Mass, Momentum and Energy Balances in Engineering Analysis
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Lecture – 20
Mathematical Solution of Macroscopic Balance Equations

Welcome. In this particular lecture we shall be learning about the Mathematical Solution of Macroscopic Balance Equations. So, far in the lectures we have seen how to make the macroscopic balances, and also in the previous lecture we have done some numerical techniques to find out the roots of some non-algebraic solutions. We considered single solution single equations and also multiple equations.

Now, from this we shall be looking into the differential equations which are also obtained when we are writing the macroscopic balances for unsteady or the dynamic situations ok. And, the solutions maybe mathematical and numerical; so, first we shall look at the mathematical solutions because, these kind of solutions give us the exact solution without any approximation. So, approximations are made during the modeling, but once we get the model equations when we apply the mathematical methods we shall be getting the exact solutions.

So, first we shall look into the mathematical solution of the various types of the macroscopic balance equations.

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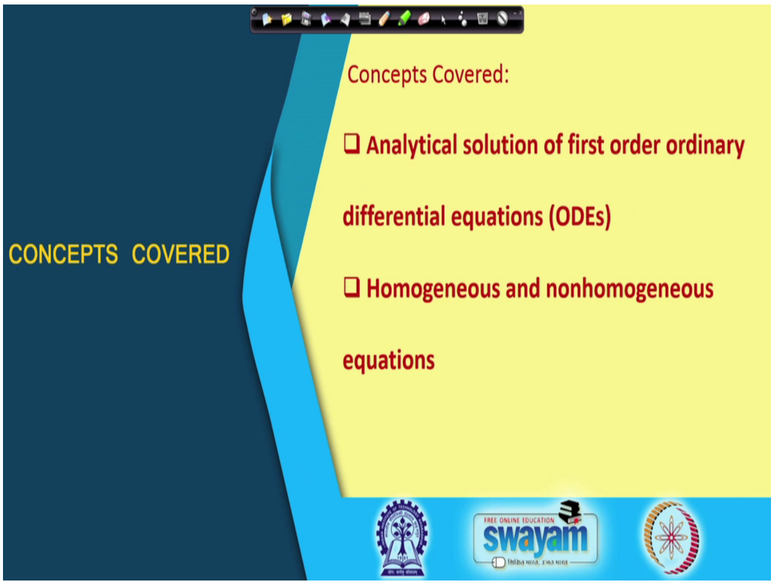

So in this, we shall look into this analytical solution of first order differential equations; why because, we have seen that in our cases whenever we are developing this non-steady state balances, in case of lumped parameters systems we found that there is no variation of the particular process variable with respect to the space. Only the variation comes with respect to time.

So, we shall we having only first order ordinary differential equations. And then we shall also look into the type of the equations. So, one type is the homogenous and non-homogenous equations. So, we will find that there are various types of the mathematical equations and depending on the type of the equation the different types of solution strategies are adapted.

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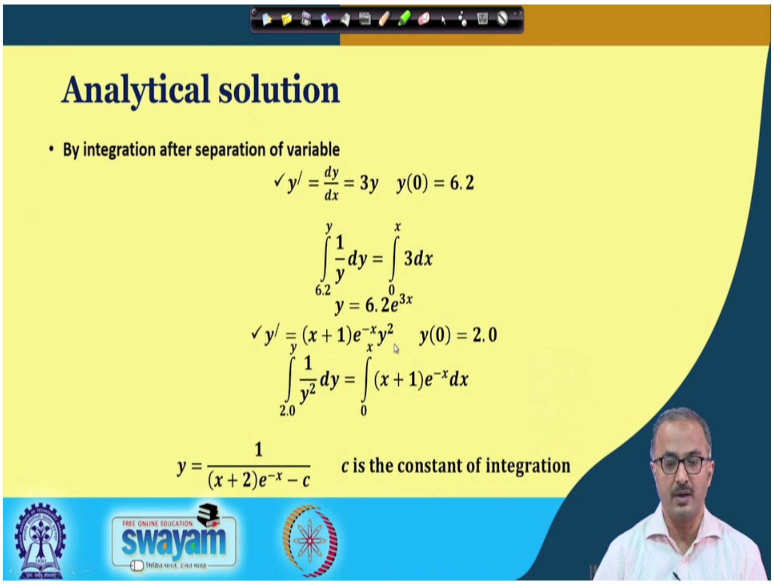

So, first let us go with the simplest case and in this we see that the solution may be obtained by simple integration after we separate the variables. Here we have a simple case. And these kind of situations are many a times we find that when we find try to find the variation of say liquid level change in a tank being filled with some liquid, or the liquid is coming out of the tank, or in case of radioactive decay or some radioactive materials. So, this kind of equations we encounter.

