Matrix Method of Structural Analysis
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Lecture – 42
Introduction to Finite Element Method
(Contd.)

Hello everyone, this is the last lecture of this week and of course, of this course. So, what we have been doing we have been just trying to understand some of the or re-introduce some of the features of Finite Element Method. So, if you recall these are the different kinds of elements that we discussed.

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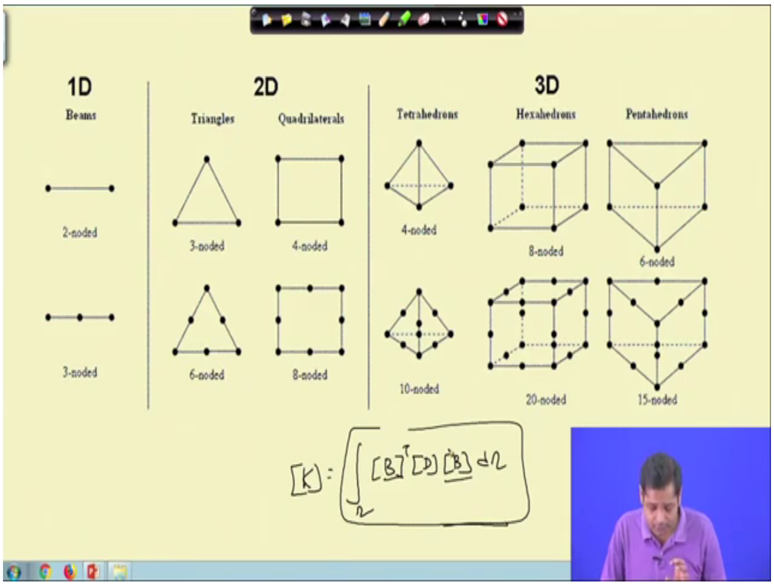

We also discussed how to approximate these nodal variables through shape functions ok. How to construct those shape function depending on the number of nodes you have, number of degrees of freedom you have ok. We also discuss the concept of convergence that we can get to the actual results by thereby reducing the number of or increasing the number of elements. And also we can get to the actual results by or we can reduce the error by having a shape functions with higher order. So, the first one is called h convergence and the second one is p convergence ok.

Now that was to be discussed, now you see if you recall this the shape function this important thing; this if stiffness matrix is obtained by this right B transpose D B. Now we also discussed what is just to name just we just named what is isoparametric element, the detail we have no discuss isoparametric element. But, we mentioned that there is a way if your elements are having different shape we can always have a map between a square element or rectangular element which is defining a parametric space and then you have a map between that parametric space to the actual space. And then you can do your, you can construct our shape functions then integrations on the parametric space keeping the map between the parametric and the actual space preserved.

So, you see this is the stiff this is the stiffness matrix and also similarly in the load vector also you have an integration right, if you recall. So, that these integrations very often this it is not possible to have this integration in a closed form. We have done it for bar element for beam, we have also done it for triangular element within your a triangular element. But most of the cases this integration is not possible using closed form.

Because, of the non-linearity associated with the model though we are talking about on a linear problem. Because, of these shapes of the, shape of the shape of these your the domain over which that integration is being done and then you have a quadratic nature here. So, because of all these things the integration may not be possible in a closed form. So, you have to look for some alternative technique.

Then you might have, you have might have studied various integration methods. Like for instance if you have a function any function suppose if I have if we before coming to the most the wider technique that is being used in finite element method for integration.

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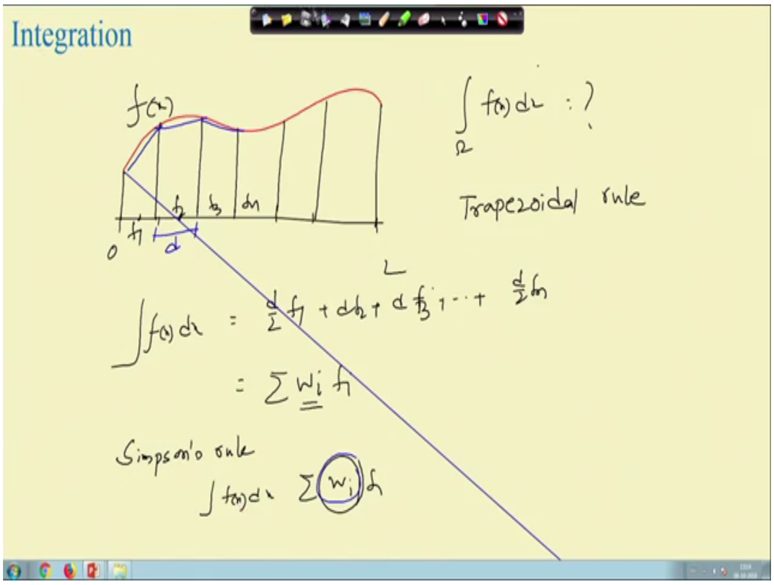

