Matrix Method of Structural Analysis

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Lecture - 40
Introduction to Finite Element Method (Contd.)

Hello everyone. This is a fifth lecture of this of this week. In the first 2 lecture, first 4 lectures we discussed the truss elements and beam element and as I say it towards the at the end of the end of the previous class, in all these formulation you could correlate with the with the with the formulation that is started in matrix method of analysis because in us in a way they are very similar,.

But now today we will see when we talk about you see there is nothing like beam or truss or something in the nature. Every object you see, every object you see there are essentially some three-dimensional object, ok. It has length, width and breadth. But yes in some cases the one-dimension is very large or two-dimension is very large compared to the third one, and in that those cases we can idealize this structure as beam as truss as plates, so as shells.

So, when you talk about truss, beams or plates they are different kinds of idealization. So, two idealizations we discussed one is beam and truss. Now, many problems we come across they are not they are they cannot be idealized in one-dimensional we need to we need to idealize them as a two-dimensional problem. And there are many cases we even cannot idealize a structure as two-dimension we have to actually deal with the three-dimensional problem.

So, if we have to now today what we see is if we if we have those class of problems then what are the options we have in finite element method, how to discretize in them, and what are the different kinds of elements we have that we are going to see today and in the subsequent classes, ok.

So, some examples of two-dimensional and three-dimensional discretization.

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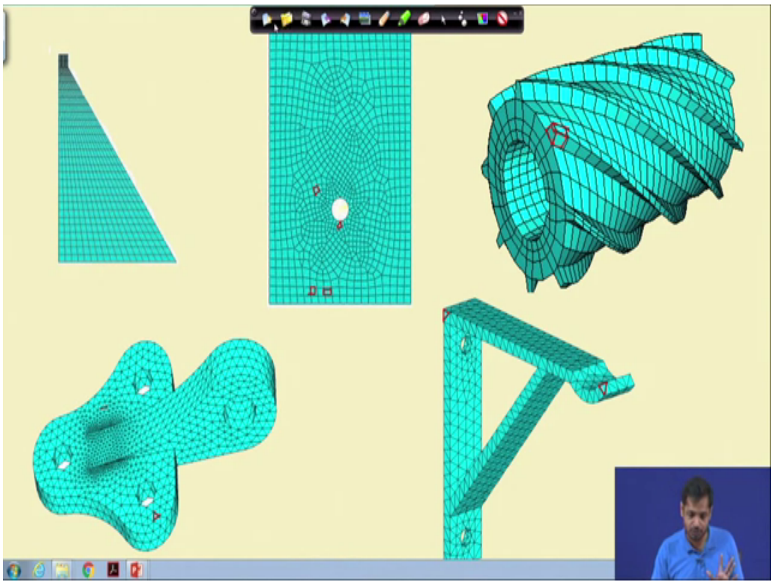

You see this is a damn problem which is discretized by a 4, let us not bother about what are the what is the characteristic of the element. But what you can see by looking at these are three-dimensional figure where this is a bracket, this is a rotor, some this is a, some machine tools, this is a plate with a hole in it.

Now, what you can look, what is important here you see yes that every elements for even you when you discretize a three-dimensional object we need a three-dimensional elements all these are all elements. These are all, so when we say this is one element. So, this is one element, this is one element. Similarly here also this is one element, when you take this is one element or this is one element. And then here also this is one these are different elements this is here also we have a different kinds of element.

So, what you can make out is these different elements are having different sizes different shapes even in a one body we can have we have, we have for instance here we have a very very smaller elements, but here we have a very large elements, ok. So, the elements are not same in shapes and sizes,, but for different structure we can have different we need different kinds of different kinds of elements, ok. So, let us see what are the options; we have broadly. So, these are different kinds of elements that we can have.