So, we take a representative case for those kind of situations, and here we have such case that we see that y prime which is nothing but the dy by dx is equal to 3 y. And to get the particular solution we also need to have the initial conditions. So, here the initial condition is given as y at t is x equal to 0 is 6.2. Now please understand here we are taking these variables y and x in general when we talk off the unsteady state equations then this x will represent the time. So, here I am not writing time, but the by replacing x with t will not change the solution strategy and the solution.

So, this kind of things is the simplest. And now, we separate the variables like this that we take d x on the right hand side and y we take on this side. So, one side we have only y term and on another side of the equality we have another x terms ok. So, once we put in this term then what we do that we put our conditions here that at x equal to 0 the y is 6.2 and x at any arbitrary x y will be having some arbitrary y value. Now you see that when you take the logarithm of this integration of this 1 by y d y we find that you get log that is natural log of y and then this will give you 3 x. So, this from here we get 3 x and here we get natural logarithm of y and when we take exponential then we are left with y on left hand side. And this particular thing goes to the exponential term. And here, we have this particular initial condition over here. So, this is a very simple way of solving this particular type of differential equation.

Now we have another case: it is a bit more involved. So, here again we have dy by dx equal to x plus 1 e to the power minus x y square. And you can see here that when this particular x goes higher and higher these sums this particular e to the power minus x goes to very small and it means we can say that it goes to a almost a 0 value, ok. Then you can see that dy by dx is of approaching a 0 value as x tends to infinity ok.

Now, you see that y equal to 0; that means, y at x equal to 0 is given to be 2. Now with this again we can see that we can easily separate the variables out, we can make the dx term on the right hand side, and this y spectrum can be taken on the left hand side. And we have a clear separation of the two variables x and y. Now we integrate it over here with initial condition that x is equal to 0, we have y equal to 2, and arbitrary x we have some arbitrary y value. And you know the integration of this it will give you that minus 1 into y to the power minus 1 ok.

And this will give it is a bit involved integration, but again you can separate this out. First you can integrate: x e to the power minus x and plus then just e to the power minus d x ok. So, I am not going into those details of how to carry out the of the integration, only I will say that if you have x e to the power minus x dx then you again take this 2 to be two different functions. And while you know the formula that once you can take the x out and take the integral of e to the power minus x and you then differentiate x and take the integral of e to the power minus x. That way you perform these integrations.

So, with this simple integrations you can obtain after some rearrangement the y in terms of the x where, c is the constant of integration that can be found easily from by knowing the value of the this particular thing ok.

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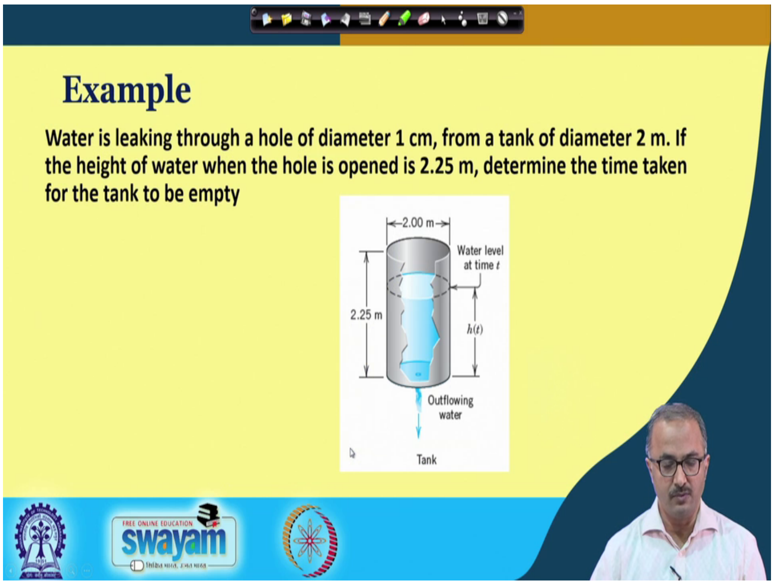

Now, we come to another the application of those form that those that method for a practical problem. What we have given is this; here we have a water tank which is leaking here ok. So, the water is leaking through a hole of diameter 1 centimeter. So, there is a hole at the bottom of the tank and water is flowing out of this hole. And this kind of system happens many at times.

Suppose, I generally we do not like to waste any kind of water, but many a times even if we keep some valve at the exit of the tank, it may so happen if either the valve was not closed by the user by mistake or the valve has given up; that; that means that the valve is not working. So, in that case we have a leakage problem ok. So, in any case of any case break down we have such kind of problem. So, you can see this every practical problem we encounter in our day to day life. And here also we will see that we can apply all the balance equations and how to solve that balance equation.

