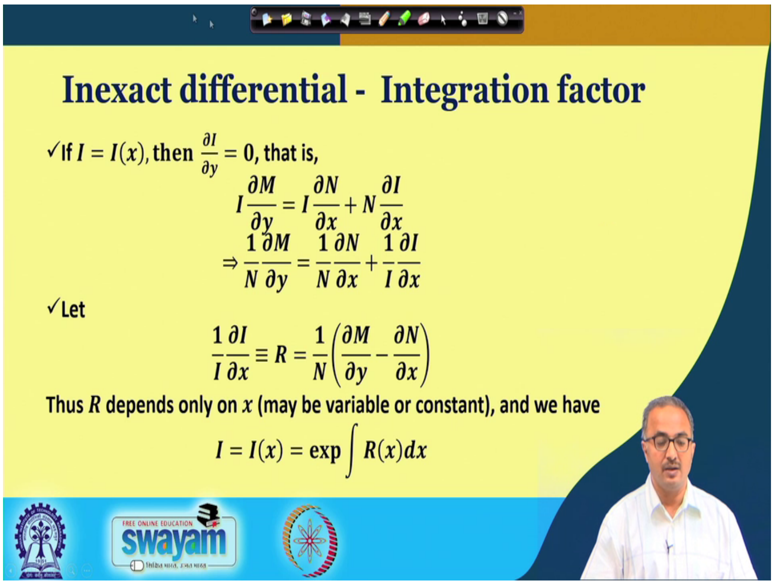
So, you see that what we learnt in my earlier lecture was that the exact differential is one, we in which we found that the when we are rearranging the equations in the in the terms of the M x y; M x y into dx plus N xy into dy equal to 0. So, we wrote this equation and there we said that when this particular in this particular equation we have dou N by dou y equal to dou N by dou x, then we have the exact differential and we said that these kind of situations happen when in the particular parameter is not a path function, but a state function ok.

But in many cases, like in case of work in case of heat transfer etcetera we find that the particular variable is not a state function, but is a path function. So, in that case we shall be having this kind of situation, where we shall find that dou N by dou y is not equal to dou N by dou x. So, in that case how do we go to solve this kind of problems? So, in this case we put some kind of an integration factor, so, that to convert the inexact differential into the exact differential form.

And here we assume that the integration factor, here I it is more correction it will be integration factor ok. So, in this case we find that this integration factor is a function of both the independent parameters that is x and y in this case. Now with this knowledge what we do is this that, if I is the integration factor, we multiply this equation this parent equation with I ok, this particular equation we multiply with I. And now you can see that this particular equation can become exact; that is let us put in this fashion like we put that dou N by dou y equal to dou N in by dou x, this is the particular condition for exactness ok.

So, with this, this is a condition for exactness ok. So, if this particular this equation becomes exact if we have this particular situation ok. Now what we do we simply make this expansion for dou IM by dou y that is I dou M by dou y plus M dou I by dou y and similarly we do if it is a put it break it up this particular differential as I dou N by dou x and plus N dou I by dou x ok.

(Refer Slide Time: 03:50)



Now, after this particular transition, what we do is this that, let us first assume that is a function of only x, in general I is a function of only on both x and y, but first we are assuming that I is a function of only x. Now if that is the case, then we find that dou y dou I by dou y will be 0. Now in that case from the previous equation what we have just find out this equation, we can put this dou I by dou y equal to 0 and then we obtain this particular equation.

And after this what we do? We just rearrange the equation like this that we simply divide this whole equation by M I, so sorry N I. Once we divide this equation by N I we will get this particular equation. And now what we do? We keep this particular term on one side and take this term on the other side and what we are achieving by doing this kind of arrangement, what we are finding that 1 by I dou I by dou x becomes purely a function of x and that we are putting in calling as R and this R is nothing, but this 1 by N dou M by dou y minus dou N by dou x.

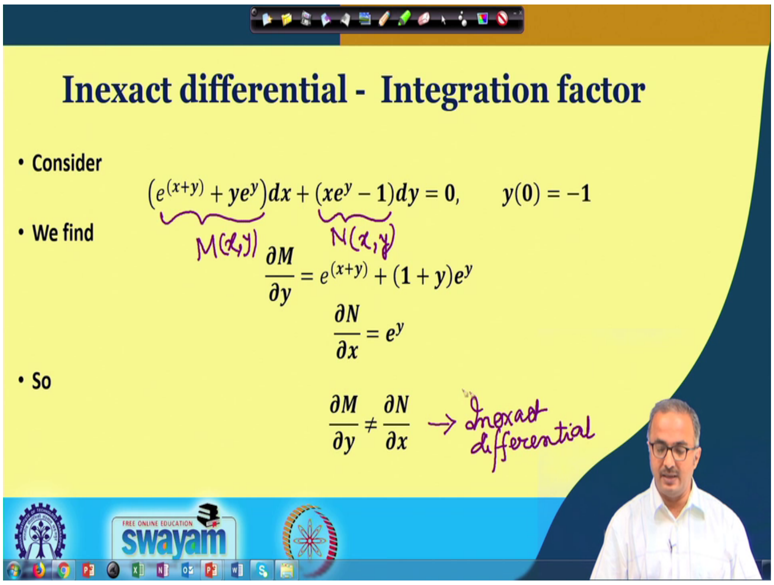
Now, because this is a function of only x; that means, this term whole term should also be a function of only x ok. Now what we do R will depend on x only and it will be variable or constant and so, we have once we integrate this particular equation, we find that we shall get logarithm of I is equal to R and then we will basically we put this as an exponential term R dx we do, so it becomes that I equal to exponential of this integration of R x into dx ok

So, once we do this particular manipulation, we would be able to get the value of the integration factor provided it is the function of only x. Now we have to also see that in all situations it may not be function of x, it may be function of only y. So, let us take that situation, it is similar to what we have done just earlier. So, here if we have I has a terms of only y, then dou I by dou x will be 0. Again going back to this particular equation, we what we do we take this particular term as 0, then we find that we have this equation ok.