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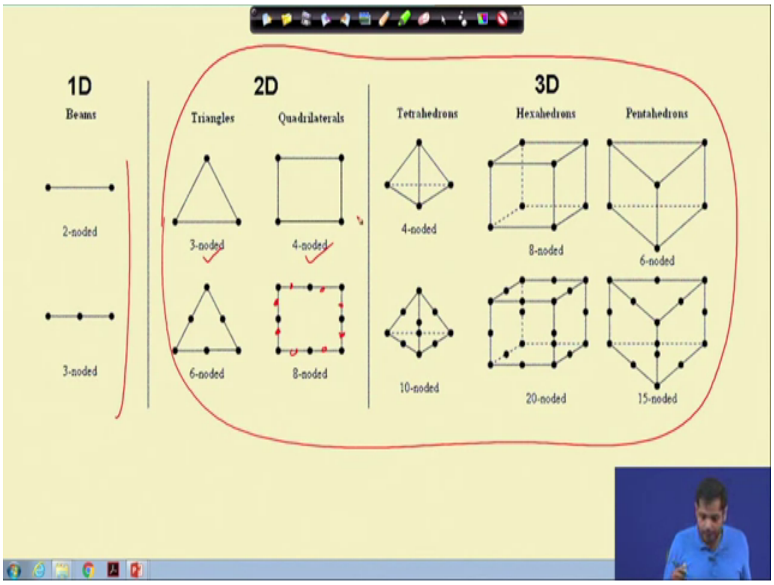

For instance if it is 1D problem then if it is 1D problem then we have 2 noded element, 2 noded it could be a bar element or it could be a beam element both we discussed in the previous classes.

We can have 3 noded element as well where for in a in a 2 noded element every node we had we had for a beam problem every node we had 2 degrees of freedom. So, total 4 degrees of freedom in a in an element, but if you have a 3 noded beam element. So, total degrees of freedom and 6 degrees of freedom. Why we need to 3 noded element or 4 noded element we will discuss at the end of this end of this week.

So, these are the things that we have in 1D problem we have we have discuss that in the previous classes. Now, if you have a 2D problem then different kinds of 2D elements are we can have this we can have a 3 noded element triangular element. Triangular element then we have a similar kind of element, but 6 noded, we can have a 4 noded element, we can 8 noded element even we can have several other different a different many other possibilities are there this is just to give you a rough idea about the different kinds of elements. Even we can have you can puts some more nodes here as well if you want, ok.

So, in that case in that case these become 16 node element, that possibility that option is also there. So, you can you can formulate such elements. Similarly if you have a 3D problem then these are the different kinds of elements that you can have, ok.

Now, you see as I as I just now said for this you have done the similar exercise in structural analysis. So, you could correlate, but these are the things that you have not done before in any structural analysis course and this we are going to discuss. So, what we do today is we will talk since it is not the it is as I said at the beginning of this week, our objective here is not to not to not even not to learn finite element method because it is anyway not possible in one week time here is to just see some of the salient features of finite element method. So, that you can so that it gives you a sufficient reason to think or to go through the books and learn this subject.

What we do here is we will take a 3 noded element and 4 noded element and then see what is the form of the, what is stiffness matrix for this and in both the cases, ok, great. So, let us start this. So, let us start first 3 noded triangular, triangular elements, ok.

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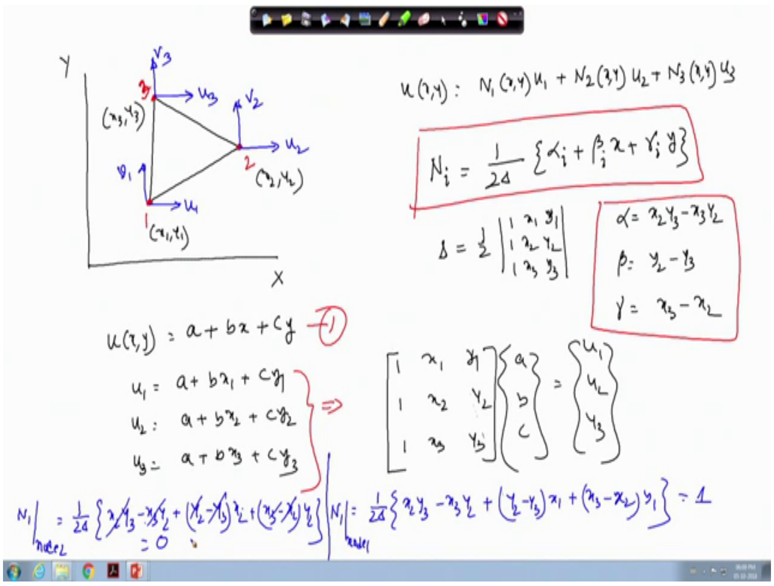

So, you see suppose this is your 3 noded triangular first, let us take one coordinate axis. So, suppose this is x coordinate and this is y coordinate, and in this x coordinate y coordinate you have an element like this which has 3 nodes. So, the, this is one node this is 1, 2 and 3, ok.