Suppose I have a function any function f x which is defined over say between x is any function if we take this is the point and this is another point this is defined over say. This is this is 0 and L and the function is suppose any functional. Now I have to integrate this function between these 2 point, between these 2 point and this is the function.

So, we have so what you have to do is, we have to integrate f x d x over say over this length omega or this is the, this is integration. So, if that we can have a closed form solution we can also do to the do to this for many functions this closed form integration may not be possible. Then we go for numerical integrations and some of the techniques that you have already studied in numerical technique. I mean for instance you might have studied trapezoidal rule.

What is trapezoidal rule? Trapezoidal rule tells you that you where you divided this entire thing into small small segments. You see the here also the concept is very similar you divide the entire domain into small small into small some steps. If suppose these steps are d for instance if these steps are d you can have equal steps non equal steps, if the steps are d then for instance in 1 dimensional case we can have this function as.

So, we can write f x d x integration over this that becomes d by 2 d by 2. Then f 1 suppose this is in between the value is say f1, this is f2 f3 f4. This is the function value at this point is a function value at this point and so on.

So, d by 2 plus f 1 plus d into f2 plus d into f into f3 and so on then finally, we have that d by 2 into fn right. And so essentially this gives you summation of w i into f i where fi at the function values at these steps and then w i is essentially the weights associated with that particular point right.

So, this is the, this is how we have seen how to integrate like this. We have also seen Simpson's rule for instance the Simpson's rules also same also very similar to that Simpson's rule. The Simpson's rules here in trapezoidal rule it is assumed that between these 2, between these 2 your function is linear. This function is this function is linear then, but you cannot have a linear functions. So, in Simpson's rule we can have non-linear assume a non-linear function, but even in Simpsons rule eventually we get function f x d x is equal to summation of w i and f i.

How you construct this w i, the essence remain same whether you use Simpson's rule, trapezoidal rule, that you are breaking into small small segments and calculating the function values for each segments and multiply it by some weights and then summing them up. So, depending on how you calculate these weights and how you divide the entire domain into small small segments there are different methods. The method that is used in the method that is used in infinite element method; this method is this Gauss quadrature method.

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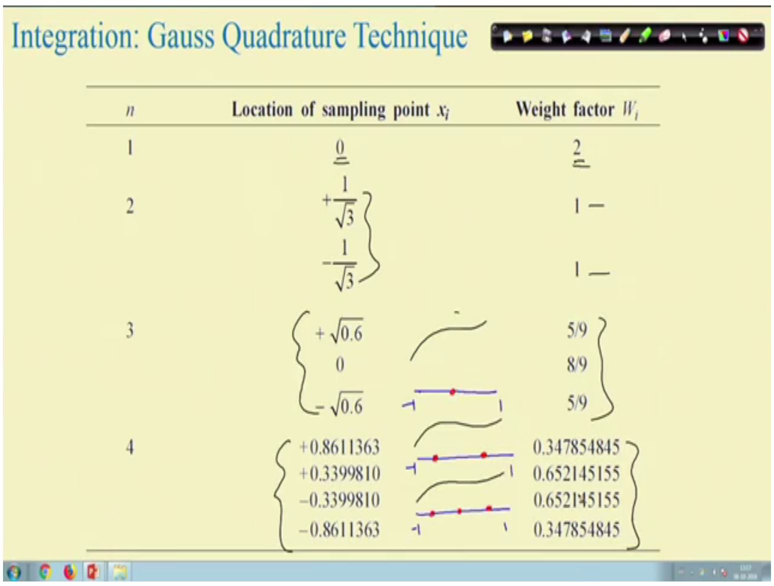

This is the method very commonly used in finite element method, Gauss quadrature method tells you what. See if you look at the other methods whether the trapezoidal rule, Simpson's rule, as I said you have several points at every points their weights are associated with those points. You calculate the function value at that point multiplied by their respective voice and then sum them. In from that perspective Gauss quadrature technique is essentially is also same. We have several weights we have several points and then you calculate those values at those particular points and multiply their weights.