Again we divide this whole equation by M I and then we get this particular equation and we retain this x my term on this reference side and take this term on the right hand side. And what we gain by this we find that this particular term is a function of only y and let us called this term as R star ok. To distinguish it from R; that is the one by I dou I by dou x ok. So, this R star is nothing, but this particular term; that means, this particular term should also be a function of only y.

So, again we do the integration as before. So, it will be logarithmic of I equal to R d y. So, I equal to exponential R star y into dy. So, this is the way we are getting this expression for I, if I is a function of only y.

(Refer Slide Time: 07:18)

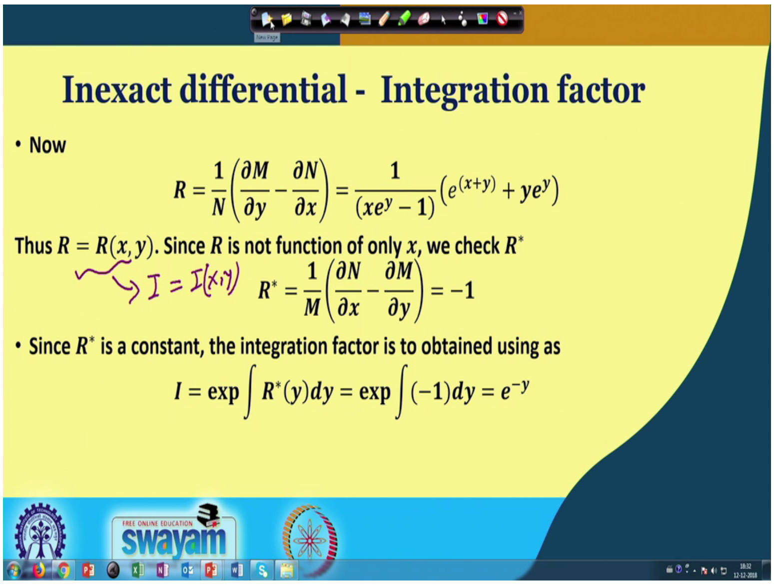


Now, let us implement these particular concepts for this given example. Here we have been given this particular equation as you can see here ok. Now we shall see that this is a first you have to check whether this is exact or inexact.

So, for that we understand that this particular term, this particular term is M M x y in general and this particular term is N x y ok. So, after identifying this M and N, then what we do we find the this differential that is dou M by dou y and we find this is coming out like this and then we see dou N by dou x that is coming like this.

Now, what we see from these 2 expressions that, dou M by dou y is not equal to dou N by dou x; that means, we have in exact differential. Now once this is the case, let us try to see whether we can find some integration factor which can make this particular system exact ok.

(Refer Slide Time: 08:43)

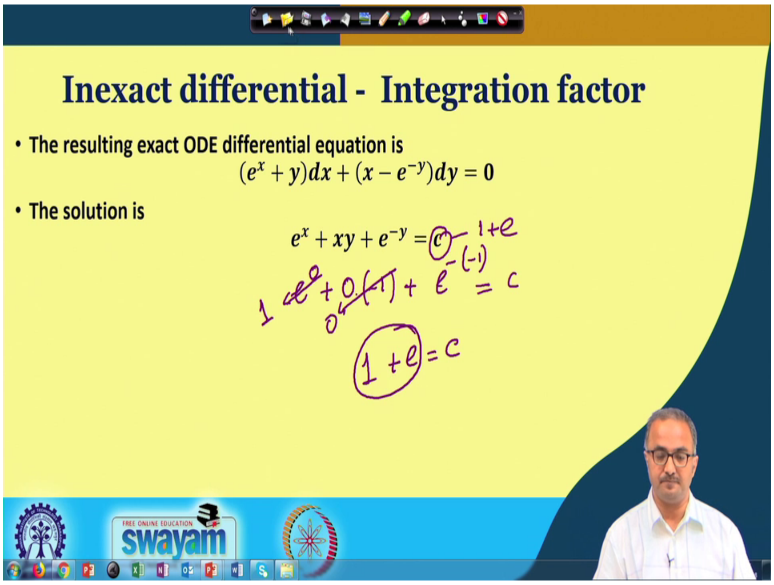


So, following our earlier derivation, now what we see that R is equal to 1 by N dou M by dou y minus dou N by dou x. So, we put the value of the N here and we find the value of dou N by dou y and dou N by dou x and we get this particular terms. Now we see that R is function of both x and y; that means, what that if I is a function of only x, then we should have R has function of only x. Now because we find that R is a function of a x and y it means it means I is also a function of x and y from this we are finding this ok.

