Matrix Method of Structural Analysis
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Lecture – 39
Introduction to the Finite Element Method (Contd.)

Hello everyone. This is the 4th lecture of this week. In the last class we derived this stiffness matrix for beam element, and then today we will see how to derive the stiffness, how to derive the load vectors for beam element and then subsequently solve a problem with finite element method, ok.

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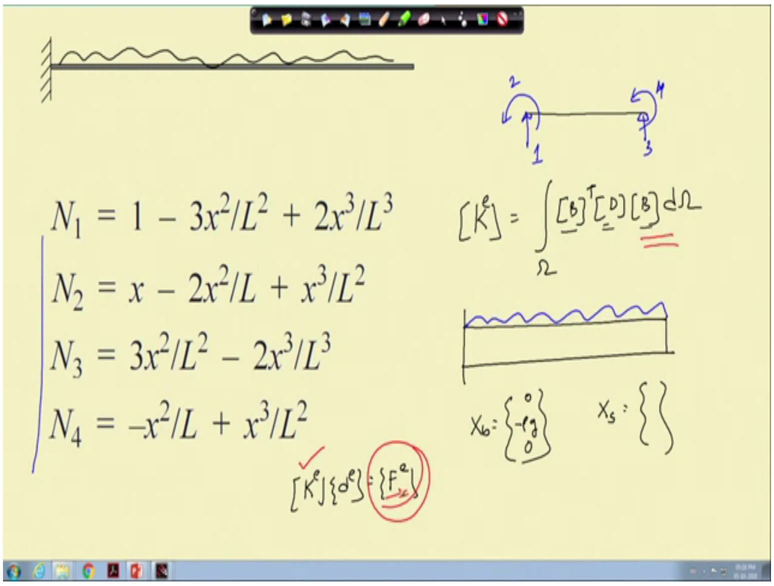

So, if you recall that was the, these are these are the same functions of a beam element. If you have a beam element like this beam element any arbitrary beam element and then if the corresponding if the degrees of freedoms of degrees of freedom are verticle translation is this, if we take a beam element like this and these are the corresponding degrees of freedom and then. So, this is 1, 2, 3, 4 and then step function associated with all these degrees of freedom 1, 2, 3, 4 are these that we derived in the last class, ok.

Then we also know that the stiffness matrix the general form of stiffness matrix element stiffness matrix rather that is equal to integration over the entire domain that depends on what kind of what kind of object you have whether it is integration over the volume, or area or line it depends on what kind of elements we are talking about. So, this is B transpose, D and B, right and D omega. And how we derive this equation? We derive by minimizing the, it is strain energy, right.

Now, so here B is the strain displacement relation and D is the constitutive relation which tells you how the stresses are related to strain. And I asked you to please draw these step functions and then and then see how these functions what and also to understand what are the properties of these safe functions, ok, great.

Now, let us find out what is the; suppose if we consider beam like this and it is subjected to say uniformly distributed load, uniformly distributed load like this. So, in order to solve, this is the stiffness matrix along with the stiffness matrix we have to find out the load vector and in matrix method we know how to do that. Let us see how to do that in finite element formulation, ok.

Now, before that there are two kinds of load that an object can have, one is the body force body force is one example of body force is say gravity. So, suppose if you have a beam like this, a beam and then the self weight of the beam is essentially the body force, ok.

And suppose that is represented by X b, ok. So, X b essentially is a vector. So, it tells you in which direction the body force is acting. For instance in this case, if we take the only the gravity then X b will be acting downward and this value will be rho g. So, this is this is minus rho g this is 0, this is 0, so this is X b in 3 dimensional case. So, it depends on the, what body force you have.

And in addition to the body force there is another force which is called surface force surface force essentially for instance if this, this beam has a load like this uniformly distributed or varying load it is acting on the surface, ok, or acting when I say surface it is essentially acting on the boundary of this domain. So, this is called surface load.

So, similarly surface load surface force is also suppose represented as s and this is also vector, ok. It gives you surface force in which direction, out reaction in which direction. Now, what we want to find out is we want that we know that for at the element level at the element level the K e is equal to K e into K e into say D e, D e is the displacement associated to for the particular element the degrees of freedom for a particular element that should be equal to F e, F e is the element load vector, for that particular element. So, we have already seen how to do this how to express this, this is the expression for this; D essentially at the unknown. So, you do not have to bother that, right. Now, we will now see what is how to express this.