So, here we have this water is coming out from a hole at the bottom of the tank which has the diameter of 1 centimeter and the tank diameter is given as 2 meters; that means, this diameter is 2 meters. And the initially the water level in the tank is 2.25 meter; when the well this particular hole was just opened. So, initially it was 2.25 meter and we have to determine the time it takes for the tank to get completely empty ok. So, let us see how we do that, and all the dimensions have been given in this particular figure. And you can see initially the water level is 2.25, and as it leaks out when we find that the water is coming down. And here, we have the situation that this h t shows us the water level at any given time instant t, and this particular thing is coming down.

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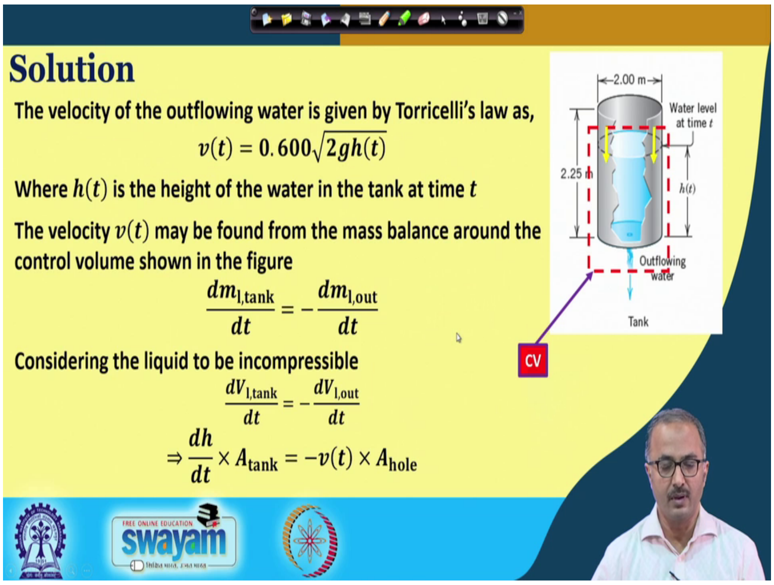

Now here we have, the first you let us see that how we do this. This Torricelli's Law has been given to for this velocity of water out of the tank ok. So, here we first choose the particular control volume. And then to find out the variation of the water level with time what we do, we do an mass balance ok; and we do this in a macroscopic scale. So, when we do the mass balance you know that the when you apply the conservation law it is input minus output plus some kind of generation is equal to accumulation.

Now because there is no input, so input term is 0. And we assume that there is no reaction or anything, so there is no generation inside the water. So, it is basically only output. So, we have minus output equal to accumulation. Now what is the meaning of negative accumulation it means that it is depletion ok. So, we find from if we apply this energy mass conservation on this particular system we find the how the mass inside the tank is changing due to the output ok.

And now we see that, if we assume that the water is in compressible that is its density is not changing as it is flowing out of the tank, then we can write the mass in terms of volume as mass equal to the product of the density and the volume on both the sides, and because density is constant so density will be getting cancelled out. And then what we are left with, we are left with only the volume. So, we can write from this particular equation assuming incompressible fluid we can write that the volume change in the tank is equal to the rate at which the water is going out of the tank; that is the volumetric flow rate of water at the exit of the tank.

Now, you see that how we write the volume of the water in the tank, it is it written in terms of the high level of the water into the area of cross section. So, we are writing because the area of cross section is constant for the tank, we can it out of the differential and we are putting this dh by dt as a change in the level of water inside the tank. And, on the other hand the outlet we are given that the velocity at the outlet is given at in terms of the height of the water level. And you can see that the velocity will be depending on this hydrostatic head or the level of the water inside the tank. And as a level goes down we will see that the velocity will also starts decreasing ok.

So, we write the symptoms of the velocity. Velocity in to cross sectional area of the exiting hole, that gives us gives us the volumetric flow rate of water at the exit hole. So, here we are writing this particular thing.

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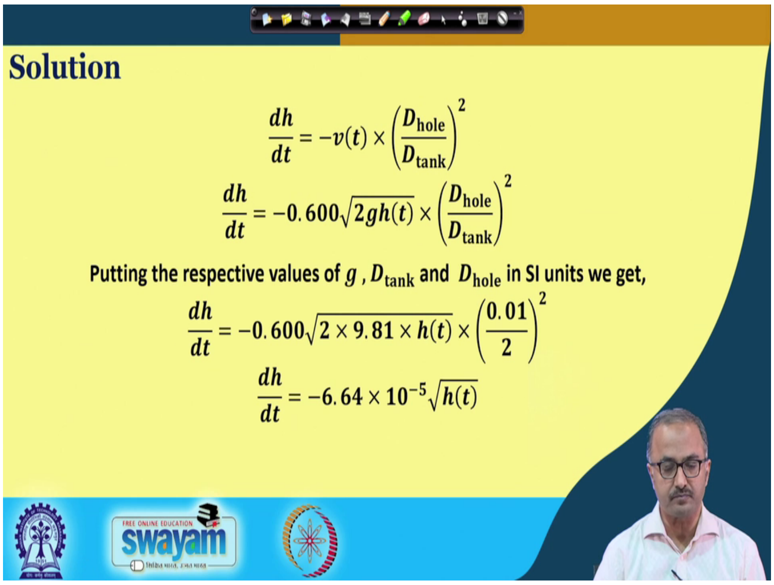