So, we should not be using this particular way of finding the integration factor. Now let us look into the other way if we assume that I is a function of only y; that is, let us check ok. Now for that what we do we find the R star that is, but as one by M dou N by dou x minus dou M by dou y and we find this is coming to minus 1 ok. So, this shows that R star is a function of only y may be considered to be that how this minus 1 may be written as this may be written as minus y to the power 0 ok.

So, we can say that R star can may be considered to be a function of only y. So, if we can write it like this then we find that here the R star is a constant. So, if we integrate it, so we find that I equal to exponential of this integration R star; that is minus 1 into d y and that is coming I to be e to the power minus y. After getting this integration factor, what we do? We go to the our parent equation and multiply both the sides with that e to the minus y.

(Refer Slide Time: 10:36)



So, we are getting this particular expression, so we can easily see that this was our parent equation if you multiply this whole equation by e to the power minus y, you have to simply multiply the e y if that is our integration factor this is the I. So, if you multiply this whole equation, this whole equation you are multiplying with ok. So, you find that you what you are having that from here you are taking out that e to the power y is e to the power x only plus y d x into here x minus e to the power minus y ok.

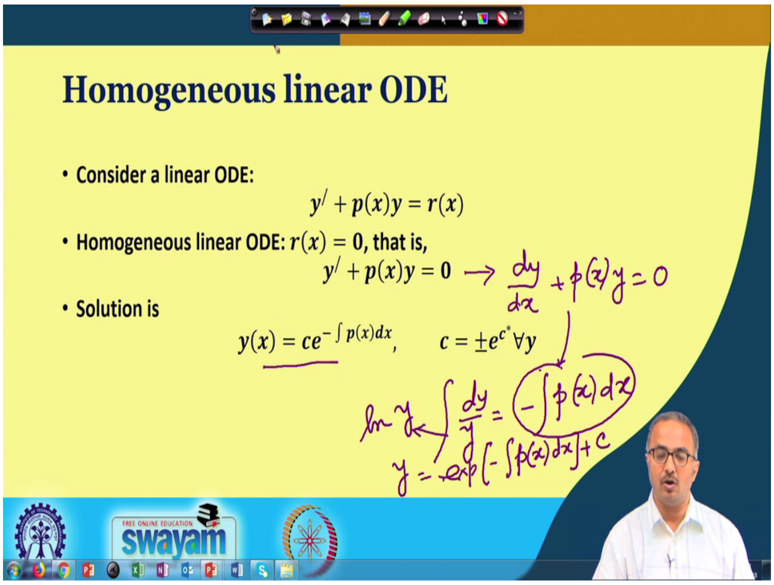
So, that is what we are obtaining here. So, this is how we are getting the exact differential and now I am not going into the solution methodology because this I have covered in my earlier lecture. So, find that this is the particular generalized solution we get. And here we have this C has the some constant of integration and this may be obtained from this particular thing this particular initial condition we are given.

So, what you can do that, you can see that we had we are given that at y is equal to 0 we have sorry at x equal to 0 we have y equal to 1. So, what we can do? If you put here x equal to 0; that means, e to the power 0 plus 0 into y is equal to e to the power plus e to the power minus y and this is also equal to this y is given as minus 1. So you put this as minus 1.

So, you can put it as minus 1 over here and plus e to the power minus of minus 1 equal to C; that means, this is going to give you 1, this is 0. So, 1 plus, if e equal to C that means, C equal to this value ok. So, simple you can put, you can replace this C with 1 plus e ok.

So, that is how you are able to get the value of c for this particular equation from the initial condition.

(Refer Slide Time: 13:14)

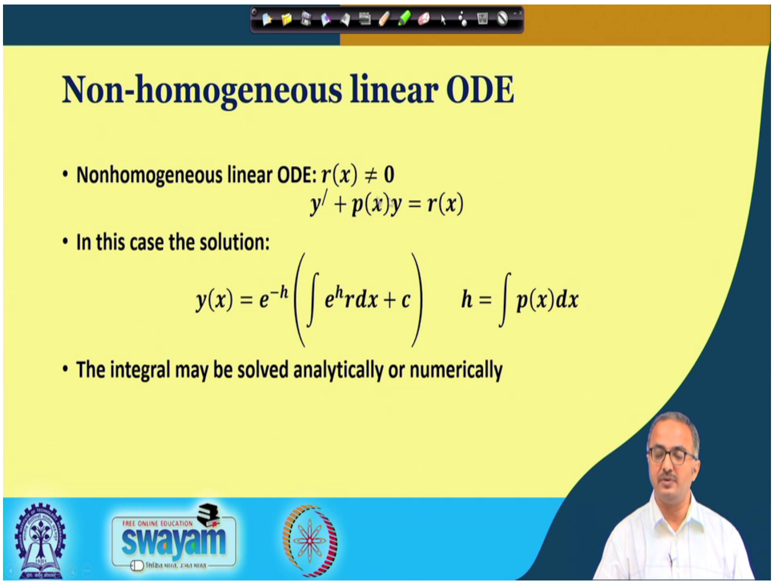


Now, let us come to the next type of these ordinary differential equations, linear odes is in this what you are seeing again the structure may be a homogenous structure or a heterogeneous structure ok. Now let us consider this particular system of the ODE, here we have y prime equal to p x into y plus r x ok. So, now, if we have r x to be 0, then we find that we are having y prime plus p x into y is equal to 0. Now this particular thing is called homogenous form and this thing you many times you get this kind of term maybe then when there are no source terms in the balanced equations you may end up with this kind of homogenous ODE ok.