Now, this any representative element, the same thing we took will while deriving the stiffness matrix or all the safe functions we took a representative element for truss and beam with the same thing we are going to do here.

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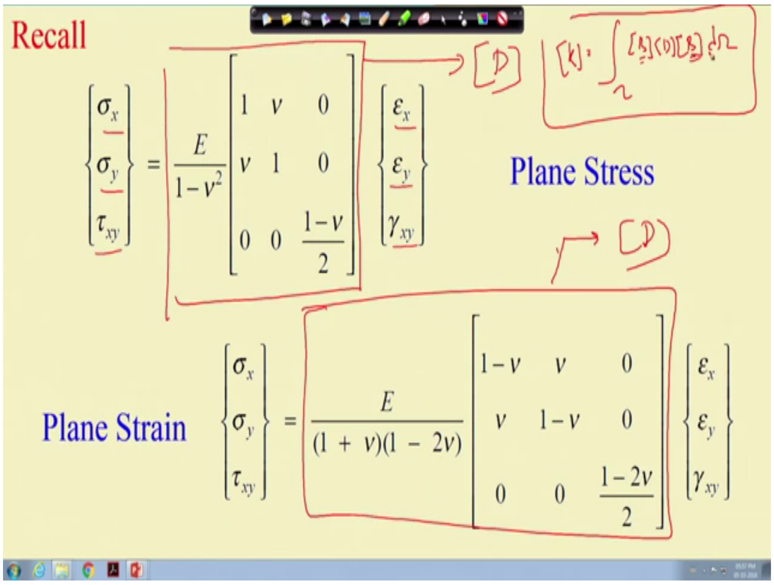

Now, what are the degrees of freedom we have? Now this is, we will come to that, but before that let us see let us discuss one important thing. That, you see if you recall you have done it in a solid mechanics course and also in strength of material course when you talk about two-dimensional problem, two-dimensional problem can have two idealization, one is a plane stress problem and a plane strain problem. Plane stress problem is where your stress at stress in a particular direction is 0 or it is so small that you can neglect it, ok.

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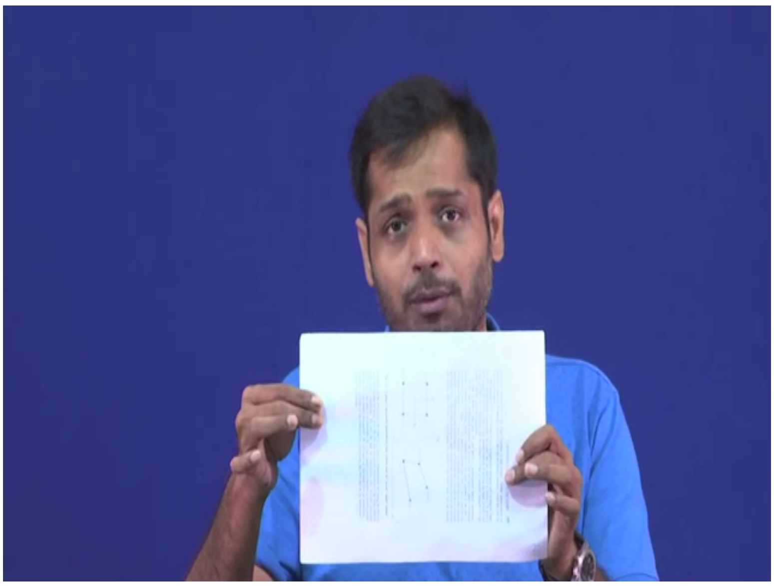

For instance, if we take a paper if we take a paper like this and apply some in plane loading here, in plane loading here, in plane loading here, then what happens. Say distribution of stress we can have stress in these direction we can have stress in these direction, but across the thickness of the plate across the thickness of this paper we can say that this stress is so small that we can neglect it we can this is so small as compared to the other stress, ok; so this is a plane this is a problem of plane stress problem.

So, what are the stress components? We have here we have that we have we have sigma in this we have sigma x, then sigma we have sigma y and then we have enormous we have a shear stress on this plane sigma x y. So, for a plane stress problem we have 3 components of stresses sigma x, sigma y and sigma x y. Similarly this is are the strain epsilon x, epsilon y and epsilon or gamma x y and this is how the stress and strain are related to each other for plane strain problem.