But you know the how to get those points and weights we are not we are not going to discuss here that is detail you might have done in numerical methods codes or if you see any numerical methods look all these descriptions. How to arrived at these points and their associated weights are given. So, in Gauss quadrature technique also we divide it into small small weights ok. It is done suppose if we have a domain between minus 1 to 1 say minus 1 to 1 and if you have to integrate this between minus 1 to 1.

So, location so you have to what cost what this technique tells you that you get a points either this point or for instance you have minus 1 to 1. You take 2 points between this say point this points and this points or you can have say 3 points. So, we have minus 1 to 1 you have 3 points and so on; so, these are 3 points. Now, when you have so you have you want to integrate a function between this so suppose this is a function this is a function. Now what this tells you calculate the function at this point and then multiplied by their weight and you get the integration. Similarly 2 point means you calculate the value at this point and at this point and multiply by their weight sum them up get the integration; similarly for 3 point integration.

Now, what are those values for 1 point integration you have just 1 value in and associated weight is 2. And then for 2 point for this case your weights the corresponding coordinates of this point are 1 then associated of associated is 1. And then for 3 point the corresponding points are these and associated weights are this. And the summation of the all the weights will be equal to 2 because in this case your domain the length of the domain is 2; it is in 1 dimensional case.

Similarly in higher dimension you can have the, you can have such Gauss point these are called Gauss points. In higher dimensional I am going to show you that. So, this is how you can similarly for 4 point integration 4 you have the 4 points and then these are the corresponding weights. And if you sum them you will see, that the summation of these weights are 2 ok.

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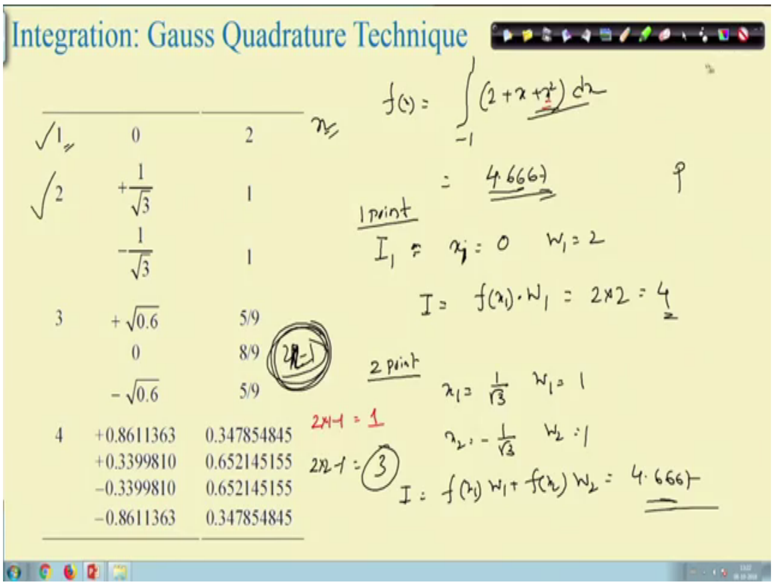

Now so, for instance for instance if I have to give you some example suppose. Suppose take an example take they suppose f x is equal to f x is equal to say integration over minus 1 to 1 ok. And then we have say 2 plus x plus x square and then d x. Now, if we integrate it if we you can if we close form if we integrate it. Then the solution what we will get is we will get 4.6667 this is the closed form solution of this integration.

Now, let us do it for using Gauss Quadrature Technique. First we use the 1 point Gauss quadrature and then we will see with 3 point, 2 point Gauss quadrature. 1 point Gauss quadrature tells you that say I 1 I 1 1 point Gauss quadrature that is equal to you calculate the values at x is equal to 0 calculate the value. So, this gives you I 1 is x i is equal to 0, and x 1 is equal to 0 and w 1 is equal to 2. So, then integration will be integration I will be that function is to be it is at x is equal to x 1 into w 1. So, this becomes if we substitute that 2 into 2 is equal to 4. Now, you see actual integration is 4.66, but we get 4. Now let us do it for this is for, this is for 1 point.