So, in that case, you can see that it is much easy to get the solution as you can separate the variables very easily. So, you can see that if you write you can write this equation as dy by dx plus p x into y equal to 0, from here to get that dy by y is equal to minus px into dx and then you integrate ok. When you integrate this will give you the this ok; that means, y will be equal to exponential to the power this whole thing; that means, this y will be equal to x this exponential to the power minus p x d x plus some constant of integration. So, that is how you are getting this particular expression ok.

Now, you see that this particular value of this integration constant may be again obtained from the initial conditions of the given problem.

(Refer Slide Time: 15:26)

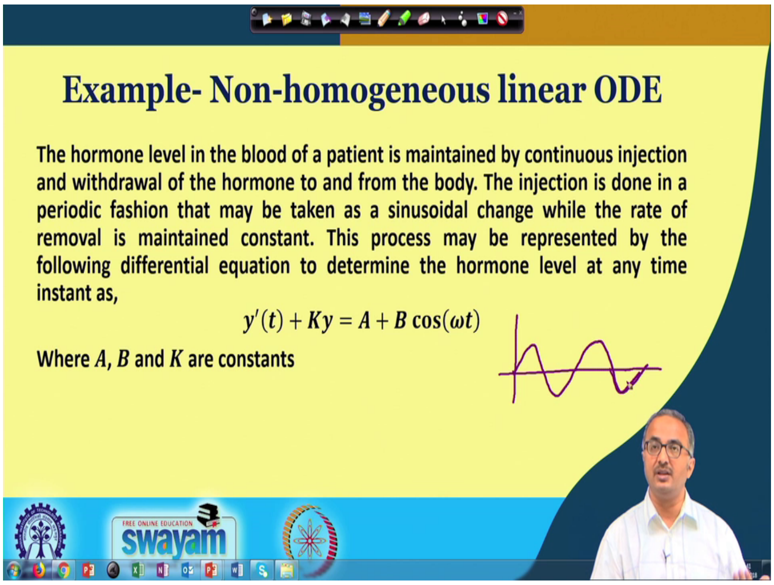


Now, let us see the another form that is the non homogenous linear ordinary differential equation. So, in this case the r x is not equal to 0. So, we have this particular form ok. Now when you have this particular form we can again solve it and for solving I am not going into the detail of the solution derivation, I am just giving you the final express final expression to solve it.

So, you what we are doing the solution will be y x equal to e to the power minus h into this integration e to the power h r d x plus c and h is nothing, but this integration of px d x ok. So, when you have this particular thing, then you can solve this for this integration if it is a simple expression, you can integrate it analytically and if you the expression is a complex then you should use any numerical technique for finding this do this integration.

So, I will shall be talking something about this numerical integration and numerical differentiation in my future lectures. Now, so this is a way you can solve this for y.

(Refer Slide Time: 16:37)



Now, let us take an example of the non homogenous linear ODE. So, here we have a question which is from some bio technological applications, here we have that hormone level in the blood of a patient is maintained by continuous injection and withdrawal of the hormone to and from the body ok.

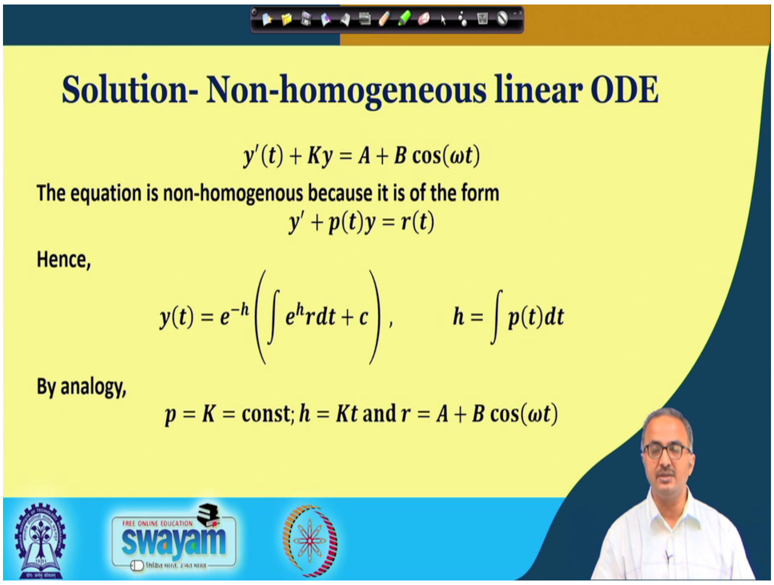
So, it is a patient somewhere to make a balance of the hormone maybe some gland is not working properly and you know that in our a biological systems, we have many many glands are involved in secreting the hormones to maintain our body activities. So, if some gland is not working you will find there will be hormonal imbalances; for example, insulin may not be situated properly, so that our sugar level in the blood will keep on fluctuating.