So, essentially if you recall the entire thing this is your D for plane stress problem. Similarly for plane strain problem the strain in a in a particular direction is 0. For instance if we talk about a damn very long retaining wall or a damn one-dimension is so long that along this direction the strain can be neglected. So, the strain is confined, confined within a plane. So, this is called plane strain problem. So, in that case so strain in a particular direction is 0. So, we are left is only 3 strain components epsilon x epsilon y and the shear strain on that plane gamma x y. At these strains are related to stresses corresponding stresses like this.

So, again this is essentially the, this is essentially the D for this is essentially D for plane strain problem. So, already we know what is D, and we know that stiffness matrix k is equal to integration B transpose D and then B, and then d omega, right this is the stiffness matrix. So, in the stiffness matrix already we have the, for two-dimensional problem.

Now, what we are going to do now is we take an element and then what is we will see what is the expression for B; now. Now, let us do that exercise.

So, this is the triangular element now. Now, what are the degrees of freedom we have? Now, at every point we have two degrees of freedom translation in x direction and translation in y direction, ok. So, suppose these node is let it get us give some node number 1, this is node number 1, then this is node number node number 2 and this is node number node number 2, and say this is node number 3, ok.

Now, what are the degrees of freedom we have? Suppose degrees of freedom we have this is u 1 u 1 and this is v 1. So, u in x direction v is in y direction. Similarly at node number 2 we have we have u 2 and then along y direction we have v 2. And similarly here we have u 3 u 3 and we have v, we have v 3, ok, we have v 3 here, v 3. So, total 6 degrees of freedom we have.

Now, if you recall what was our first step? First step was to see these values are u 1, u 2, u 3 and v 1, v 2, v 3 these values are the values of displacements at the nodal points, right. Now, what we did? We need to we need to find out the values if you have to if you have to find out values at the intermediate point within an element then how to do that. In order to do that we need a functional representation of this of this entire displacement field, right. So, in the case of truss or in the case of beam, in the case of truss we assume that your u is equal to x plus beam a linear function then in a case of beam we assume it is a it is a cubic function similarly here also you have to start with an assumption.

Now, what form you take? Whether it is a linear quadratic cubic what form you take that depends on what are the information you have and whether those information sufficient to find out the constant associated with that approximation, ok. Now, what are the information we have? We have we know say here you see u and v are two independent displacement field, ok. So, here we have to approximate u separately and we have to approximate v separately. Unlike in the beam case where we approximated only the v and the derivative of the v d, v d x gave us slow it is not like that yet u and v are both different independent degrees of freedom and in the an they have to be they have to be they have to be represented separately, ok.

So, essentially for a given degrees of freedom say for instance you we have only 3 information. So, 3 information means 3 values at the 3 nodes. Similarly for v we have 3 information 3 values at the 3 node. So, what we take is we assume a polynomial which as 3 constant, which has only 3 constant, ok. So, let us let us assume u is equal to say u is equal to at any point x and y that is equal to say a say a plus b x plus since it is it is in two-dimensional we should have a function of y as well it is c y, ok.

Now, some of the important point we will discuss at the at the last class one point of course, as I was mentioning I was mentioning how to calculate the error between the prediction and the and the and the actual thing this is one point. And another important point is this one that suppose instead of a plus bx plus cy if I take a plus bx plus c x square there is no it is not a function of x or instead of that if I take a plus bx square plus cy square, this is also an approximation which has 3 constants.

So, there are many possibilities as of now, because we do not know what are the conditions need to be satisfy while choosing a particular polynomial that condition also we will discuss at the last class of this course, ok; now.

So, this is u. So, now, what information we have? That u 1, so at x is equal to x 1 this is suppose coordinate of this is x 1 y 1 and this is x 2 y 2 and similarly here it is x 3 and y 3, right. Now, then what information we have? We have that at x is equal to at x 1 y 1 u is equal to u 1 x x 2 y 2 u is equal to u 2 x 3 y 3 u is equal to u 3 let us let us put that expression.

So, u 1 is equal to a plus bx 1 plus cy 1, and u 2 is equal to a plus bx 2 plus cy 2 and u 3 is equal to a plus bx 3 plus cy 3. Similarly we can do it for v let us not do it because it will be the similar exercise. Now, from this expression from this entire expression what we can write is, from these expression what we can have is we can have that the constants a b c, that constant a b c is essentially or we can write it like this u 1 or a no not that let us yes, ok.