Now, we find that this dh y dt is now minus v t d hole by d tank to the power whole square; why, because the area of cross section is pi by 4 d square ok. So, pi by 4 will get cancelled out from the numerator and the denominator, so we shall be having only the diameter of the hole and the diameter of the tank.

Now, what we do that we have this is the final equation, we put the expression for the velocity in this expression in this equation and we find that this diameter of hole, diameter of the tank the g are all constants. And we if we put SI unit then what we find that this is the thing we are putting. So, here this is the value of the g that is 9.81 meter per second square. And this is the hole diameter that is 1 centimeter or 0.01 meter and the tank diameter is 2 meters ok.

And after this computation we find dh by dt is equal to minus 6.64 into 10 to the power minus 5 under root h t ok. So, you see that the level of the water is changing as square root of the level inside the tank.

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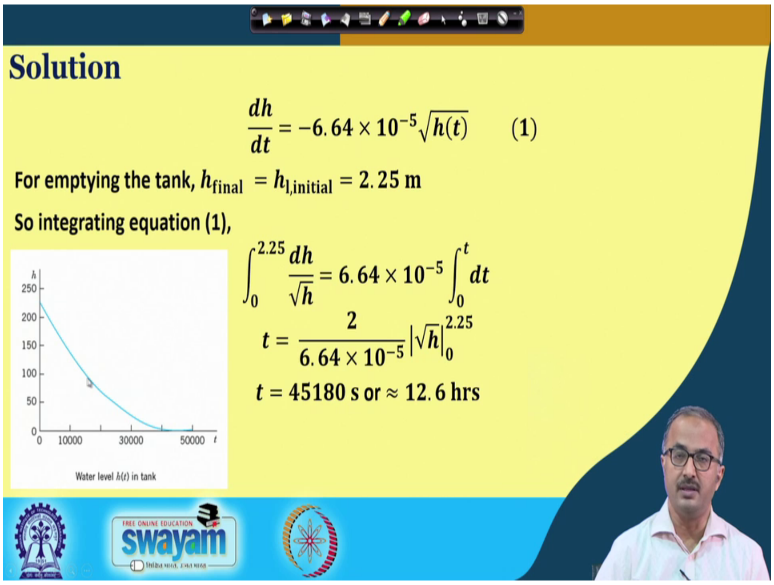

Now what we do that, the final finally, what we are having that means it says that it has to be made 0. That means, finally, the height of the tank should be the 2.25 meter; that is a initial level of the height of the liquid. So, what we do that we put these integration constant, we have to integrate it and we put 0, 0 on the same side and t at this 2.25. And what we find then after this integration, and we find that we have the time is coming out to be about 45000 second that is approximately about 12.6 hour.

So, you can see that it takes a long time to drain out a water tank from a height of about 2.25 meters through a small hole of 1 centimeter. That means, the whole day you can see the 12.6 hour is how much of whole day goes to empty the tank. And it is a practical way of looking at is that; we have very plenty of time if you can detect the leak in time then we have plenty of time perhaps go for any kind of repair work ok. And this is we have taken for water, but this may be extended for any kind of liquid in a storage tank, we encounter in any industry. So, any problem you are looking at here should be always correlated with some practical problem which we find in our day today life because, I am just teaching you the ways to solve the material or the mass balance or the energy balance equations.

Now here you see that if you plot the variation of the water level with time you will get typically this kind of nature with the things ok. So, you see that it first comes very fast and slowly and slowly the slope is changing. Slope is changing means, the initial rate of the change of the water level is now going slower down it is taking getting a bit slow as with time ok. So, initially the drop is faster and then drop is slow, because the velocity decreases due to the decrease in the level of the water in the tank ok. So, it is not a constant slope line.