So, for treatment of all these things what is being done that in you are injecting and also withdrawn the hormone from the body. The injection is done in a periodic fashion that maybe taken as a sinusoidal change where the rate of removal is maintained constant ok. So, this kind of a situation we are considering that the removal rate is constant, but the injection rate is periodically valid and for simplicity sake we are assuming this periodicity may be represented as a sinusoidal wave.

And perhaps you know when we talk of sinusoidal wave; what you mean is, this we have some kind of and amplitude and a time period ok. So, this kind of a sinusoidal wave, so this is the kind of thing we are talking about ok. So, this is a kind of a we are assuming that is a periodically getting injected to the system..

So, that this process, may be represented by the following differential equation to determine the hormone level in the blood at anytime, so this is the petrol expression again we are not going to the equation how it has been derived? But these are final expression to know the hormonal level in the blood and here this A B K A B and K are taken to be constant and this omega may be changed by the user.

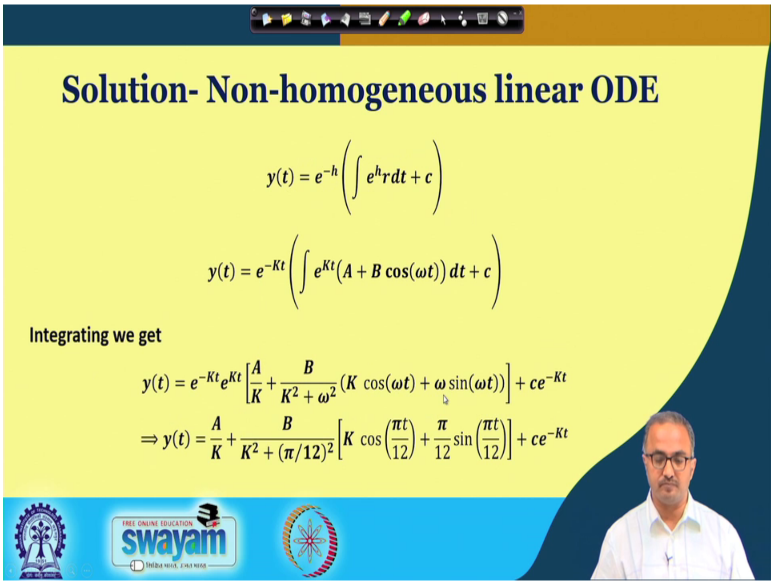
(Refer Slide Time: 18:48)



Now, let us see that how to solve this kind of a problem. Here we have this is a expression given and we when you put this in terms of the generalized expression we find this is y prime plus p t into y plus rt.

And now if you see that here pt is nothing, but K and this rt is nothing, but the whole of the right hand side ok. Now when we have this y t equal to e to the power minus h and this integration e to the power h are dt plus c and h is this now as wise as I said that by a comparison we can see p is a constant and h is a K t and r is A plus B into p cos omega t.

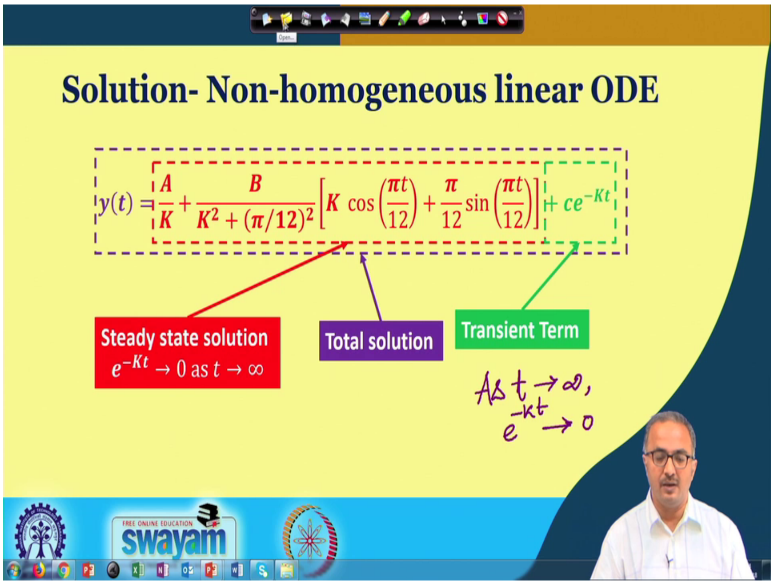
(Refer Slide Time: 19:37)



Now, once we know this, what we do? Now we simply take this M h to be this particular thing from our expression and we put all these things over dt e to the power h into r we do here ok and now because this is a quite a simple expression for integration, so we can carry out the integration and we find that we will be having this kind of an expression ok.