So, we can have a matrix which is 1 x 1 x 1 y 1; then 1 x 2 y 2; 1 x 2 y 2 and then finally, finally, 1 x 3 y 3 that is into a b c, a b c is equal to u 1, u 2 and u 3, ok. So, if we solve this we know what is if we solve this equation then we can get a b c as a function of x 1 as a function of nodal coordinate and the nodal displacements, ok.

Now, once we have once we calculate this a b c and substitute that a b c in these expression a in these in these expression in its in equation number 1, say equation number this equation number 1. Then what will happen? We get an expression you please do that exercise we get an expression as u x y that is equal to your N 1 x 1 there is the same thing we have N 1 x 1 N 1 which is a function of x and y both, u 1 plus N 2 function of x y both, u 2 plus N 3 function of x y and u 3, ok.

And what would be N 1? What is the expression for N 1? The expression for N 1 will be I just give you the final expression for N 1 N i will be 1 by 2 delta. What is delta, I will tell you, then alpha i alpha i plus beta i x plus gamma i y, where delta is equal to delta, where delta is equal to the area of this triangle, ok.

Now, what is the area of this triangle? If you recall the area of this triangular will be half of the determinant of this determinant of 1 x 1, 1 x 1 y 1, 1 x 2 y 2, and 1 x 3 y 3. So, determinant of this essentially gives you delta.

So, this is delta and what is alpha 1, alpha 2, alpha 3? Then alpha 1 is equal to this gives you that your this is this is alpha 1 is equal to say alpha is equal to your alpha is equal to x 2 y 3 minus x 3 y 2. And then beta is equal to y 2 minus y 3 and gamma is equal to x 3 minus x 2, ok. So, this is the information we have, or it is alpha i you can put, essentially for all i this is same, ok. So, this is alpha beta gamma and this is your final expression of N, ok.

So, let us if you recall the safe functions you can do this exercise please. If you recall there is there are certain properties that safe function must satisfy the one property was the safe function satisfies the Kronecker delta property. Kronecker delta property means that the safe function, suppose if you are talking about N 1 then N 1 is 1 at the first node number 1, but all other node these values are 0. Similarly if you take N 2 then N 2 will be 1 as node number 2, but all other node these values will be 0. So, that is the Kronecker delta property let us see whether this is being satisfied or not.

So, you can quick you can sub state for instance N 1, for N 1 suppose N 1, if I calculate N 1, if I calculate N 1. So, N 1 will be 1 by 2 delta and if I if I do it for say if I do it for this is x 2 minus y 3, x 2 y 3 minus x 3 y 2 and then plus beta is y 2 minus y 3 y 2 minus y 3 and if it is if I am calculating at node 1 node 1 then this x will be x 1 and then plus gamma is x 3 minus x 2 and again this will be y 1. So, if I do that then you will see that this is essentially the determinant of this. So, these become 1, ok.

Now, if I do the same exercise if I calculate same N 1 let us do it here N 1, but node number node number 2. If I do that these becomes 1 by 2 delta, then x 2 y 3 minus x 3 y 2 then plus y 2 minus y 3 these become x 2 and then plus x 3 minus x 2 and these becomes y 2 and then please check what is this, ok. So, these values will be that x 2 x 2, x 2 y 2, these x 2 y 2 and these x 2 y 2 will get cancelled; y 3 x 2, y 3 x 2 and y 3 x 2 will get cancelled x 3 y 2 and x 3 y 2 this will also get cancelled this will be 0.

So, N 1 at node number 1 is 1, N 1 at node number 2 is 0; similarly you can have N 1 at node number 3 will also be 0. So, similar thing you can do to check whether this satisfy this Kronecker delta property or not. And then if we have to if I have to draw the draw the safe function the safe function will be suppose if I draw it if I draw it if it is node number 1, node number 2 and node number 3 then for node number 1 this safe function will be this is something like this, this will be the safe function. So, this is 1 and this is 0, this is 0, ok. Try to visualize in a three-dimensional space, ok.