Now do it for 2 point integration 2 point integration, 2 point integration is what x 1 is equal to 1 root 1 by root 3 and w 1 is equal to 1, x 2 is equal to minus 1 by root 3 and w 2 is equal to 1. Then integration will be f of x 1 into w 1 plus f of x 2 into w 2 if you do that you will get 4 point you will get 4.6667 which is close to this.

So, what point rule what rule you apply that depends on what is the function you are going to integrate. In general if you are using n rule suppose, n is equal to 1 in this case n is equal to 2 in this case n is equal to 3 in this. So, if n is if you have a polynomial suppose in this case if you integrate a polynomial which is which order is p. So, in this case your, in this case your for a constant p is equal to 1 for a linear function p is equal to p for points constant p is equal to 0 for a linear function p is equal to 1. For a quadratic function p is equal to 2 and so on. So, generally n point n point rule can integrate a function of 2 p minus 1 2 p minus 1.

So, if you are using n point rule then it can integrate it can gives you exact integration of a polynomial up to order 2 p minus 1 ok. For instance in this case in this case it is for in this case n is equal to 1. So, if I use first point in 1 rule 1 n is 4 n is equal to 1, 1 point integration. Then it can integrate it can in order to suppose in order to get a linear function. Suppose, I have a linear function then for linear function what order of what order what rule into what integration rule I have to use 2 into 1 minus 1 means 1 point integration rule. But if you have a quadratic function for instance it is quadratic function. So, we have to use so, for so yes so, 2 n minus no there is the it is not 2 n minus 1 it is 2 n minus 1 this is 2 n minus 1.

So, if your n is equal to whatever rule you have. So, if you are using first rule then first rule can integrate up to integrate a polynomial up to 2 n minus 1. Similarly if you are using second rule then it can integrate a polynomial up to 2 into 2 minus 1 means up to cubic polynomial. Now, we have this quadratic polynomial that is why with 2 point rule we got the results ok. So, similarly now if we have similarly if we have say if we have an expression like.

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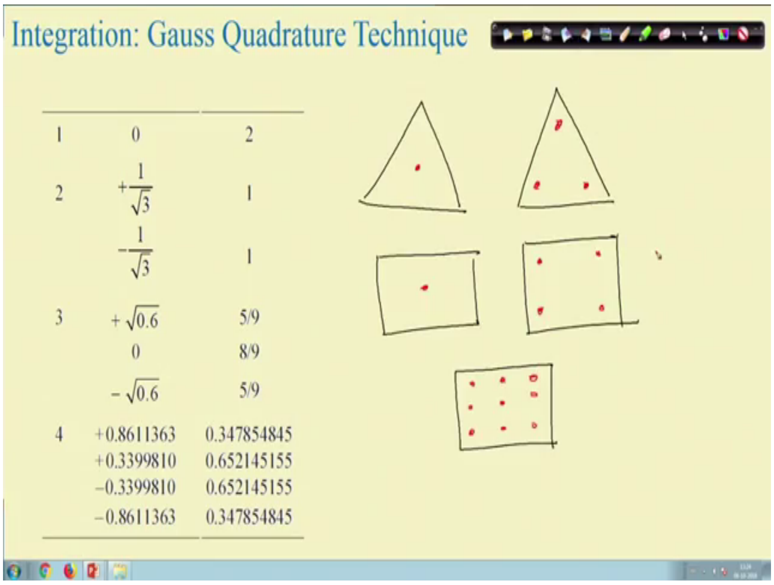

If we have a triangular element like this, even in triangular element you can have a Gauss points this is for 1 point for a triangular element then you can have a Gauss point like this so these are 4 different rules. So, this is your you can have a Gauss point like this these are called Gauss points.

So, if you want to integrate a function over this triangular element then this rule says that you calculate the value at this point multiply by their weights. This points says that if calculate the value at these points and multiply by their weights and sum them up. So, similarly for rectangular element for this kind of domain we have we can have this we can have a rule like this or we can have a rule like this. We can have a rule like this this 2 point 2 by 2 rule and so on. Now the thing is we can have a rule like this or say 3 point rule if you want to use then so these are the 3 by 3 rules ok.

So, depending on that what function you are going to integrate over them over this domain we have to choose the Gauss what rule of Gauss point what integration rule you use. Now you see here your Gauss points are if you if you look at the previous slides if we look at the previous slides your Gauss points are already it is between minus 1 to 1 right; that is what we it is always between minus 1 to 1.