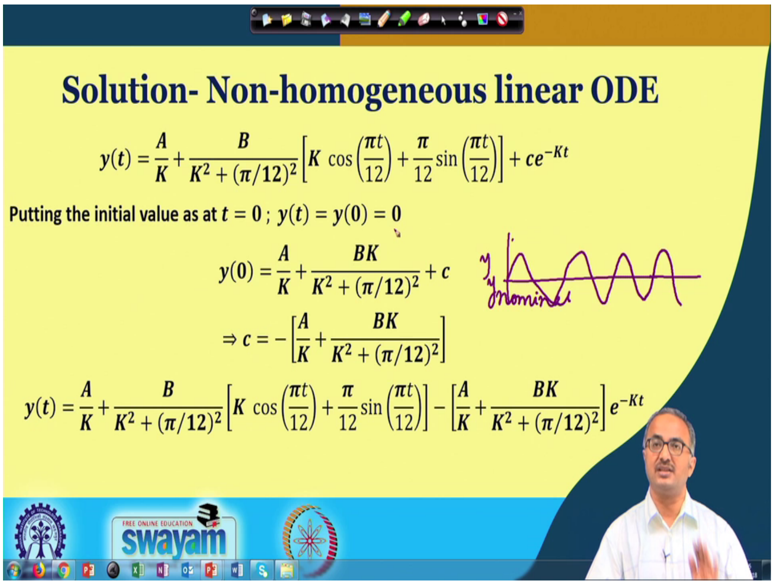
And when we rearrange it, we will get this party expression. Now what you find in this expression is this that, this particular term is the sinusoidal term and this particular term is going to get diminished with time ok. So, this particular term is kind of giving us the transient change in the hormone levels in the blood and if this term goes whatever we are left with will be giving us some steady variation of the hormonal level in the blood.

(Refer Slide Time: 20:42)



So, what we say here is this with transient term because as this is the total solution is this ok. So, this by this particular violet line, we are saying that we get the total solution and when as we see that for this one we see that as, as time goes to infinity, we find that e to the power minus K t tends to 0 ok; that means, what, that at long time this term will vanish. So, this thing we shall be left with and this is giving rise to a sinusoidal form. So, this particular is red coloured one is giving us the steady state solution ok.

(Refer Slide Time: 21:35)



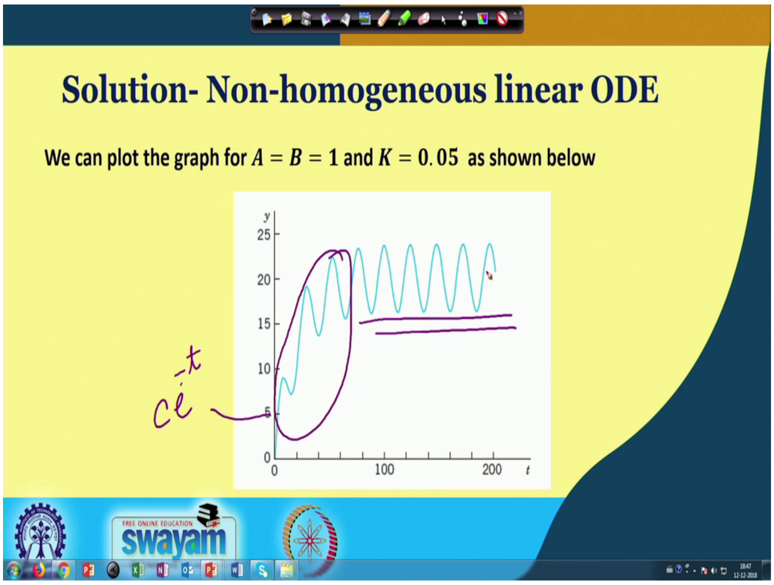
So, now if we plot this particular expression, we can see that how we shall be able to this what can kind of variation is this, this the variation of things.

And now what we are doing this we are trying to find out the value of the integration constant. So, from our problem, we take that initial value of wire that at time t equal to 0 we say that there is no hormonal things. This 0 means for that it is not that we have 0 hormone, it is something some best value because, we are here we are not really interested to know the exact value of the hormone in the blood here we are trying to see that some base value, how the hormone level is fluctuating ok..

So, you can see, you can say like this that we have some base value over here. Suppose we take this base value, this measure we call it nominal value, this one and over this how the things are fluctuating. So, we can put it like this and many a times we do this kind of analysis instead of getting the exact value of the particular variable, we take what we call the difference variable and this concept, you can also see in the enthalpies etcetera and in control systems we also talk about the difference variable rather than the exact variable to check that how much we are fluctuating or deviating from the nominal value or some kind of set point for the given parameter.

So, here we are using the same kind of concept here, that is why we are putting that y equal to 0 and it should not be interpreted as if there is no hormone at t equal to 0, but should be taken as the hormone level is exactly what is desired in the blood ok. So, with this particular initial condition we can see this is a value of C and here we are putting the value of the constant of integration.

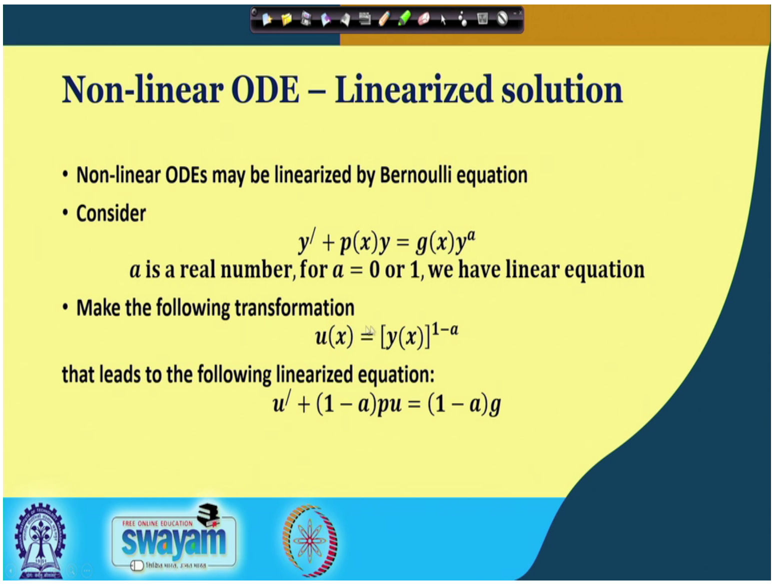
(Refer Slide Time: 23:48)