Now, in order to do that we have to go back to the same the starting point how we derived these expression, we follow the same approach to derive the expression for F e. Let us do that.

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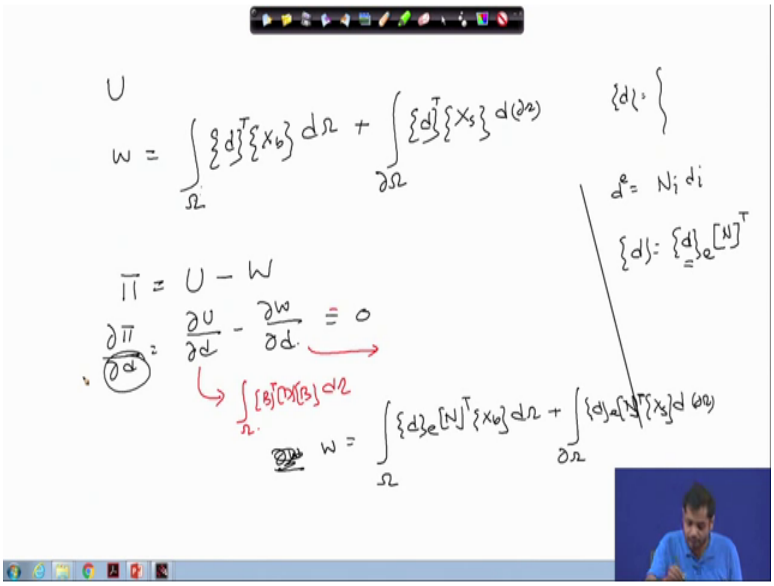

So, if you recall what was our approach you see I suppose if you our approach was we use U e, U was the strain energy suppose U e is the strain energy associated with a particular element, ok, so U e. So, we minimize, so we minimize U e. Let us forget about U. So, the U minimize U, right U can be written as after minimizing U minimizing U means differentiating this strain energy with respect to, with respect to displacement we get force and that from that from that from that from that step we got that expression for stiffness matrix.

Now, suppose a body is there is some load acting on a body. So, if there is some load acting on a body then that that load is some work done some work is done by that load. And what is that load? Suppose that is W p, W that is W, ok. So, W will be what? W will be it is the work done by the by the surface load and as well as it is the work done by the body force, right. So, what would be the contribution of the body force? Contribution of the body force will be since it is acting on the acting on the volume of the body let us, ok.

Let us write the, it is acting on the entire domain omega and then the X b is the suppose displacement is D displacement is D multiplied by the force is X b, ok. And then it is integrated over the entire domain that is the contribution from the, that is the contribution from the body force, right.

Now, similarly we have a contribution from the surface load, but the surface load is reacting on the boundary of this boundary of the domain. So, that integration will be has to be over the boundary note on the on the domain. For instance if you if you have a 3 D object then these, the first is integration over the volume and the second is integration over the surface, ok. So, this will be again D and then this is multiplied by X s, X s is the corresponding displacement, corresponding body force and this is integration over say d s or d del omega, right, ok. Now, this in order to be consistent this will be transpose. So, that this becomes a scalar. So, this is the expression for expression for the work done.

Now, what we know is, we know that pi is equal to the potential energy the potential energy is equal to the strain energy minus W, ok. Now, while considering the now then, so now before that; so d is equal to what? d is equal to the this vector is equal to you have a component x direction component y duration and component z direction let us use, fine.

Now, then, so when we minimize it when we minimize this pi with respect to the we respect to the displacement, then we from this term the del W del d term we get the stiffness matrix then we have. So, if I if I differentiate with respect to say del pi del d. So, this will give you del U del d del d minus del W del d, ok.

So, this gave us this gave us the integration B transpose D B, right which was the stiffness matrix d omega this was stiffness matrix. And that has to be equal to 0 that is the that has to be equal to 0. Now, so for the minimum potential energy, so this gives us the stiffness matrix and this part the this part will give us the load vector.

Let us see what is now if we now these has two components, one component is this, another component is this, another component is this. Now, you recall that the how these displacements are how these displacements are expressed. Displacements are expressed using like this if you recall, that if we have suppose if d for any element for element d this is expressed as the N i d i, right N 1 d 1 plus N 1 the d 2 and so on through the safe function.