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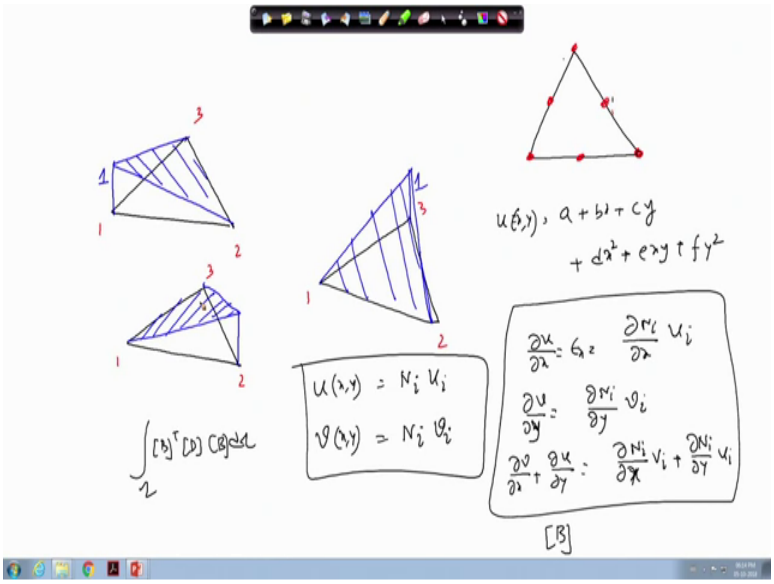

Similarly if I have to draw if I draw node same figure, if I draw if I draw for node number it is node number 1, 2, 3; 1, 2 and then 3 if I draw safe function for node number 2 this will be this and this, ok. So, this will be the safe function, this will be the safe function.

And similarly if I draw it for node number other node. So, this is node number 1, 2 and 3 for this node this will be it is 1, this is 0, and this will be 0. So, this will be this is the safe function, this is 1, ok.

So, it will be linear because our approximation was linear this approximation was linear this is a linear function, ok. But, for instance if your if your node is will not discuss that just to for the completeness of the discussion we have I am talking about suppose you are you are if you are using a 6 noded element. So, we have a node here, we have a node at 4 3 corners and similarly we have a node at the size.

So, in that case what will happen how many how many how many unknowns you have 1, 2, 3, 4, 5, 6. So, you can have an expression, you can have an expression for instance like this where u is equal to u x y is equal to a constant plus bx plus cy plus say dx square plus exy plus fy square. So, a b c d e f are 6 unknown you have, the values at the 6, 6 nodes you can get these corresponding safe functions.

In that case your safe function will not be linear, in that case your safe function will be quadratic, but still they satisfy that property at the at that particular node your safe function will be 1 and raised other nodes it will be 0 this satisfy that property, ok.

So, now, once we have calculated the safe function we are almost done. So, similarly we can. So, u is equal to now can be written as u at x y can be written as N i u i N i u i means summation over i, N 1 u 1, N 2 u 2 and N 3 u 3. Similarly v can be written as is equal to the same safe function we can use for v as well. So, this is how the displacement is approximated.

Now, once you approximate the displacement then, then how we can write the displacement vector if you look? So, displacement vector can be written as. So, once you have this we have to calculate this strain, strain can be obtained as suppose if I have to phi (Refer Time: 24:59) del u del x del u del x is the epsilon x. So, epsilon x can be obtained as del N i del x u into i, ok.

Similarly, del v del x can be obtained as del N i del v del y del y into v i, and then shear strain suppose del v del x plus del u del y. They can be obtained as del N i del y del N i del y del N i del x into v i plus del N i del y into d y and so on. So, we have; these gives us strain displacement relation.

Now, see so once you have the strain displacement relation, means from these expression we get what is the expression for B, from this relation we get the expression for B. Once we get the expression for B then we have to substitute that B in this B transpose D B and integrate it over d omega.

Now, in the case of beam we could integrate it in a close form, but many times it is may not be possible to integrate it because now it has to be integrated over the entire over the over a over a triangle, right. So, it depends on what is the size, what is the shape of the triangle. So, in many case it may not be possible to integrate them in a close form in that case we have to do integration in a in a integration numerically, ok.

So, you might have done a course on numerical methods and you studied there are many techniques for many numerical methods for integration, ok. Again just name of the methods we will discuss at the at the at the last day of this of this of this week, ok. So, this is how you can calculate B and the corresponding stiffness matrix as far as the load and load vector is concerned. Again it is the same way you can calculate the loaded load vector, ok.