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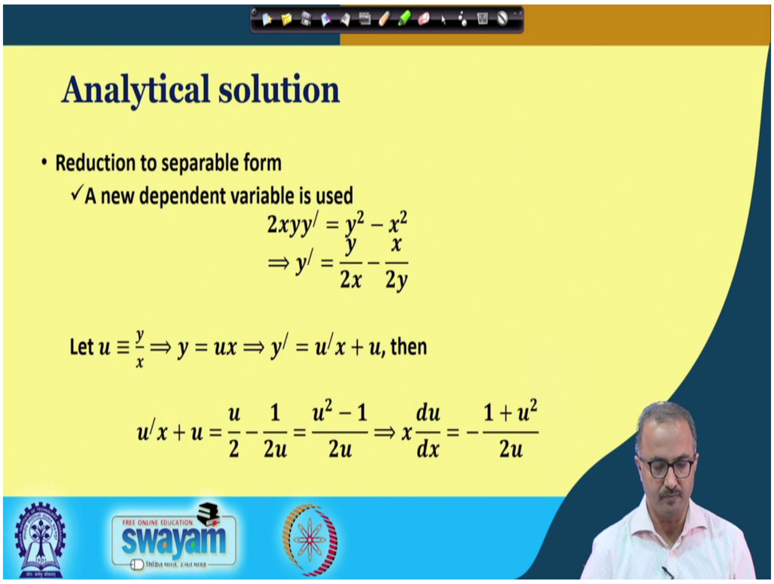

Now, we go to another form of the differential equation which is not separable at first instant, but we can reduce it to a form by some kind of manipulation to get into a separable form ok. So, let us go with an example. So, we are given this particular equation you can see that 2 xy dy by dx equal to y square minus x square. Now what we do, we put the y prime time on the left hand side and take this 2 xy turn on the right hand side.

Now, after doing this division by 2 xy we have y by 2 x minus x by 2 y. So, by simple observation you can see there is a common factor like either you can say y by x or x by y; that means, y by x is here and reciprocal of y by x is over here. So what we do. we take this y by x to be another variable u. Please understand this x and y maybe the variables which are coming from the balance of equations ok.

So, we are manipulating or we are taking some combining these variable process variables to get a third variable u which is not coming in the model equation naturally. This is just you can say a mathematical convenience we had a hours of devising so that we can separate the variables. So, u is defined as y by x, so y is equal to ux.

Now, if we with this definition what you do you write that y prime equal to u prime x plus x y prime is dy by dx. So, dy by dx is equal to du by d x into x plus u, because d x by d x is 1. So, here you have this particular thing. Now what I would do that these we equate with this here. So, u prime x plus u is equal to for y by x we write u and for x by y we write 1 by u. So, we have u by 2 minus 1 by 2 u and then you just do these mathematical manipulations. So, you get x into du by dx equal to minus 1 by u square by 2 u ok.

Now you can see easily here that once we have reduced it to this form you can easily separate out the u and x. Now, if you take the this x and this is x on the right hand side and all this u’s can be brought to the left hand side, you got seen that you are able to separate these two variables.

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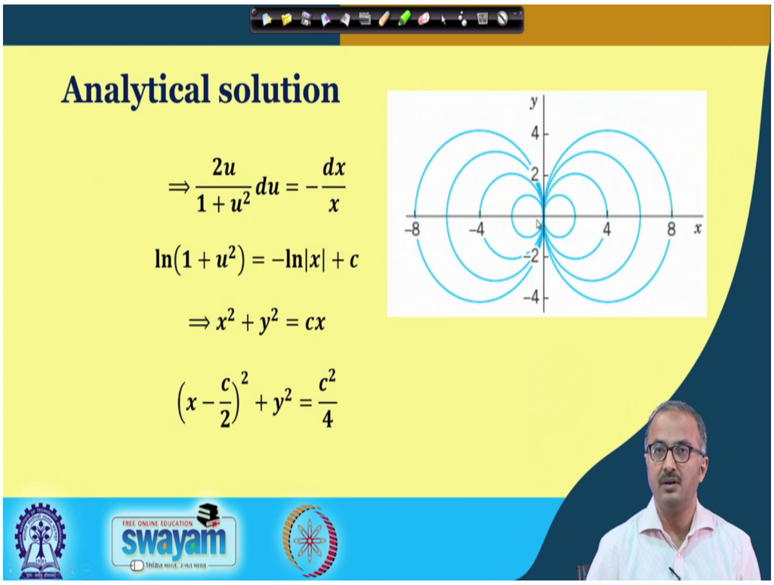

So, here I have done just that. So, again I am not going into the integration of it. So, once you integrate it you will get this kind of a thing and then you manipulate it using that this equation ok. Now, this equation again may be reduced to this particular equation. And you can see that this particular equation is quite known to us, and this equation represents a circle. And you can see the depending on the value of the c, the circle will have different type of radii ok. So c may be 0, c may be positive, c may be negative, but c will be some kind of integer value, but it may be also some real value also. After identify that this represent a circle we can see how it will look like.