So, naturally, so these displacement these vector these vector we can write as like this if d is equal to, so this vector this is equal to we can have we can have a displacement which is at the element level displacement at this say element level and then we have a matrix for N, that is transpose. So, this is the, this is how, so this d e essentially at the nodal values of the displacement, right. So, if I substitute that here in this expression in this expression. So, what we get? We get that we get two terms, one term will be. So, let us consider only these parts separately.

So, this one term will be one term will be let us write it here. So, del W del W or del first let us write W will be what W will be integration of omega and then this is d e and then N transpose which is only for this displacement part. And then we have X b and then integration over the entire domain and then plus we have integration only on the boundary and these displacement is written as d e and then N transpose, N transpose and then we have X s and then d del omega, ok. This is we have.

Now, if we now this is for W. Now, if we differentiate it with respect to del (( )), then what will happen? Then we have if we differentiate it with W, then we have del W, del W del d will be integration over the domain then we have N transpose the d term will go and X b which is the body force term and then d omega and then plus integration over the boundary. Then N transpose and then x the boundary term, right and then d del omega, ok.

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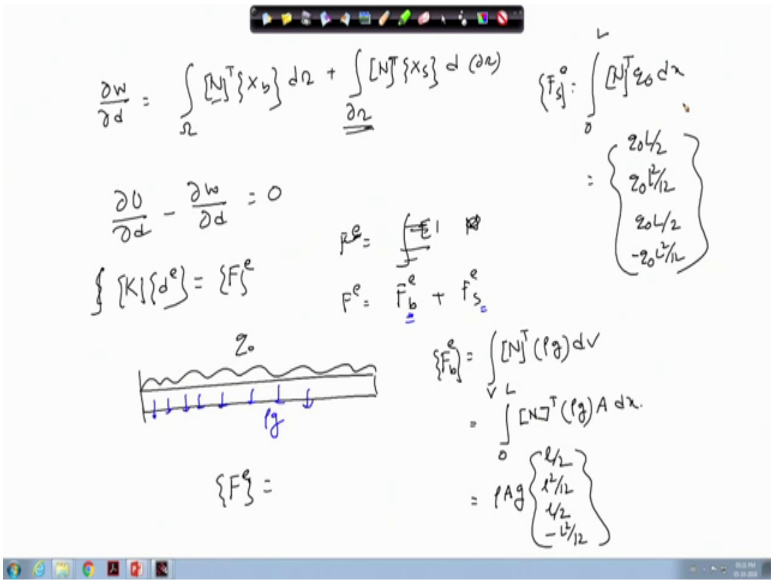

Now, this is this is we have, right. So, this is so, we have then del we have del U del d minus del W del d that is equal to 0. So, this is the stiffness matrix as I we have already seen this is the stiffness matrix. This part will gives us what? This parts will gives us K into d e, and this part will give us load vector you can take it out take it other side and this load vector will be F e. And this F e is F e is this integration of N transpose or this F e is two part, one is f this F e has two parts, say F e is one is F e B body force term plus F e s which is the surface force surface part.

Now, for a given element, so we know the, we know the safe functions and then if we substitute safe functions and integrate it over the domain over the entire body or over the in the in the case of beam over the length of the beam we get the corresponding forces. Just to give you an example, suppose consider this same example consider this example where you have you have a cantilever beam which is subjected to say subjected to uniformly distributed load and suppose the uniformly distributed load intensity is q, q 0 is. And in addition to that we have body force term and the body force term is suppose this is the body force term which is downward acting which is rho g the gravity term.

So, we have to find out what is the so what is F b term and what is F s term. So, what will be F b term? F b term will be if we, so F b will be F b e element level that is equal to integration over the volume over the volume and then N transpose, N transpose then rho g rho g and then d v this will be this part.

Now, this integration can be written as again integration if the uniform cross section is uniform. So, this can be written as integration over length 0 to L. And then N transpose N transpose rho g rho g and then A into d x, ok. We can take A outside.

Now, we already know what is the expression for N, these are the these are the expression for these are the expression for N, right. These are the expression for these are the expression for N, and if we substitute that in a in that what is N then if when we write N is equal to when we write; when we write N, N is equal to is essentially N 1, N 2, N 3 and N 4, right.