And now with this manipulation if we take the value of the A and B to be 1 and K to be 0.05, we have this typical fluctuation of the y with respect to time and you can see that initially we can see that this region is giving us the transient. And this is being this is coming from the c e to the power minus t term ok. So, this thing is coming from this and ultimately finds this term dies down and in this region we are getting the steady state solution, where we are finding this has assumed a periodic fluctuation of the hormone level in the blood ok.

So, similar kinds of problems are also then many other fields you may find and you have to treat these problems in this particular fashion.

(Refer Slide Time: 24:39)

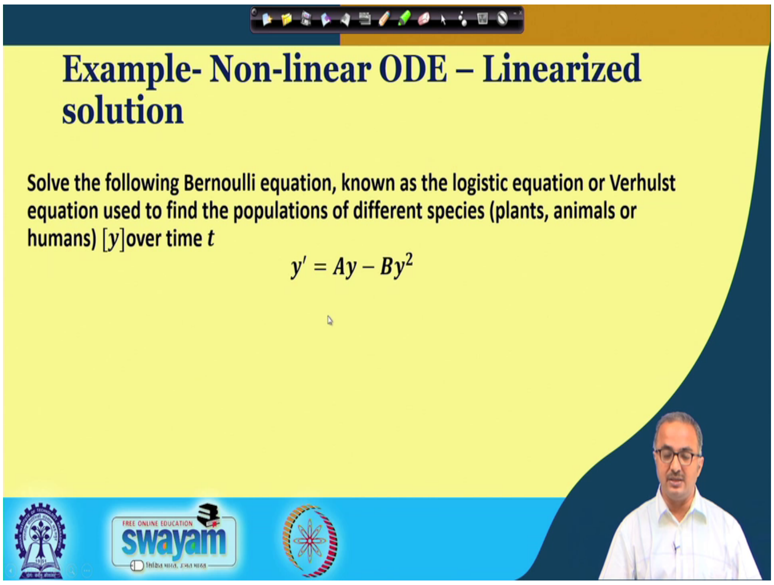


Now, we shall look into non-linear ODE and in this case what we do that we get the solution by linearizing the non-linear ODE ok. And one of the methods is to do by Bernoulli equation please do not confused this particular with the energy balance we have derived earlier for and we named it Bernoulli equation.

Ok now, this particular method was proposed by Bernoulli the same scientist ok. Now here we have this particular expression for the non-linear ODE and in my earlier lecture I told you what you mean by linear and non-linear equation ok. Now a is a real number it is here and if a equal to 0 or one we have the linear ODE linear equation. So, without again going into the detail of the solution, here we have the final Bernoulli equation, he suggests that you make this particular transformation. So, you put another variable u and put that as y to the power 1 minus a.

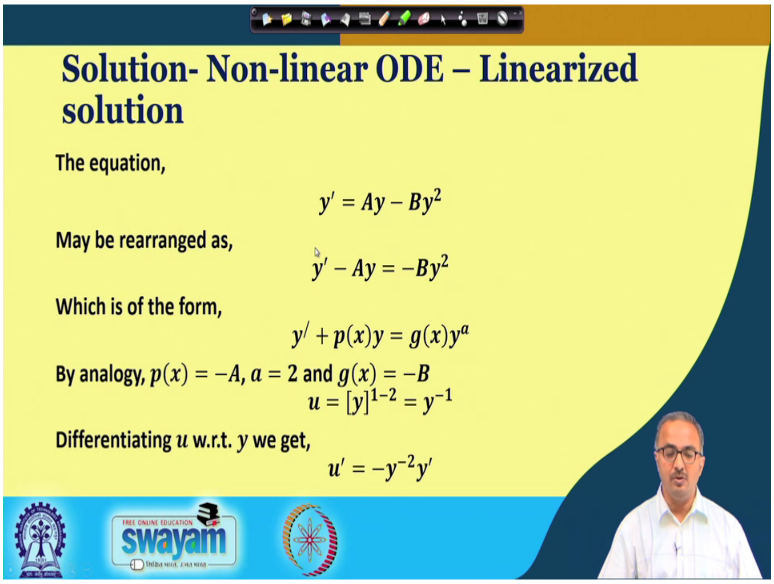
And with this particular thing, you just replace in this equation and you get for this equation. Now you can see that on the right hand side, we have totally taken of the any dependency on the u and we have linearized the original non-linear equation in at in terms of some transformed variable ok. So that means, what we shall be solving for will be u and once we know the value of u, we can now solve for y ok.