Here, you see that we have by depend the by changing the value of c what we are getting here, we are getting the family of circles here ok. And this family of circles are also shown to be that they are kind of a reflection along this particular axis. And this is happening because the c may be positive or negative and because of this nature of these things you will find that this, depending on the sign of the c either I have the circle on the right hand side or a circle on the left hand side for the same value of c, but with different sign.

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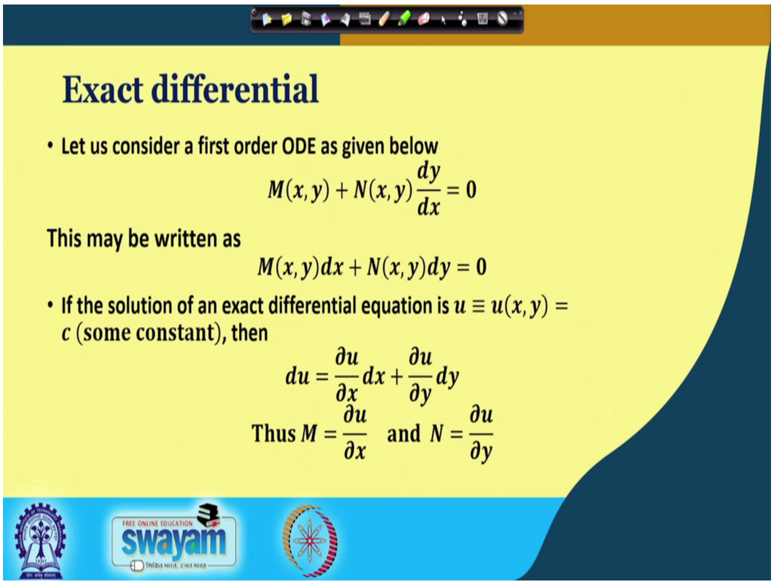

Now, we come to another form of the differential equations and we call this form as exact differential. So, before we go on to the solution of this kind of equations let us understand what we mean by the exactness of the differential. Again I am not going into the theoretical details I will just give the final results for all these kind of analysis. So, here we have the equation given like this. So, here two terms which are again functions of both x and y. And we can rearrange the equation in this form ok. So, we get M x y d x plus N x y dy.

Now, let as assume that the equation has a solution u which is interned a function of u x y, and that is a constant. So, if that is a constant, then what we can see from here is that we get du as dou u by dou x into d x plus dou u by dou y into d y. And thus you can see if we take a similarity over here, that we find that that M equal to dou u by dou x and N equal to dou u by dou y. And because, that right hand side is constant, so we have also have the dou u this dou c by dou x or dou y equal to 0. So, it exactly matches with this kind of this particular form of the solution matches exactly with this kind of the equation.

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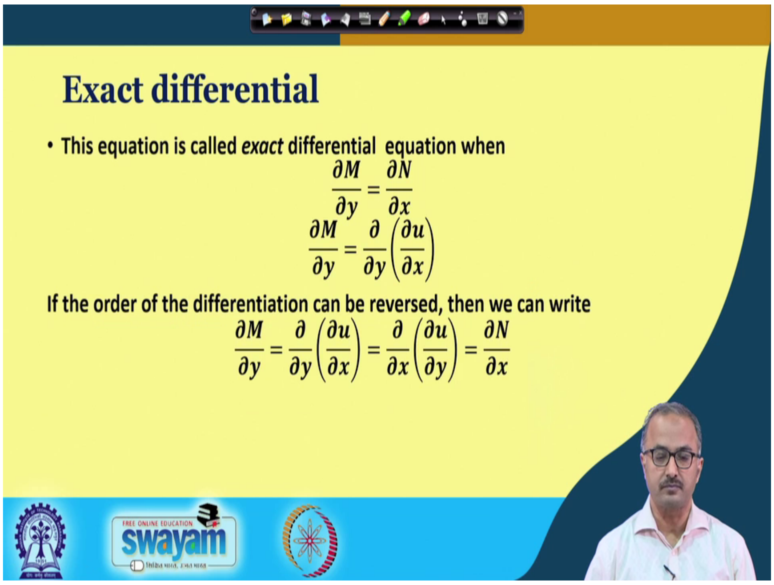

Now, let us see that to call it exact we have this thing that dou M by dou y equal to dou N by dou x; if this particular condition is made we call the given equation as homogeneous equation. Now how we do that let us see. That if you put dou M by dou y then you see that it is dou by dou y into dou u by dou x. And if the order of the differentiation can be changed it is not so always, but it can be changed then we can write we try to manipulate it and we can see that it will be dou u by dou x into dou u by dou y and then this dou u by dou y is nothing but N, so it comes as dou N by dou x.