And if you are having 2 dimension it is x minus 1 to 1 y minus 1 to 1, but your, but at the end, my at the end, my element may not be may not be as I said element may not be may not be a perfect rectangle, may not be a perfect square. But again we all the Gauss points are defined in a parametric space. And then we have a map between parametric space to the actual domain and that map needs to be preserved.

This is defined by that map and that map needs to be put into the integration to get the integration over the actual domain with the information about the parametric domain ok. So, this is how the integration can be done in (Refer Time: 16:38). You see though if you recall then there are 3 major things we have done 1 is ok; 1 is we have we have discretized the we first is we have we have a we have discretize the domain in to set of elements and then next thing is this is called the domain discretization discretization of the domain.

Then next is you have done is for every element we have an approximation of the field variable. And that approximation depends which kind of element we are using that is called approximation of field variable that step. And then once we have approximated the field variable calculated the strain displacement relations and so on. We substitute that relation in that expression of in the in that expression of stiffness matrix and then we get this stiffness matrix. And for the stiffness matrix we have to do the part from the integration.

So, there are 3 major things 1 is decomposition of the domain, the discretization of the domain, and then approximation of the field variable, and then finally, integration. At every stage you have some error, you introduce some error into the formulation. For instance: at the domain decomposition stage if you are discretizing, you are discretizing a circular domain for instance.

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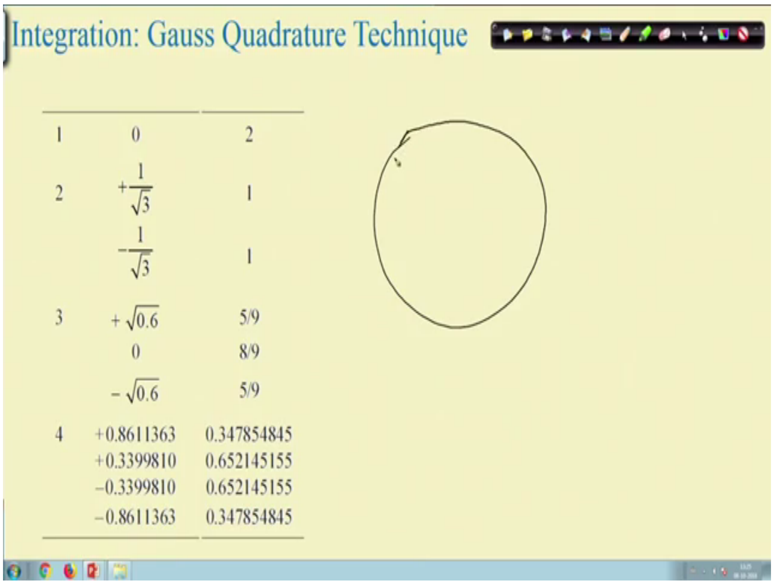

Suppose your domain is a circular domain now if we want to discretize this circular domain with a triangular element or a quadrilateral element. What you can do is you can this exact domain cannot be your discrete domain may not be an exact representation of the actual domain. In limited sense yes if you keep on increasing the number of elements reducing the size of the elements all these curved boundary can be closely approximated.

But for that you have to in if you have to increase the size you have to decrease the size of the elements. But even we decrease the size of the elements whatever formulation you have we have studied so far that based on that we can say that even with that we introduce some error that is error is called domain decomposition error. And that error can be reduced by choosing by choosing by choosing large smaller number of elements right.

Another thing is second thing is the discretization error discrete approximation at the approximation of field variable. We assume over each element your approximation linear or quadratic, but your approximation mean your actual solution may not be linear may not be quadratic it may be some higher order it may it may be your solution may be may not be consistent with that approximation.

So, there we introduce some error and finally, at the integration stage. Because you have to do integration numerically that integration also we introduced some error. The total error will be the cumulative of the total error will be the sum of all these errors, but there are way to reduce all these errors. There are mathematical basis to quantify those errors that we are not going to discuss here.