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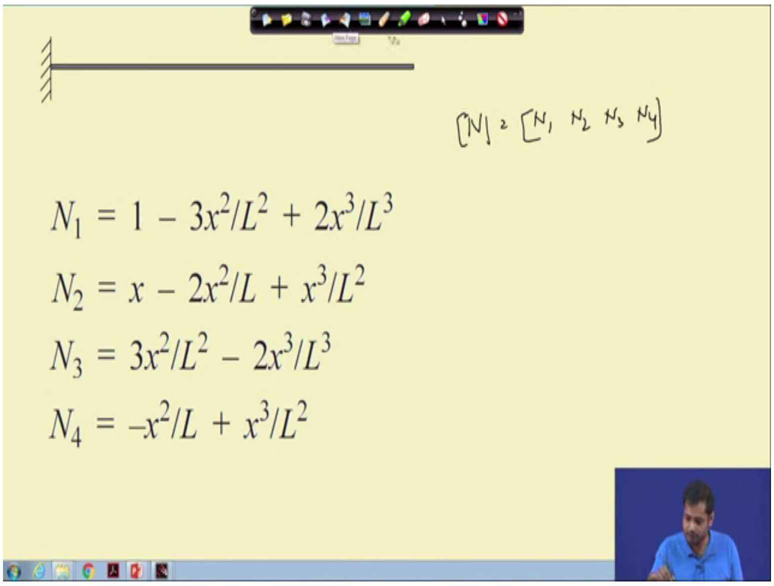

Now, if we substitute this if we substitute this now in this expression in this expression and then integrate it over the length we get this is equal to rho A g, rho A g and then this term will be L by 2 and then L square by 12 you can check it and then L by 2, L by 2 and then minus L square by 12. So, for this beam this contribution will be this, ok.

So, in addition to that we have one more contribution that is F s term. Then what will be F s term? Then F s term will be similarly we can calculate, F s will be F s element level that will be integration here it is acting on the as I said this is this will this integration will be over the surface in this case this integration will be over the length, over the length where on which these forces force is acting. So, 0 to L and then we have N transpose N transpose and q 0 and then d x, right

Now, we already know what is the expression for N 0 for beam element we can substitute that and if we substitute that then this will be integration q 0 L by 2 and then q 0 L square by 12, if you recall q 0 L by 2 again and minus q 0 L square by 12, ok.

Now, if you recall in the matrix method of structural analysis we first calculated the fixed end moments. And from the fixed end moments at every joints we satisfy the equilibrium to get the nodal load vector and if you recall for a uniformly distributed load the fixed end actions were essentially this, is not it. But here we can obtain we do not have to bother about what is the fixed end moments and what satisfy the equilibrium just by this expression the entire thing minimization of potential energy will give us both the stiffness term as well as the force term, ok.

So, then in this case, so what will be the total force? The total force will be F e our total force term will be this term plus this term, ok. Now, so for a given element we have the, we already have the stiffness part and we also have the loading part, right. Now, we have not yet discussed if there is a concentrated load then what is to be done will come to that point shortly.

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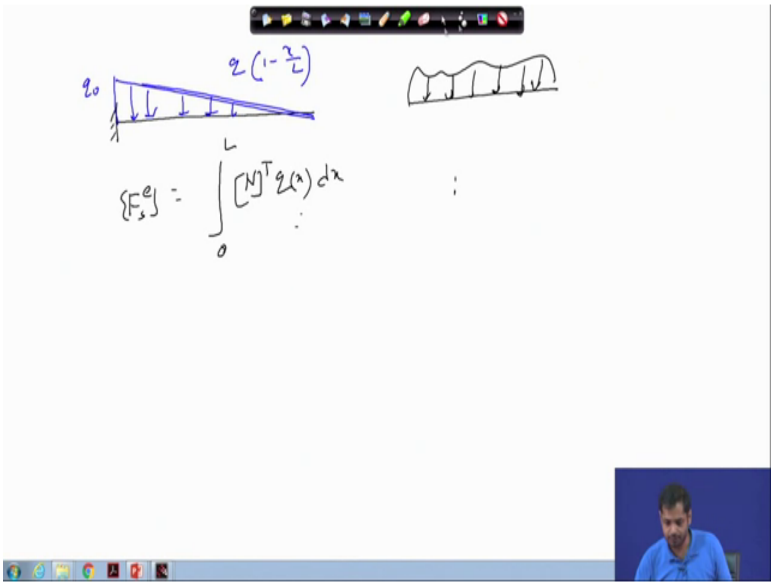