So, if you see that for this particular reversal of the differentiation order of differentiation we are getting dou M by dou y equal to dou N by dou x; that means, what whether when we are changing any particular variable in one direction and then in another direction it does not matter which path we are taking. Whichever path we are taking it will go to the same point ok. So, it is getting a path independent ok.

So, you can say these are the kind of straight function we are talking about which are not path functions. In that case we are talking about this kind of differential they will lead to these kind of differential, but when we have the path functions we will find that they are in exact differential ok. And you know that any state of a system like temperature, pressure, density, specific enthalpy all these are state functions they are not path functions. So, they will lead to this kind of exact differential.

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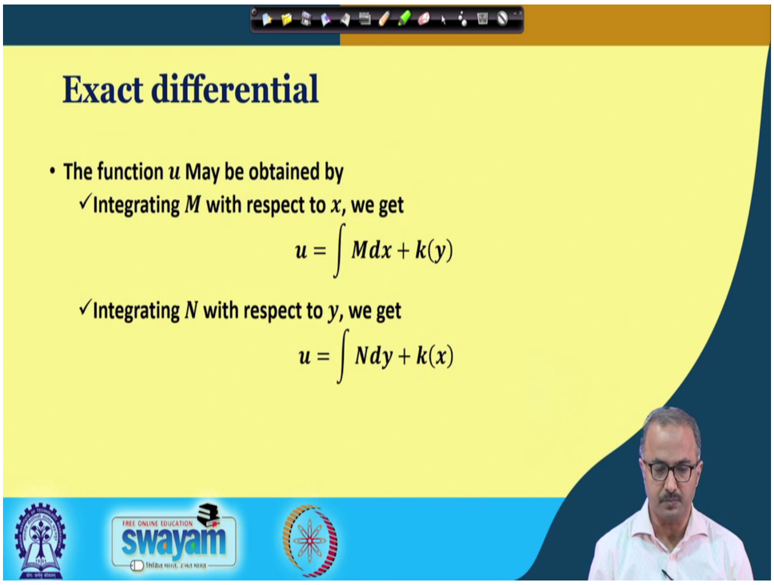

Now, you can now let us see how we can obtain u; u may be obtained in two ways ok. Now either we integrate this M with respect to x to get the value of the u. And you can see that here thus integral M d x plus k into y or we can also integrate the N with respect to y and to get the value of the u. In this case the coefficient which will come out of the integration is the function of only x.

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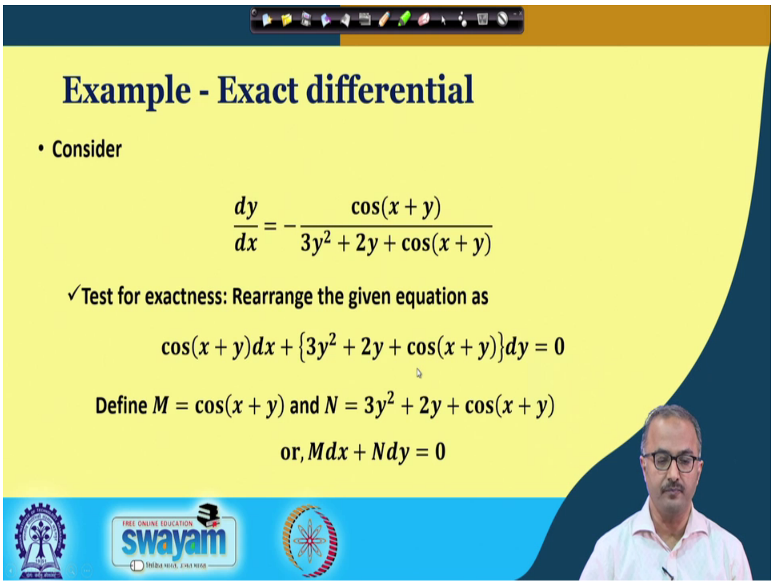

Now, let us take an example of this. So, here we have the given equation. And what we do again we check for the exactness first. So, we put this equation in this particular form now we can identify that M is this and N is this thing.

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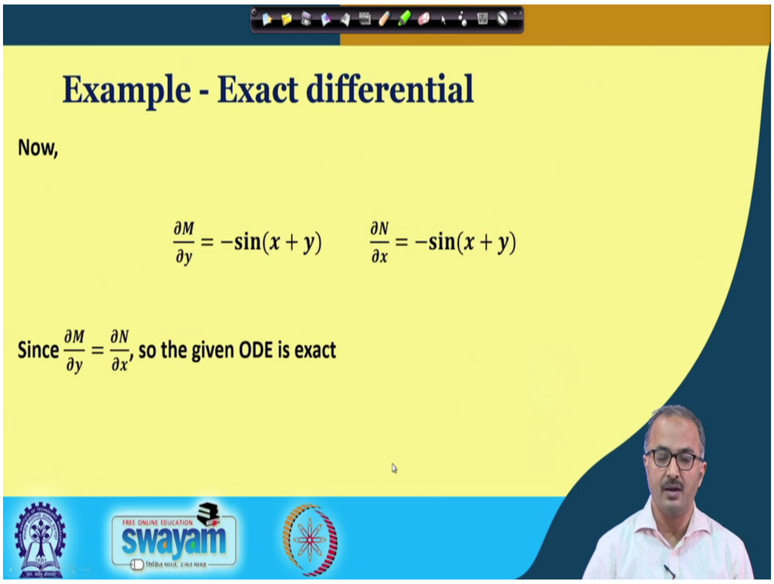