(Refer Slide Time: 26:19)



So, let us take one example problem. So, here we have one equation which is called a logistic equation or verhulst equation, which is used to find the populations of different types of species in the world like plants animals or humans and how this population varies with time. So, this is the equation ok. Now you can see this particular equation is a non-linear equation.

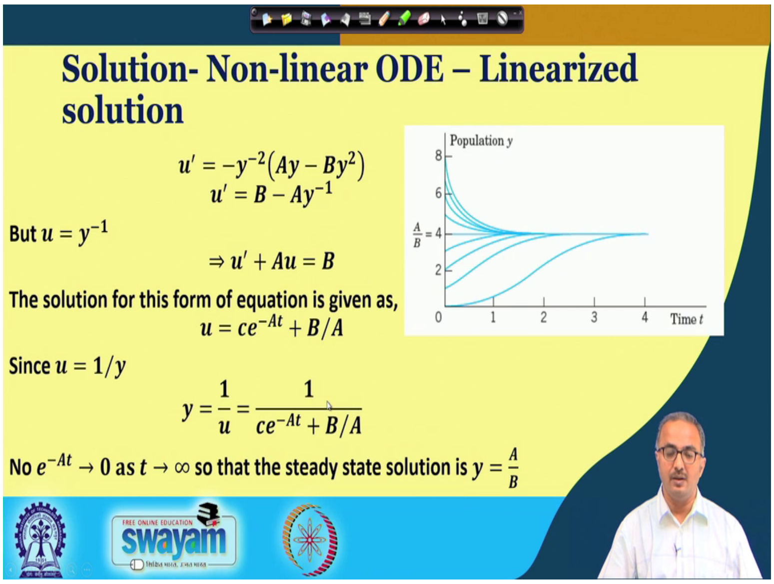
(Refer Slide Time: 26:46)



So, we rearrange the equation like this and if we put in the form we just presented in earlier, we find that y prime plus px y is equal to g x, but your a and now we compare these thing we find that px is nothing, but minus A and a is 2 here and gx is minus B.

So, incidentally px and gx are constants. And now what we do the transformation as we suggested that u is equal to y to the power 1 minus a, that is 1 minus 2 I equal to 1 y to the power minus 1 that is equal to 1 by y and this is the we when we differentiate with respect to this u with respect to y, we get u prime equal to minus 1 by y square into y prime ok.

(Refer Slide Time: 27:39)

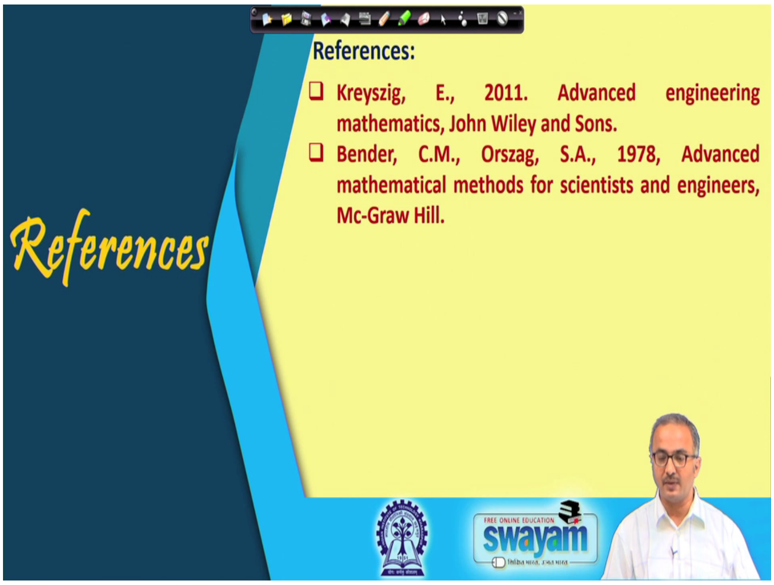


Now; that means, u prime equal to this particular expression and we have this simply we are doing the mathematical manipulations with respect to the original equation. So, here this you can easily see that u prime will B equal to B minus A to the A by y and this we know that this is equal to u. So, we simply replace this 1 y with u and we get ultimately this particular expression which is a linear equation and we know that we know we have learnt how to solve this linear equation and this is the particular solution of the linear equation.

And again we are having this particular constant c, whose value may be obtained by the initial condition prescribed in the particular problems ok. Now we know that we have obtained u, now we can find the value of y by simply taking the reciprocal of u and this is the final solution for y ok. And here again we shall see that this particular is showing a transient state and as t goes to infinity we find this particular term tends towards 0; that means, that as steady state this y value will be A by B; that means, all the populations whatever may be the initial state, they always be converging to this steady state value and this has been shown in this particular graph, you can see that we take choose different value of A by B.

So, whatever value of A by B we choose a over a period of time we find that they show different type of the variations; that means, some of them are decaying and some of them are showing growth ok, but ultimately whether its decaying or growing, ultimately we find all them tend to converge to the value of the particular A by B ok. So, this is a y and this is a time ok. So, this is one for some given A by B ratio, you can change the value of A by B ratio, you will find that you are converging to different values of the population and of course, this A by B will be determined by the local factor, as the geographical factor or a demographic factors, so will be coming into action whenever you are deciding the population of any species in the world.

(Refer Slide Time: 29:58)



Now, you can look into these references for more detail and more examples on these particular topics.

Thank you.