It is for uniformly distributed load; now suppose if your load is not uniformly distributed for instance if the load is like this. So, if we take a beam if we, if you take a beam and the load is suppose if we take a cantilever beam or any beam if the load is something like this uniformly distributed it is like this, this is like this. So, what is the load? This q 0 is, q is equal to essentially. So, this is q 0 this is q 0. So, q is essentially your, it is it is 1 minus x by L, right.

Now, if q is this then what will be F s, F s will be the body force the force F e term the F s term the because surface load this is integration of if you recall 0 to L, 0 to L N transpose N transpose into x into q, but then q is a function of x. So, this has to be written as a function of x and then d x. You substitute that function and then integrate it we get the corresponding vector.

So, we can have any orientation any, we can have any arbitrary orientation of arbitrary distribution of load on the on the on the beam element. But if we know that distribution if we know that function we can substitute that function in this expression and get the corresponding load vector, ok. This load vector is called consistent load vector, ok. Now, once we have obtained the stiffness matrix at the element level and also the load vector at the element level.

Next we have to do is we have to assemble this stiffness matrixes not only the stiffness matrixes we have to assemble the nodal load vector as well exactly the same way we assemble the stiffness matrix, ok, but now it is just in a vector form. So, once we assemble them what we get is we get a global stiffness matrix and the global load vector then you have to do partitioning. And once we do the partitioning we with reduce stiffness matrix applying the, with the information of boundary condition we reduce the stiffness matrix and calculate the calculate the unknown displacement, right.

So, let us take one example and then and then we have we have translated the entire process the way we have done it for matrix method. We have translated the entire process into a program. I am just going to go in I am just going to show you that code and just we taken one example and then see how the various steps and the results what we get and we compare these results with other methods, ok. So, take an example of a cantilever beam cantilever beam subjected to say for instance, ok. So, first let us find out the code. So, the code is yes, ok.

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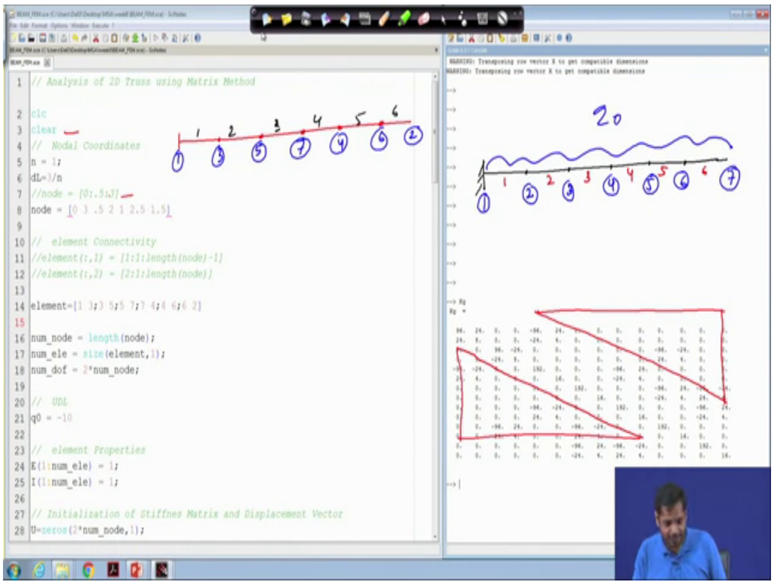

So, this is the code. So, consider a problem like this problem we are considering is this, ok. Take a beam, cantilever beam then it is subjected to uniformly distributed load we will come to the point later. Suppose the length of the beam is suppose the length of the beam is 3 meter, entire length is 3 meter, ok.

Now, let us divide that a beam into say 6 segments. So, we have just 1 point, 2, 3, 4, 5 and then 6 we call in matrix method. We the 1, you see it is a cantilever beam we have just one support at the end now in recalling matrix method how we discretize discretization is an important step in both the cases.