Now, quickly as I was mentioning quickly just if I give if I if I give you the if I take you these another element, these 4 noded quadrilateral element and give you how to construct the safe functions of this element. So, this will be again quickly take you have a you again you this define x and y coordinate, x y coordinate and then we have a 4 noded quadrilateral suppose this and these are 4 nodes these are the 4 nodes.

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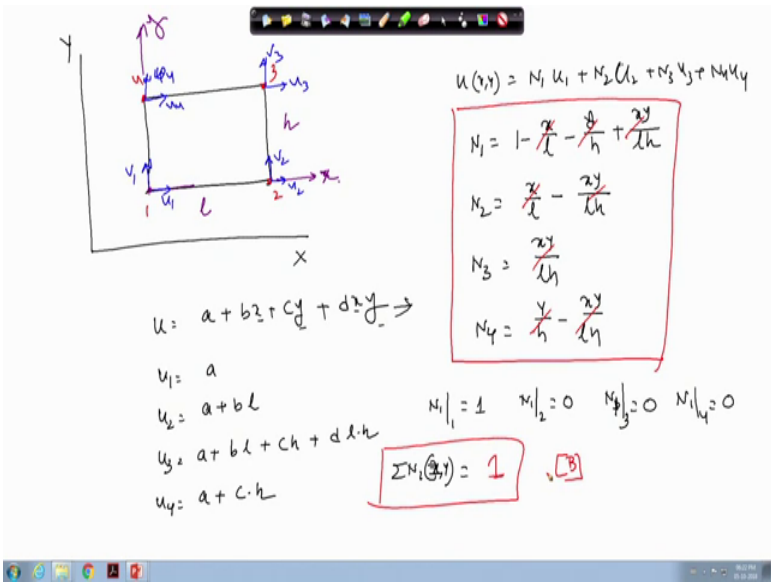

And then 4 nodes are named as suppose this is node number 1 node number 2 node number 3 and node number 4. And suppose the suppose this is your x local x coordinate this is your x and this is your y direction, small y and this is small x, ok.

So, node number 1 with respect to small x small y node number what coordinate is 0 and suppose this length is l, this length is l and this length is h, ok. So, node number 2 coordinate will be l and 0, node number 3 coordinate will be l and h and so on, ok. So, now at every node we have 2 degrees of freedom, here also 2 degrees of freedom u 1 and then v 1 and then similarly u 2 v 2 and u 3 v 3 u 4 and then v 4, right v 4.

So, if I the first step is to have an expression for u and v the field variables here the expression will be it is a two-dimension. So, expression will be say a plus bx plus cy plus this is very important.

The last term for the time being you take it for granted as I said what how many terms and what are the terms to be considered in their expression we will discuss in the subsequent week, but for the for the for the time being you take the last term for granted. So, d into x y, ok; it could be say d into x square it could be d into y square, but particularly we have taken d into x. We will discuss that why.

So, this is u. So, what are the information we have? We have the nodal values at the 4 nodes. So, u 1 will be a plus, now let us let us now you see the expression is written in terms of the local coordinate small x and small y. So, at node number 1, so essentially x 1 y 1 the coordinate will be 0. So, u 1 will be if you substitute 0 u 1 will be 1, u 1 will be a, ok. So, and then similarly then u 2 u 2 will be u 2 will be here x is equal to l, y is equal to 0. So, it will be a plus b into l and y is equal to 0; and u 3 will be it will be a plus b into l plus c into h plus d into l into h and finally, u 4 will be x will be 0 and y will be h, so a plus c into h. So, this part is 0. So, these are these are these are 4 equations we have.

Now, we have to find out these 4 equations. We have to find out we have to find out these constants a b, a b c d, 4 constants and then substitute that constant in these expression and if you if you do that then we get finally, x y is equal to N 1 u 1, plus N 2 u 2 plus N 3 u 3 plus n 4 u 4, ok. Similarly we can we can have an similar expression for v.

And what are the expression for N? The expression for N is I will just write the expression for N, N 1 will be N 1 will be 1 minus x by l minus y by h plus x y, x y by l into h. And then N 2 will be x by l, please verify these values x y by l h and N 3 is x y by x y by l h and finally, N 4 will be y by h minus x y by l into h. So, these are the values of this, ok.