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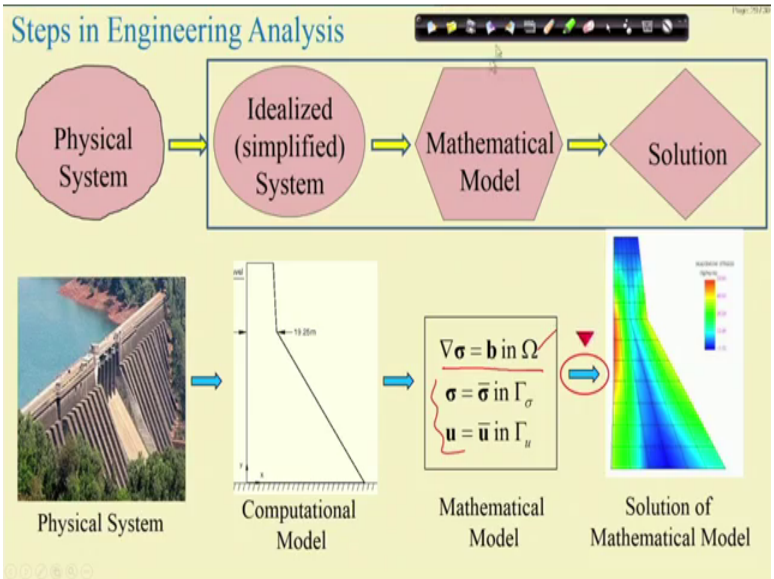

Now, finally, as a final remark if you before we close you see as I started this week by saying here in this week our objective is not to objective is not to even introduce finite element method because that is also not possible in just a week time. And whatever discussion we have made in finite element method the discussion we try to make that discussion in line with the methods that we learnt in matrix. The line with the philosophy of matrix method of structural analysis, we try to view finite element method in the same way that we viewed matrix method of analysis. That is why in our discussion we fought we portrait or we as I said we try to understand finite element method as an extension of matrix method of structural analysis.

So, that we did because that to that the objective here to. So, that better comprehension can happen, but yes as a finite element method either it was started in 50's and in the last 70 years time it has been extensively developed it has a very very sound mathematical basis; the errors can be proved the. And today any branch of science and engineering; not even in science and engineering everywhere many places finite element method can be used. You see there is two way you can look at finite element methods one is the ingenious way, and another one is mathematicians way.

Ingenious way that just now we have studied we have we have tried to see that, but then finite element methods is a methods it can be viewed as a numerical technique for solving partial differentially equations right that is the mathematicians way. So, wherever you have partial differentially equations you can have in mechanics problem, you can have heat conduction problem, you can have electricity, electric engine, electrical engineering problem, you have physics chemistry, or even you have some other fields which are non engineering or non sense field there also you can have partial differentially equation.

Wherever you have that yes finite element method can be applied that it is so huge so vast. So, therefore, now so, as I said the objective here is to open the door to tell you there is the method out there and which needs to be explored and that is the finite element method ok. Before we close as concluding remarks of the entire course you see this is the basic steps in engineering analysis. If you remember these steps we discussed in one of the earlier classes of this way. We start with a, we have a physical system, we idealize the physical system, and then write the mathematical model of that physical system, and then and then solve them and find out the solution of that.

It is an example of the example is given like this; for instance, if you have a dam problem this is the physical system, this is the idealized system, and this is the mathematical model, for this physical system. In this mathematical model we have three components one is the governing equations here, governing equations here then the definition of the domain over which this equation is valid in this case it is omega and then set of boundary conditions and initial conditions. That describe the entire mathematical model and then we solve it.

Now, and this solution technique this what we are studying. Whether it is a we are studying in structural analysis one or matrix method of analysis and so on or even in finite element our main focus is essentially this step how to solve this governing equation two different methods and finite element method is just finite element method is a one method for solving this equation. Now, just to make our discussion for the completeness of the discussion, this is the thing.

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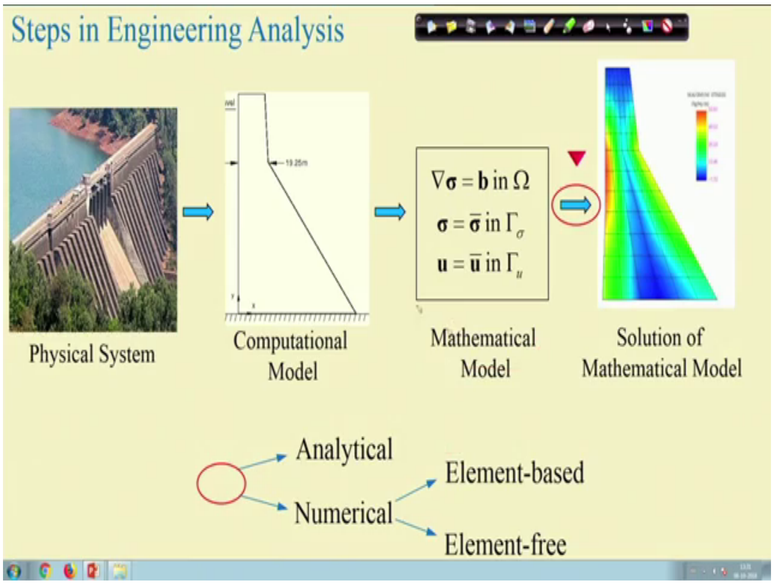