The matrix method also we discretize the entire structure here also you have to discretize the entire structure. But when we discretize in a matrix method of structural analysis we took the segments between the between the between the two supports as one segment as one member, but not necessarily here in finite element method you have to consider that as one member that is why we do not call it member it is an element. Even just one span of the beam one member of the beam can be divided into several parts. For instance this is just one beam, but it can be divided into several parts and what is the consequence of that we will we will come to that shortly.

So, there are how many suppose we start with we give numbering like this. So, this is node number 1, this is node number 1, this is node number 2, 3, 4, 5, 6, and then 7, right

7. And these are the elements number corresponding elements number 1, 2, 3, 4 5 6, now what happens is you see.

So, first we have to give the connectivity of nodal information. So, node coordinates are this is the entire length is 3. So, that is equally divided into 6 parts. So, whatever length of the length of each element we can obtain. But again not necessarily that all the elements all the elements have to be of same length we can have elements with different lengths as well, for here for the demonstration purpose let us take all the elements having same length, ok.

So, these are the elements now we have element connectivity. So, these gives you the node let us slightly increase the font size, yes that is makes that is better; why not, yes. So, this gives the, this is the node number. So, we have node different nodes and this is the element connectivity. So, node 1 is connected between, node 1 element 1 is connected between node 1 and node 2, ok. Now, then the load is here load is q 0 it is uniformly distributed load, we have uniformly distributed load like this. And suppose this value is q 0. So, this is q 0 here. So, this is q 0.

So, EI is again you can for the demonstration we are taking is one you can change the values of e 1, e 1 for the same element we can have for the same member you can have different e is, ok. So, these are almost same as matrix method this is the stiffness matrix element stiffness matrix and recall this is the load nodal load vector the consistent load vector that is just now we derived this is there is no body force here. So, body for there is no body force term it is only the surface load term.

Now, this is the assembling of the load vector the assembling of the stiffness matrix, but we have to assemble the load vector as well this is the assembling of the load vector. So, finally, once we have this then this is the K 11 is a partitioning of the matrix that we already discuss in matrix in the in the in previous classes. And then finally, we are solving for unknown, once you solve for unknown then we can do we can obtain the support reactions and so on. So, let us find out the solution of this. So, solution is this of this will be this.

So, let us run it. So, we run it here forget about this blue line because these are the red one is the, red one is the actual length actual deformation of this of this of this beam and the blue dots you can see these are the obtained from the finite element solution. The blue line is not important here because blue line just those points joining by the line, but that is not the real representation of solution. The real representation of solution is this is the red one is the actual solution and the dots you can see these are the obtained these are obtained from obtained from the obtained from the from the analysis, ok.

Similarly, you can increase the increase the size say for instance you can increase the number of point, you can make 0.5 and then run it and then we get then we get different solutions. So, you can see the solutions are close to what is the area between these solutions the predicted results and the actual (Refer Time: 27:07) let us not bother about, right now. We will discuss the nature of the error and how to how to how to how to how to measure that error, how to represent that error and what are the what are the difference way that we can minimize this error those we will discuss in later in the subsequent classes, ok.

So, you, but you see the reason why you could draw the draw the deformed shape because in even in one element, one beam you have several such segments. It is just not in matrix method of analysis when you solve it you we had the we had the displacement at the nodes displacement at the supports, right. But here just not the displacement at the support we can divide the beam as many times as many as many as many with as many number of elements we want and that every discretization we get the intermediate values of the displacement even in a single member. So, that so we can draw the, deform a deformation, ok.

Now, one important point let us let us see the let us see the, what is the what is the Kg what is the, this is the stiffness matrix of this element, ok.

Now, you see this is an interesting observation here. The stiffness matrix you see you recall some time backs we are talking about band deadness in a stiffness matrix all these elements in the stiffness matrix is 0, this is 0 and then this is also 0, this is 0. So, only nonzero, nonzero elements in the stiffness matrix is the off diagonal term and the sorry diagonal term and this few of diagonal terms, ok.

Now, so the advantage before advantage of having this is when you when you have a bandage stiffness matrix there is some of the some of the when you actually try to solve it at the end of the day we have to solve some system of equations linear equations. That solution process the operations while solving these equations are much lesser. Now, suppose you take a take an example of the same beam, but let us we discretize slightly different way. For instance suppose this is the beam again once again you take this beam and this is divided into 7 6 parts 1, 2, 3, 4, 5, 6 these are the element number one element number two element number 3, element number 4, element 5 and element 6, ok.