Now, now you can quickly verify what is N 1, N 1 suppose say N 1 at node 1 is equal to x is equal to 0 y is equal to 0 this is become 1. N 1 at node 2 is x is equal to l y is equal to 0, if I substitute if I substitute x is equal to l and y is equal to 0 it is 1 minus 1 this becomes 0. Simply N 3 at sorry N 1 at node 3 become it is x is equal to l, y is equal to h, x is equal to l and y is equal to x is equal to l and y is equal to h if you substitute this becomes this is also become 0 and finally, N 1 at node 4 this becomes this is also become 0.

So, similarly you can see that these satisfy the Kronecker delta property. Now, not only that there is another property called partition of unity property at any intermediate point if you sum the all the safe functions these safe functions will be 1. So, you sum them all these safe functions at any point, you will see this will be 1.

So, let us do this exercise: If we take summation of N i summation of N i, so you sum them at any arbitrary point at any arbitrary point. If you sum them then what will happen? So, these x l these x l will get canceled, this will get cancelled, this will get canceled and this will get cancelled. So, sum of all the N i is equal to 1 always, ok. So, by construction itself it is you do not have to do that exercise here itself it is 1, ok. So, this is a partition of unity property that is an important property.

Now, quickly; so once we have n the next is we have to calculate this strain, strain is a function we can differentiate this expression with respect to the coordinate axis we get the corresponding strains. And once we have this strain we get this strain displacement relation that is B. So, once we have B, D is already discussed what is the expression for D for plane stress and plane strain problem. So, once we have B substitute that in the expression we get the stiffness matrix for a 4 noded corollary element.

Now, quickly if I have to a quickly let us let us see what is the shape of this shape of this of the of the safe functions and a these are the shape of the safe function, ok.

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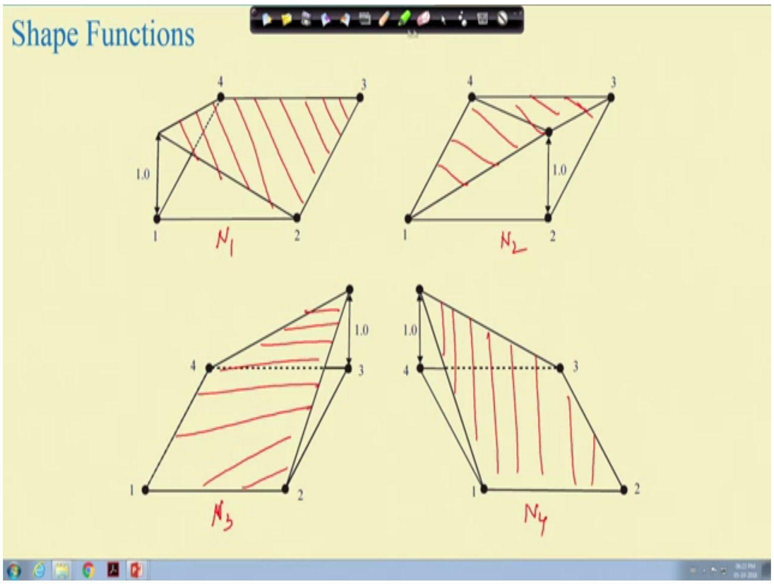

So, this is these are the so this is this is one and all the places it is 0. So, your safe function is essentially this. And safe function is linear not linear it is actually bilinear, linear in x and linear in y both. So, it is bilinear.

So, again it is it is for N 3. So, this is for N 1, this is for N 2 this is for N 3 and finally, this is for N 4, ok. So, this is the safe function. And this is the safe function, just a plane, and finally, this will be the safe function what is that yes, yes. So, this will be the all are 0. So, this will be the safe function, this will be safe function.

So, this is how we can; now if we have a 6 noded if we want to if you want to have this kind of element suppose element of this kind, if we want to have a element of this kind set h noded element, the same exercise you have to do we have to start with an approximation of u. And then find out and substitute the boundary conditions at every point and get an expression for N. Once we get N we can calculate the stiffness matrix subsequently, ok.

Next class what we discuss is next class will, again the similar to 1D problem beam problem we translate the 2D beam element, 2D formulation into a code and solve on 2D problem. And then, through that solution we will try to understand some of the some more important aspects of finite element method. So, I stop here today. See you in the next class.

Thank you.