Now first what we do that we again take this differential of M which respect to y, here we find which is minus sin x plus y and with when we take the differentiate N with respect to the x we get again minus sin x plus y. Now from these two we can see that the dou M by dou y is equal to dou N by dou x; that means it is an exact differential equation.

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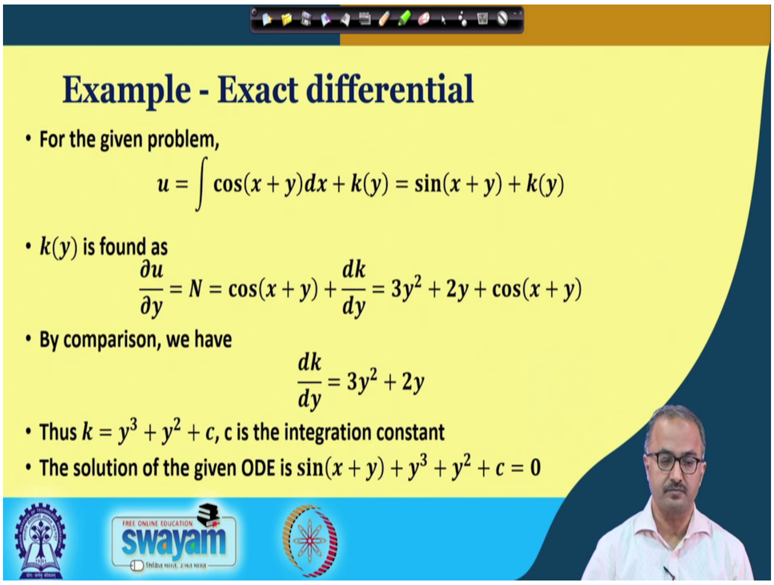

Now the solution part: so let us see that if we go with the u, u equal to cos x plus y into d x plus k y. So, this is we are putting that sin x y is coming that the intuition of this sin to x plus y and k y. And this k y we can find easily that if we find that dou u by dou y equal to N and this is giving us this particular cos x x plus y into d k by d y. And this is a thing we are getting and by comparison we can see d k by d y is nothing but 3 y square plus 2 y.

So, up to this is the one what is the solution and here we are just again going back to the original solution to and comparing the things to get the value of the d k by d y. And now you can see that k is only a function a solution y that can be integrated by separation of variables which we have just learned. So, once you do that you get finally this as the solution of the equation, and this c is the integration constant, and this may be found out from the initial condition of the particular problem.

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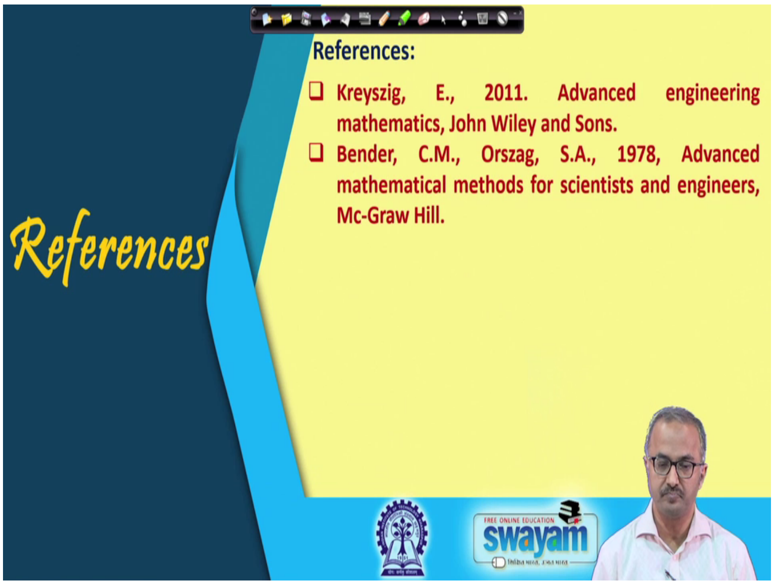

Now, these are the two reference books you can refer too to more about these particular methods I have taught today.

Thank you.