Suppose the node numbers are these node is node number 1, these node is node number 3, these node is node number 5, this is node number 7. If I give node number arbitrarily node number 4, node number 6 and then finally, node number 2, node number 2, ok. So, here the node numbers are in a in a sequence in a in a particular order, but here we have given the node number arbitrary. Then let us see what happens if the node numbers are like this, ok.

So, this is the node number. So, this is the different nodes this is the different nodes and then corresponding connectivity is this. So, here element, element 1 is connected between 1 and 3 connected between 1 and 3, element 2 is between 3 and 4, 3 and 5 between 3 and 5; 3 is between 5 and 7 between 5 and 7 and so on, ok.

Let us solve it. So, f 5 again forgets about the blue line blue lines. So, if we see the solution solutions a solution will be 7 it does not depend on the, your node number here in this case. But what is the important thing you see that is this one. Let us let us let us see what is the stiffness matrix. This is the stiffness matrix.

You see these stiffness matrix if no longer banded stiffness matrix. If this is no longer a bandage stiffness matrix you see somewhere in some of the elements at 0 here, here some of the elements at 0, then we have some of the 0, means 0 here, some are 0 here, and then a some are 0 here some are 0 here. So, this stiffness matrix is not banded.

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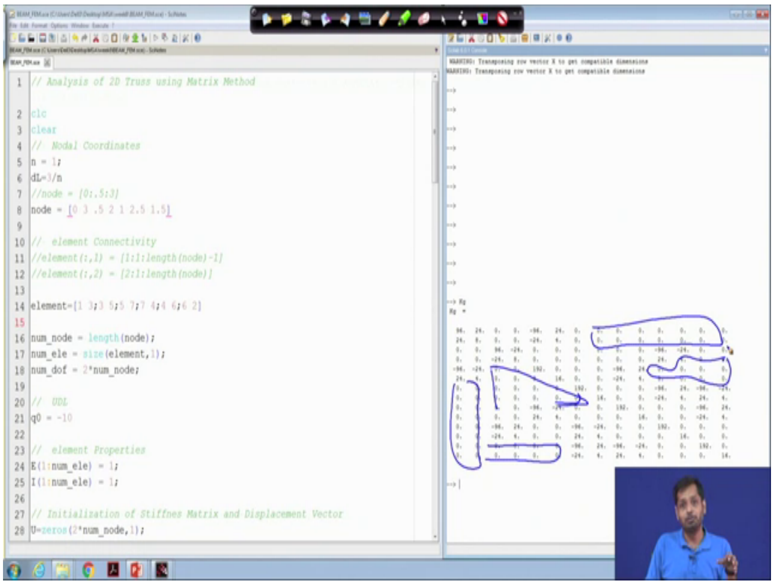

So, how you it is just not the discretization just that does not just mean that you have a structure and divided into parts. How you divide into parts? How you number them your computation, the intensive the intensiveness of the computation? How many numerical operations that you have to perform that significantly depends on these aspects, ok.

So, when you when you when you do this, when you when you write your own code then please keep that in mind and there are literature which tells you how what is the best possible way to number elements in which order; so, that you can get a banded stiffness matrix, ok.

So, this is the solution of solution of beam. Now, you see the beam is one structural component. So, we have discussed how to what is the formulation for truss, what is the formulation for beam. But then next class we will see if our problem is a two dimensional problem, domain is a two dimensional domain then what would be the formulation, what were the different kinds of elements that we have to use to discretize those two dimensional object. For truss and beam the advantage of truss and beam is you directly can correlate with the matrix method the, that you studied in matrix method of the analysis, because you did this similar thing in matrix method in your structural analysis direct stiffness matrix, direct stiffness method.

So, you could correlate between what you learnt what we discussed there and this finite element formulation for truss and beam. But then next class we move we see what is the formulation for 2D element and that will be probably something new that you have not yet done in your previous structural analysis courses.

Now,, but there is a sense of these steps will remain same we get the stiffness matrix the general from the stiffness matrix will be same, general from the load vectors will be same. But the since it is in different dimensions our different the B matrix or the D matrix those are will be different. That exercise will do in the next class. So, I stop here today. See you in the next class.

Thank you.