So, this solution these equations can be solved either analytically or numerically. Analytical solution are most of the practical cases not possible. We have to solve it for numerically and finite element is a numerical methods. The finite element we discretize the domain into set of elements and then write the equations for each element assemble them get these solutions. There is another class of method which are called element free method where we discretize the entire domain, but not into elements. We have a set of points and all the equations and everything writes based on those set of points not there is no element and that class of method is called element free method.

So, now even in this category element based method and element free method finite element is one element based method. So, in this category element based method or element free method again there are many such methods which can be coined which can be put either in this broad category either based on elements or which are do which do not rely on elements.

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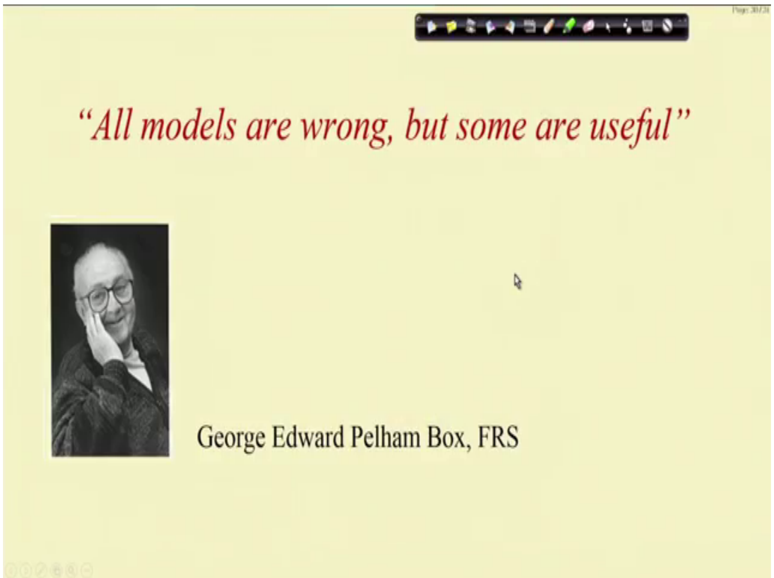

Now, before we close you see essentially when we talk about there is a famous quote by mathematician George Box that "All models are wrong, but some are useful". See when we say a model it is a very word model itself says that we are dealing with an approximate, its approximate system right. Right from the physical system to idealization then writing the mathematical models of that idealized system and then finally, solving this mathematical model. At every stage we assume something right every idealization is based on every step is based on certain assumptions right.

And therefore, and with every assumption we deviate slightly deviate from the actual system right. So, the final solution what we get that is not the solution of the actual physical system rather the solution of the idealized system or the idealization that we have met throughout the process ok. In that sense that then there is nothing like which one is right model or which one is wrong model right approximation or right wrong approximation, the very word approximation tells you it is not right it is not the actual thing. What is important is whether that approximation is useful or approximation is not useful.

Whether some approximation whether some method is useful or not that depends. That that you have to decide based on what information you want to extract from the analysis. What is the purpose of the analysis? What is the outcome of the analysis? How the structure or any object may behave under certain circumstances? What is what kind of behavior we are expecting from the analysis from the structure? Depending on that we have to choose a numerical methods right.

So, now this is the starting point that you will be, it is the first step towards that first step towards finite element method. And, as you learn finite element method we will also see there are different branches in finite element method. Branches in the sense difference where you can enrich the basic formulation of finite element methods. Then there are methods which do not rely on elements and even in that category there are hundreds of methods ok. Every method has their own advantages and disadvantages.

Now, depending on your requirement of the end requirement from the analysis you have to see which method will be useful for you. And with this I stopped here today and this close this entire course so, with this thank you. And I believe that you have enjoyed the entire journey; as in the beginning of the every lecture we will show you the front slides where our email ID's are given. If you still have, if you have any doubts, any confusion, you want to ask, you want to interact, you are most welcome to write to us; with this thank you.

Thank you very much.